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Stability of capillary free surfaces of viscous incompressible fluid

庵原隆雄 (大阪大学 理学研究科)
Takao Iohara
Department of Mathematics
Graduate School of Science
Osaka University
Osaka 560-0045, Japan

1 Introduction

We consider the motion of viscous incompressible fluid moving in some region $O \subset \mathbb{R}^N$ with free surfaces. The motion can be described by the Navier–Stokes equations on the time-dependent fluid domain $\Omega(t) \subset O$ with certain boundary conditions and an equation describing the motion of free surfaces. We show existence of solution for initial data close to a (not necessarily flat) stationary free surface which restores exponentially to the stationary solution.

2 Governing equations

We consider a mass of fluid moving in a domain $O \subset \mathbb{R}^d$ under the effect of potential force $F = -\nabla V$. We assume that the boundary of O and the given potential V are smooth. Its physical state at some instance t is represented by the domain $\Omega(t) \subset O$ occupied by fluid and the velocity vector field u defined on $\Omega(t)$. We assume that the fluid is incompressible and viscous and its surface has surface tension.

We consider the usual Navier–Stokes equations

$$u_t + u \cdot \nabla u + \operatorname{div} \mathsf{T} = F, \ \operatorname{div} u = 0 \ \text{ in } \ \Omega(t), \tag{2.1}$$

where the bulk stress are given by $\mathsf{T} = p\mathsf{I} - 2\nu\mathsf{D}(u)$. Here, $\mathsf{D}(u) = \frac{1}{2}(\nabla u + {}^t\nabla u)$ is the deformation tensor of u and $\nu > 0$ is the viscosity coefficient of the fluid.

The boundary of the fluid consists of two parts, $\partial\Omega(t) = B(t) \cup \Gamma(t)$, where $B(t) := \partial\Omega(t) \cap \partial O$ and $\Gamma(t) := \partial\Omega(t) \cap O$. On the part $B(t)$ which is contained in the boundary of container bounding the fluid, we require the slip condition, $u \cdot \vec{n}|_{B(t)} = 0$ and $\vec{n} \cdot \mathsf{T} \cdot (\mathsf{I} - \vec{n} \otimes \vec{n})|_{B(t)} = 0$.

On the moving part $\Gamma(t)$, we consider the stress balance

$$\mathsf{T}\cdot\vec{n} - p_{\text{air}}\vec{n} = \sigma H\vec{n}. \tag{2.2}$$

Here, p_{air} is the constant of the pressure of the surrounding air, $\sigma > 0$ is the surface tension coefficient, which we assume to be constant and $\vec{n}$ and H is the outer normal unit vector and the sum of the principal curvature ($= (d-1)\times$ the mean curvature of $\Gamma(t)$). Furthermore, we require that the normal vector $\vec{n}$ at the points in $\partial O \cap \overline{\Gamma(t)}$ is tangent to ∂O.

The motion of $\Gamma(t)$ is described by the kinetic boundary condition

$$(\text{normal speed of } \Gamma(t)) = u\cdot\vec{n}|_{\Gamma(t)}. \tag{2.3}$$

This equation is just an expression of mass conservation for incompressible fluid.

We need to supplement these set of equations with initial conditions $\Omega(0) = \Omega_0$ and $u|_{t=0} = u_0$ on Ω_0. From the momentum conservation in $\Omega(t)$ and the stress balance on $\Gamma(t)$, by using integration by part formula

$$\int_\Gamma \sigma H\Phi\cdot\vec{n}dA + \int_{\overline{\Gamma}\cap\partial O} \sigma(\Phi\cdot\vec{n})\vec{n}\cdot\vec{n}_{\partial O} = \int_\Gamma \sigma(\mathsf{I} - \vec{n}\otimes\vec{n}) : \nabla\,\Phi dA, \tag{2.4}$$

