Vapor flows with evaporation and condensation in the continuum limit: Effect of a trace of noncondensable gas

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Abstract

Steady flows of a vapor with evaporation and condensation on the boundary consisting of the condensed phase of the vapor are considered in the following situation: (i) the boundary is of arbitrary smooth shape; (ii) the Knudsen number Kn, the ratio of the typical mean free path of the vapor molecules to the characteristic length of the system, is small; (iii) a small amount of a noncondensable gas is contained in the system; more specifically, the amount is such that the average concentration of the noncondensable gas is of the order of Kn in the case of a closed domain (the case of an infinite domain is also discussed). The steady behavior of the vapor and the noncondensable gas, in particular, that in the continuum limit where·Kn vanishes, is investigated by means of a systematic asymptotic analysis based on kinetic theory. In this situation, the average concentration of the noncondensable gas becomes infinitely small in the continuum limit in the case of a closed domain. However, it is shown that the noncondensable gas accumulates in the infinitely thin Knudsen layer on the boundary where condensation is taking place and has a significant effect on the global vapor flow in the continuum limit. An example demonstrating such an effect is also given.

1 Introduction

Vapor flows with evaporation or condensation on the boundary have been one of the important subjects in rarefied gas dynamics. For single-component systems consisting of a pure vapor and its condensed phase, many successful results have been obtained. For example, a new type of gas dynamics (i.e., fluid-dynamic equations and their boundary conditions) describing the vapor flows around the condensed phase of arbitrary shape in the continuum limit has been established, together with its correction in the near continuum regime, by means of a systematic asymptotic analysis of the Boltzmann equation for small Knudsen numbers (Sone and Onishi [1], Onishi and Sone [2], Sone [3, 4, 5], Aoki and Sone [6]). As for the vapor flows at intermediate and large Knudsen numbers, we refer to Sugimoto and Sone [7], Sone and Sugimoto [8], and Takata et al. [9] as typical examples and to Kogan [10], Ytrehus [11], and Rebrov [12] as review papers.

In practical situations, however, evaporation and condensation often take place in the presence of other gases that neither evaporate nor condense (noncondensable gases). Such two- or multi-component systems (vapor–gas mixtures) have also been investigated in the literature (e.g., Pao [13], Matsushita [14], Onishi [15, 16], Bedeaux et al. [17]). But, because of the complexity of the systems, the level of understanding is still unsatisfactory. For instance, the behavior of the mixtures in the continuum limit has not fully been understood yet.

In a series of recent papers (Aoki et al. [18], Takata et al. [19], Takata and Aoki [20], Aoki [21]), we investigated the continuum limit of a mixture of a vapor and a noncondensable gas in a simple one-dimensional problem. More specifically, we considered the mixture in the gap between two parallel plane condensed phases of the vapor with different uniform temperatures (the condensed phases may be moving with a constant speed in their surfaces) and clarified the features of the continuum limit by means of a systematic asymptotic analysis as well as an accurate numerical analysis based on kinetic theory. Let n_r be an appropriate reference number density of the vapor molecules (e.g., the saturation number density of the vapor molecules at the temperature of one of the condensed phases), n_{av}^B the average number density of the noncondensable gas in the gap, and Kn the Knudsen number with respect to the vapor, namely, the ratio of the mean free path of the vapor molecules in the reference equilibrium state at rest to the reference length (the width of the gap). According to Aoki et al. [18], Takata et al. [19], Takata

and Aoki [20], and Aoki [21], there are two different situations in the continuum limit, where Kn goes to zero, depending on the amount of the noncondensable gas contained in the gap, i.e., (i) the case of $n_{av}^B/n_r = O(1)$, and (ii) the case of $n_{av}^B/n_r = O(Kn)$. In case (i), evaporation and condensation stop. However, the vanishing (or nonexisting) evaporation and condensation have an important effect on the flow field (i.e., the profiles of the temperature, density, and flow velocity) in the continuum limit. This is an example of the ghost effect first pointed out by Sone et al. [22] and then discussed in Sone et al. [23], Sone [24, 25, 5], and Bouchut et al. [26] for single-component systems. In case (ii), a uniform flow of the pure vapor is caused from the evaporating to the condensing surface. Because n_{av}^B/n_r vanishes in the limit, the amount of the noncondensable gas becomes infinitely small compared with that of the vapor (or the average concentration of the noncondensable gas becomes infinitesimal). However, the trace of the noncondensable gas still has a significant effect on the vapor flow. This seemingly paradoxical result is due to the fact that the infinitesimal amount of the noncondensable gas is concentrated in the Knudsen layer with an infinitesimal thickness on the condensing surface by the vapor flow, so that its local number density there becomes high enough (comparable to n_r) to affect the vapor flow (see Aoki et al. [18]). Thus, the continuum limit is nothing obvious even in such a simple one-dimensional problem.