we obtain

$$\begin{aligned} 0 &= \int_0^\infty dt \int_{\Omega(t)} -u\cdot\Phi_t - u\otimes u : \nabla\,\Phi + 2\nu\mathsf{D}(u) : \nabla\,\Phi dx \\ &\quad + \int_0^\infty dt \int_{\Gamma(t)} V\Phi\cdot\vec{n} + \sigma(\mathsf{I} - \vec{n}\otimes\vec{n}) : \nabla\,\Phi dA + \int_{\Omega_0} u_0\cdot\Phi|_{t=0}dx \qquad (2.5) \\ &=: \int_0^\infty (I_{\text{bulk}} + I_{\text{surface}})dt + I_{\text{initial}} \end{aligned}$$

where $\Phi(x,t)$ is an arbitrary divergence free vector field defined in O which satisfies $\Phi\cdot\vec{n}|_B = 0$.

We assume that u satisfies $\operatorname{div} u = 0$ and $u\cdot\vec{n}|_B = 0$. Then, the equation (2.5) is equivalent, for sufficiently regular $\Omega(t)$ and u, to (2.1) and (2.2). We consider the existence of solution of this set of equations for a given initial condition (Ω_0, u_0).

This set of equations with $O = \mathbb{R}^d$, $V = 0$ and Ω_0 close to the sphere was studied in [So], and he obtained a global existence result for small initial conditions. Another study was [B], which consider the case of horizontally infinite free surface $O = \{x_d > b(x_1, \cdots, x_{d-1})\}$, $V = -x_d$ and Ω_0 close to $\{x_d < 0\}$ and obtained some result on the existence. For the horizontally periodic case, global existence result can be proved (see [I]). Our result is a generalization of this result.

Remark . *Due to incompressibility of the fluid, the volume of $\Omega(t)$ is a constant of motion of our equations. Energy equality*

$$\frac{d}{dt}\Big(\int_{\Omega(t)} \frac{u^2}{2}dx + \int_{\Omega(t)} Vdx + \int_{\Gamma(t)} \sigma dA\Big) + \int_{\Omega(t)} 2\nu\mathsf{D} : \mathsf{D}dx = 0$$

can be obtained by substituting $\Phi = u$ to (2.5).

3 Stationary solutions

The equation for the stationary solution without fluid motion($u = 0$) reads

$$0 = I_{\text{surface}}[\Omega; \varphi] = \int_{\Gamma} V\varphi \cdot \vec{n} + \sigma(\mathrm{I} - \vec{n}\otimes\vec{n}) : \nabla \varphi dA$$

where φ is an arbitrary vector field defined on O satisfying $\operatorname{div}\varphi = 0 \operatorname{in} O$ and $\varphi \cdot \vec{n}|_B = 0$. In this section, we consider a solution $\Omega = \Omega_s$ of this equation.

Using (2.4), we obtain

$$0 = \int_{\Gamma_s} \varphi \cdot \vec{n}(V + \sigma H)dA + \int_{\overline{\Gamma_s}\cap\partial O} \sigma(\varphi \cdot \vec{n})\vec{n} \cdot \vec{n}_{\partial O}.$$

Since $\varphi \cdot \vec{n}|_{\Gamma_s}$ can be any function on Γ_s with vanishing average, we obtain that potential force and surface tension must be balanced

$$V + \sigma H = \text{const. on } \Gamma_s$$

and that Γ_s and ∂O must meet at right angles. This problem is known as the capillary surface problem, which is extensively investigated in [F]. (When the potential is absent ($V \equiv 0$), a stationary surface Γ_s is a hypersurface with constant mean curvature.)

Capillary surface problem can be formulated as the variational problem for the energy functional

$$E_{\text{surface}}[\Omega] = \int_{\Gamma} \sigma dA + \int_{\Omega} V dx$$

in $\{\Omega \subset O : \text{the volume of } \Omega \text{ is prescribed}\}$ as stated in the following proposition.

Proposition 3.1. *Let Ω be a domain in O with finite perimeter and φ be a C^1-vector field satisfying $\varphi \cdot \vec{n}|_{\partial O} = 0$. We define deformation Ω^ε of Ω by φ by $\Omega^\varepsilon = X_\varepsilon(\Omega)$ where X_ε is the flow map generated by φ. Then,*

$$I_{\text{surface}}[\Omega; \varphi] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{\text{surface}}[\Omega^\varepsilon].$$

From this proposition, $I_{\text{surface}}[\Omega_s; \cdot] \equiv 0$ is equivalent to its stationarity with respect to the energy functional E_{surface} under the constraint of volume prescription and, in particular, a region with minimal energy with prescribed volume is stationary.

In the following, we assume that Ω_s is smooth and bounded. Stability of a solution of a variational problem can be examined by investigating the second variation of the functional at the solution. For E_{surface}, the second variation at a capillary surface Ω_s can be expressed as([Si])

$$\delta^2 E_{\text{surface}}[\Omega_s; \psi, \psi] = \int_{\Gamma_s} (\sigma| \nabla_{\Gamma_s} \psi|^2 + (-\sigma S(x) + \vec{n} \cdot \nabla V)|\psi|^2)dA =: b(\psi, \psi),$$

where ψ, a function defined on Γ_s, represents the infinitesimal normal variation of Γ_s and $S(x)$ is the sum of the square of the curvature of Γ_s. We assume that $b(\cdot, \cdot)$ is positive definite on $\dot{H}^1(\Gamma_s)$. We refer to this as geometrical stability. We note that, for smooth Ω_s, the positive definiteness of $b(\cdot,\cdot)$ implies the local minimality of Γ_s with respect to E_{surface}.

Our result is that, under some additional assumption on asymmetry of Ω_s, geometrical stability of a capillary surface implies its linear stability and then its nonlinear local stability as a stationary solution of our fluid mechanical system.

Assumption. B is nonempty and is not contained in a hypersurface with translational or rotational symmetries.

Under this assumption, Korn's form

$$\langle u, \varphi\rangle = \int_{\Omega_s} 2\nu \mathsf{D}(u) : \mathsf{D}(\varphi)dx$$

is positive definite on $H^1_\sigma(\Omega_s) = \{u \in H^1 : \operatorname{div} u = 0, u \cdot \vec{n}|_B = 0\}$ ([SS]).

The above assumption excludes, for example, (a) $O = \mathbb{R}^N$, Ω_s is a sphere, and (b) $O = \{x_d > -1\}$, $\Omega_s = \{0 > x_d > -1\}$. In such cases, uniform translations or rigid rotations obviously violate Korn's inequality.

4 Reduction to a problem on a fixed domain

In this section, we describe the procedure of reducing our moving boundary problem to a problem on a fixed domain, which is basically a generalization of that used in [B] with some modifications. A major difference to [B] is that we directly work in integral formulation of equation without getting back to the local differential equations.

First, we will choose a coordinate (ξ_h, ξ_n) on a tabular neighborhood of Γ_s and write $\Omega(t)$ as $\{\xi_n < \eta(t, \xi_h)\}$ by a function $\eta(t, \xi_h)$ defined on Γ_s. We choose a smooth divergence-free vector field $\vec{a}$ defined in a tabular neighborhood of Γ_s in O which is parallel to ∂O and satisfies $\vec{a}|_{\Gamma_s} \cdot \vec{n}_s = 1$ where $\vec{n}_s$ is the outer unit normal vector of Γ_s. Then, we define a coordinate (ξ_h, ξ_n) $(\xi_h \in \Gamma_s, |\xi_n| < \epsilon)$ in a tabular neighborhood of Γ_s by

$$x(\xi_h, \xi_n) = \text{integrating } \vec{a} \text{ in time } \xi_n \text{ from } \xi_h.$$

We represent $\Gamma(t)$ as a graph of a function η on Γ_s in this coordinate, that is $\Gamma(t) = \{x(\xi_h, \eta(\xi_h)) : \xi_h \in \Gamma_s\}$.

We choose a map $X : \Omega_s \to \Omega(t)$ by $(\xi_h, \xi_n) \mapsto (\xi_h, \bar{\eta}(\xi))$ where $\bar{\eta}(\xi)$ is an extension of $\eta(\xi_h)$ to Ω_s vanishing outside the tabular neighborhood of Γ_s, by some extension operator compatible with Sobolev space estimate $\|\bar{\eta}\|_{H^{r+1}(\Omega_s)} \leqslant C\|\eta\|_{H^{r+1/2}(\Gamma_s)}$. We need to transform fields defined on $\Omega(t)$ to Ω_s. We transform vector u to $\tilde{u}$ on Ω_s as a $(d-1)$-form : $u_i = J^{-1}X_{i,\alpha}\tilde{u}_\alpha$, where $X_{i,\alpha} = \partial X_i/\partial \tilde{x}_\alpha$, $J = \det(X_{i,\alpha})$. (We also transform the vector test function Φ as a $(d-1)$-form.) This transformation preserves divergence freeness and

$$\int_{\Gamma(t)} \psi(x) u \cdot \vec{n} dA_{\Gamma(t)} = \int_{\Gamma_s} \psi(X(\xi))\tilde{u} \cdot \vec{n}_s dA_{\Gamma_s}$$

holds for any ψ (see [H] p.80). Since the normal speed of surface is $(\vec{n} \cdot \vec{a})\eta_t$, (2.3) can be expressed as

$$\eta_t = \left.\frac{u\cdot\vec{\mathrm{n}}}{\vec{a}\cdot\vec{\mathrm{n}}}\right|_{\Gamma(t)}.$$

The area elements of $\Gamma(t)$ and Γ_s are related by $(\vec{\mathrm{n}}\cdot\vec{a})dA_{\Gamma(t)} = dA_{\Gamma_s}$ due to the incompressibility of $\vec{a}$. Using the above formula and $u\cdot\vec{\mathrm{n}}dA_{\Gamma(t)} = \tilde{u}\cdot\vec{\mathrm{n}}_s dA_{\Gamma_s}$, we obtain

$$\eta_t = \tilde{u}\cdot\vec{\mathrm{n}}_s|_{\Gamma_s}.$$

The integral I_{bulk} becomes through the above transformation the sum of the linear part

$$L_{\text{bulk}} = \int_{\Omega_s} -\tilde{u}\cdot\partial_t\tilde{\Phi} + (\tilde{\nabla}\tilde{u} + {}^t\tilde{\nabla}\tilde{u}) : \tilde{\nabla}\tilde{\Phi}d\tilde{x}$$

where $\tilde{\nabla}$ is the gradient in $\tilde{x}$, and the quadratic part

$$Q_{\text{bulk}} = \int_{\Omega_s} A_0(\partial_t X)\tilde{u}\tilde{\Phi} + A(\tilde{\nabla}X - \mathsf{I})(\tilde{u}\partial_t\tilde{\Phi} + \tilde{\nabla}\tilde{u}\tilde{\nabla}\tilde{\Phi}) + B(\tilde{\nabla}X)\tilde{u}\tilde{u}\tilde{\nabla}\tilde{\Phi}d\tilde{x},$$

where A_0, A, B are some function satisfying $|A_0(\partial_t X)| \leqslant C|\partial_t X|$, $|A(\tilde{\nabla}X - \mathsf{I})| \leqslant C|\tilde{\nabla}X - \mathsf{I}|$, $|B(\tilde{\nabla}X)| \leqslant C(1 + |\tilde{\nabla}X|)$.

The fact that the surface integral could be expressed as the sum of

$$L_{\text{surface}} = \int_{\Gamma_s} \sigma\nabla_{\Gamma_s}\eta\cdot\nabla_{\Gamma_s}(\tilde{\Phi}\cdot\vec{\mathrm{n}}) + (\vec{\mathrm{n}}\cdot\nabla V - \sigma S)\eta\tilde{\Phi}\cdot\vec{\mathrm{n}}dA.$$

and

$$Q_{\text{surface}} = \int_{\Gamma_s} O(|\nabla_{\Gamma_s}\eta|^2 + |\eta|^2)(\nabla_{\Gamma_s}(\tilde{\Phi}\cdot\vec{\mathrm{n}}) + \tilde{\Phi}\cdot\vec{\mathrm{n}})dA$$

can be shown by using $I_{\text{surface}}[\Omega;\Phi] = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} E_{\text{surface}}[\Omega^\varepsilon]$.

We also need to rewrite I_{initial} as $I_{\text{initial}}[\Omega_0, u_0; \Phi|_{t=0}] = \int_{\Omega_s}\tilde{u}_0\cdot\tilde{\Phi}|_{t=0}d\tilde{x}$ where $\tilde{u}_0$ is determined by u_0 and Ω_0. This $\tilde{u}_0$ is small in H^{r-1}_σ when u_0 is small in $H^{r-1}(\Omega_0)$ and η_0 is small in $\dot{H}^{r-1/2}(\Gamma_s)$.

We have reached the equations on time-independent domain

$$\eta_t = \tilde{u}\cdot\vec{\mathrm{n}}|_{\Gamma_s},\ \eta|_{t=0} = \eta_0,\ \int_0^\infty L[\eta,\tilde{u};\tilde{\Phi}] + Q[\eta,\tilde{u};\tilde{\Phi}]dt + I_{\text{initial}} = 0 \quad \forall\tilde{\Phi}$$

where $L = L_{\text{bulk}} + L_{\text{surface}}$, $Q = Q_{\text{bulk}} + Q_{\text{surface}}$. Using the result on the linear system with homogeneous initial conditions

$$\eta_t = \tilde{u}\cdot\vec{\mathrm{n}}|_{\Gamma_s},\ \eta|_{t=0} = 0,\ \int_0^\infty \mathcal{L}[\eta,\tilde{u};\cdot]dt = F(\cdot)$$

for given $F(\cdot)$ in the next section, we can show our existence result.

Theorem . *We assume $r > 1 + d/2$ and $(r-1)/2 \notin \mathbb{Z}$ and that Ω_s is geometrically stable. Then, (i) there exists $\gamma < 0$ so that the linearized system has a unique solution $(\eta,\tilde{u}) \in K^{r,1/2}_\gamma(\Gamma_s)\times K^r_\gamma(\Omega_s)$ for data $F \in K^{r-2}$. (ii) We assume that initial condition $\eta_0 \in H^{r-1/2}(\Gamma_s)$ and $\tilde{u}_0 \in H^{r-1}(\Omega_0)$ is small. Then, there exists a exponentially decaying solution $(\eta,\tilde{u}) \in K^{r,1/2}_\gamma(\Gamma_s)\times K^r_\gamma(\Omega_s)$.*

The function spaces in the statement of the theorem are defined in the next section.

5 The linear problem

The linearization of our system of equations is the evolution for η defined on Γ_s

$$\eta_t = u \cdot \vec{n}|_{\Gamma_s}$$

complemented with an initial condition $\eta|_{t=0} = \eta_0$ coupled with the equation for $u(t,\cdot) \in \{\operatorname{div} u = 0, u \cdot \vec{n}|_B = 0\}$,

$$\int_0^\infty \int_{\Omega_s} -u \cdot \Phi_t + 2\nu \mathsf{D}(u) : \mathsf{D}(\Phi) dx dt$$
$$+ \int\int_{\Gamma_s} (\sigma \nabla_{\Gamma_s} \eta \cdot \nabla_{\Gamma_s}(\Phi \cdot \vec{n}) + a_1(x)\eta\Phi \cdot \vec{n}) dA dt$$
$$= \int_0^\infty \int_{\Omega_s} F \cdot \Phi dx + \int_{\Omega_s} u_0 \cdot \Phi|_{t=0} dx$$

for all $\Phi(t,x)$ satisfying $\operatorname{div} \Phi = 0$ and $\Phi \cdot \vec{n}|_B = 0$. For the following results, absence of inhomogeneous terms in the equation for η is crucial. We assume $\nu > 0$ and $\sigma > 0$.

We assume that the initial condition satisfies $\int \eta_0 dA = 0$, then, due to incompressibility of the flow, $\operatorname{div} u = 0$, $\int_\Gamma \eta dA = 0$ holds.

In the rest of this section, we omit subscript s for stationary state and write Ω and Γ for Ω_s and Γ_s.

Remark . *This system is equivalent, for smooth η and u, to the Stokes equations*

$$\operatorname{div} u = 0, \quad \partial_t u - \operatorname{div} \mathsf{T} = F \quad in\ \Omega$$

where $\mathsf{T} = p\mathsf{I} - 2\nu\mathsf{D}(u)$, with boundary conditions,

$$\partial_t \eta = u \cdot \vec{n}|_\Gamma,$$

$$u \cdot \vec{n}|_B = 0 \ and\ \vec{n} \cdot \mathsf{T} \cdot (\mathsf{I} - \vec{n} \otimes \vec{n})|_B = 0 \ on\ B$$

and

$$\mathsf{T} \cdot \vec{n}|_\Gamma = (-\operatorname{div}_\Gamma \sigma \nabla_\Gamma \eta + a_1\eta)\vec{n} \ on\ \Gamma.$$

At this point, we define some function spaces to state our result of this section. We use spaces of functions defined on $(0,\infty) \times \Omega$

$$K^s = K^s(\Omega) := H^0((0,\infty); H^s(\Omega)) \cap H^{s/2}((0,\infty); H^0(\Omega))$$

used in [B] and its weighted version

$$K^s_\gamma = K^s_\gamma(\Omega) := \{f : fe^{-\gamma t} \in K^s(\Omega)\}$$

and similar spaces $K^s_\gamma(\Gamma)$ of functions defined on $(0,\infty) \times \Gamma$. Their norms are denoted as $|||\cdot|||_{\gamma,s}$ and $|||\cdot|||_{\Gamma,\gamma,s}$. We denote $K^s_{\gamma,(0)}$ the closure of $C^\infty_c((0,\infty) \times \overline{\Omega})$ in K^s_γ. When $(r-1)/2 \notin \mathbb{Z}$, this subspace is expressed as $\{f \in K^s_\gamma : \partial^k_t f|_{t=0} =$

$0 (0 \leqslant k < (s-1)/2, \text{integer})\}$. We also use $K^{s,1/2}(\Gamma) = H^0((0,\infty); H^{s+1/2}(\Gamma)) \cap H^{s/2}((0,\infty); H^{1/2}(\Gamma))$ and denote its norm $|||\cdot|||_{\Gamma,s,1/2}$.

Our result of this section is the following proposition for our linear system with homogeneous initial conditions.

Proposition 5.1. *We assume $r \geqslant 2$ and $b(\cdot,\cdot)$ be positive definite on $\dot{H}^1(\Gamma)$. There exist $\gamma < 0$ so that, a solution $\eta \in K^{r,1/2}_{\gamma,(0)}(\Gamma)$, $u \in K^r_{\gamma,(0)}(\Omega)$ exists for data $F \in K^{r-2}_{\gamma,(0)}$.*

We use spaces of vector field defined on Ω

$$\mathcal{D}_\sigma = \{\varphi \in C^\infty(\overline{\Omega}) : \operatorname{div}\varphi = 0 \text{ in } \Omega, \varphi\cdot\vec{n}|_B = 0\},$$

$$H^s_\sigma = (\text{the closure of } \mathcal{D}_\sigma \text{ in } H^s) = \{u \in H^s(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega, u\cdot\vec{n}|_B = 0\}$$

and we use over-dot notation as $\dot{H}^1(\Gamma)$ to indicate that this space consists of functions with vanishing average. The inner products of $H^0_\sigma = L^2_\sigma$ and $\dot{L}^2(\Gamma)$ are denoted as (u,φ) and $(\eta,\psi)_\Gamma$, respectively. We denote by R the restriction operator $Ru = u\cdot\vec{n}|_\Gamma$, which send a divergence free vector field to a function defined on Γ with vanishing average. (In fact, we interpret the definition of R as $\int_\Gamma Ru\psi dA = \int_\Omega u\cdot\nabla\bar{\psi}$ where ψ is any smooth function on Γ and $\bar{\psi}$ is an extension of ψ defined by $\Delta\bar{\psi} = 0, \psi|_\Gamma = \psi, \psi|_B = 0$.) We denote Q the adjoint operator of $R : L^2_\sigma \to \dot{L}^2(\Gamma)$. R and Q are bounded as $R : H^s_\sigma \to \dot{H}^{s-1/2}(\Gamma)$ and $Q : \dot{H}^{s-1/2}(\Gamma) \to H^s_\sigma$ $(s \in \mathbb{R})$. P is the orthogonal projection to $\{Ru = 0\}$ in L^2_σ. We use the following notations for Korn's form and bilinear form of surface terms:

$$\langle u,\varphi\rangle = \int_\Omega 2\nu \mathsf{D}(u) : \mathsf{D}(\varphi)dx, \quad b(\eta,\psi) = \int_\Gamma \sigma\nabla_\Gamma\eta\cdot\nabla_\Gamma\psi + a_1\eta\psi dA.$$

With these definitions, the Stokes system can be written as

$$(\partial_t u,\varphi) + \langle u,\varphi\rangle + b(\eta, R\varphi) = (F,\varphi) \quad \forall\varphi \in \mathcal{D}_\sigma.$$

$\partial_t\eta = Ru$ can be rewritten as $b(\partial_t\eta,\psi) = b(Ru,\psi)$, where ψ is an arbitrary test function defined on Γ with vanishing average. By summing these, we has reached the final formulation : $(\eta,u) \in L^2(0,\infty; \dot{H}^1(\Gamma)\times H^1_\sigma)$

$$(\partial_t u,\varphi) + b(\partial_t\eta,\psi) + \langle u,\varphi\rangle + b(\eta,R\varphi) - b(Ru,\psi) = (F,\varphi) \tag{5.1}$$

where test functions φ and ψ runs through $\mathcal{D}_\sigma$ and $\dot{\mathcal{D}}(\Gamma) = \{\psi \in C^\infty(\Gamma) : \int_\Gamma\psi = 0\}$ respectively.

Under the assumption on Ω_s in section 3, there is no affine φ in $\mathcal{D}_\sigma$, thus, Korn's inequality $\exists\delta > 0, \langle u,u\rangle \geqslant \delta\|u\|^2_{H^1}$ $(\forall u \in H^1_\sigma)$ holds (see [SS]).

The equations for the Laplace transforms $\hat{\eta}(\lambda)$ and $\hat{u}(\lambda)$ of η and u in t reads as follows : $\hat{u} \in H^1_\sigma$, $\hat{\eta} \in \dot{H}^1(\Gamma)$,

$$\lambda\{(\hat{u},\varphi) + b(\hat{\eta},\psi)\} + \langle\hat{u},\varphi\rangle + b(\hat{\eta},R\varphi) - b(R\hat{u},\psi) = (\hat{F},\varphi)$$

$$\forall\varphi \in \mathcal{D}_\sigma, \forall\psi \in \dot{\mathcal{D}}(\Gamma).$$

We can prove the existence of solution and estimates for this spectral problem. The above result for the evolution equation is the direct consequence of the following result for the spectral problem.

Proposition 5.2. *We assume $r \geqslant 2$. There exist γ so that, when $Re\lambda > \gamma$,* *there exist a unique solution* $(\hat{u}, \hat{\eta}) \in H^r_\sigma \times \dot{H}^{r+1/2}(\Gamma)$ *holomorphic in* λ *for data* $\hat{F}(\lambda) \in H^{r-2}$ *holomorphic in* $Re\,\lambda > \gamma$ *which satisfies estimates*

$$(|\lambda|^{r/2}\|\hat{u}\|_0 + \|\hat{u}\|_r) + (|\lambda|^{r/2}\|\hat{\eta}\|_{1/2,\Gamma} + \|\hat{\eta}\|_{r+1/2,\Gamma})$$
$$\leqslant C(\|\hat{F}\|_{r-2} + |\lambda|^{(r-2)/2}\|\hat{F}\|_0).$$

When $b(\cdot,\cdot)$ *is positive definite on* $\dot{H}^1(\Gamma)$*, the above* γ *can be taken to be negative.*

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