The continuum limit of type (i) is discussed for the general geometry by Takata and Aoki [27]. That is, the mixture in contact with arbitrarily shaped boundary at rest, consisting of the condensed phase of the vapor, is considered (the mixture is assumed to be at rest at infinity in the case of an infinite domain), and the correct fluid-dynamic-type equations and their boundary conditions for the continuum limit are derived from the Boltzmann equation and its boundary condition for hard-sphere molecules. From this fluid-dynamic system, the cause of the ghost effect is clarified in the case of the mixture.

The aim of the present study is to clarify the continuum limit of type (ii) for the general geometry. That is, we are going to carry out a systematic asymptotic analysis for small Knudsen numbers on the basis of kinetic theory to derive an appropriate fluid-dynamic system that describes the effect of a small amount (or an infinitesimal average concentration) of a noncondensable gas in the continuum limit. Actually, Aoki et al. [28] performed a Monte Carlo simulation of a two-dimensional vapor flow for small Knudsen numbers in the case corresponding to type (ii) and found that the small amount of the noncondensable gas changes the stream lines of the vapor flow significantly from those in the pure vapor case. This result also supports the necessity of the present study.

2 Formulation of the problem

2.1 Problem

Consider a vapor in contact with an arbitrarily shaped smooth boundary consisting of the condensed phase of the vapor, and suppose that steady evaporation or condensation of the vapor is taking place on the boundary. For the sake of clarity, we restrict ourselves to the case of a closed domain (we will discuss the case of an infinite domain in Sec. 3.6). We investigate the steady behavior of the vapor for small values of the Knudsen number Kn, especially that in the continuum limit where Kn vanishes, on the basis of kinetic theory in the case where a small amount of another gas that neither evaporates nor condenses on the boundary (noncondensable gas) is also contained in the system. To be more specific, the ratio of the average number density of the noncondensable gas n_{av}^B to the typical or reference number density of the vapor n_r is of the order of Kn $[n_{av}^B/n_r = O(Kn)]$. There is no external force in the system. For simplicity, we assume the conventional condition for evaporation and condensation (the so-called complete condensation condition) for the vapor and the diffuse reflection condition for the noncondensable gas on the boundary [see the sentence below Eq. (6b)]. In the following, we assign the label A to the vapor (it will also be called A-component) and B to the noncondensable gas (it will also be called B-component).

2.2 Basic equation

We first introduce some reference quantities in addition to the reference number density of the vapor n_r: L is the reference length of the system, T_r is the reference temperature, p_r is the reference pressure of the vapor defined by $p_r = kn_rT_r$, where k is the Boltzmann constant. Further, m^α is the mass of a molecule of α-component ($\alpha = A, B$).

Let us denote the rectangular space coordinates by Lx_i, the molecular velocity by $(2kT_r/m^A)^{1/2}\zeta_i$ [or $(2kT_r/m^A)^{1/2}\zeta$], and the velocity distribution function of α-component by $n_r(2kT_r/m^A)^{-3/2}\hat{F}^\alpha$

$(\alpha = A, B)$. Then the Boltzmann equation for a binary mixture (Kogan [29], Chapman and Cowling [30], Hirschfelder et al. [31]) in the present time-independent problem is written as

$$\zeta_i \frac{\partial \hat{F}^\alpha}{\partial x_i} = \frac{2}{\sqrt{\pi}} \frac{1}{\mathrm{Kn}} \sum_{\beta=A,B} \hat{J}^{\beta\alpha}(\hat{F}^\beta, \hat{F}^\alpha), \qquad (\alpha = A, B), \tag{1}$$

$$\hat{J}^{\beta\alpha}(f, g) = \int [f(\zeta'_*)g(\zeta') - f(\zeta_*)g(\zeta)] \hat{B}^{\beta\alpha}(|\mathbf{e} \cdot \hat{\mathbf{V}}|, |\hat{\mathbf{V}}|) \, d\Omega(\mathbf{e}) \, d^3\zeta_*, \tag{2}$$

$$\zeta' = \zeta + \frac{\hat{\mu}^{\beta\alpha}}{\hat{m}^\alpha}(\mathbf{e} \cdot \hat{\mathbf{V}}) \mathbf{e}, \quad \zeta'_* = \zeta_* - \frac{\hat{\mu}^{\beta\alpha}}{\hat{m}^\beta}(\mathbf{e} \cdot \hat{\mathbf{V}}) \mathbf{e}, \tag{3a}$$

$$\hat{\mu}^{\beta\alpha} = \frac{2\hat{m}^\alpha \hat{m}^\beta}{\hat{m}^\alpha + \hat{m}^\beta}, \qquad \hat{m}^\alpha = m^\alpha/m^A, \tag{3b}$$

$$\hat{\mathbf{V}} = \zeta_* - \zeta, \qquad d^3\zeta_* = d\zeta_{*1} \, d\zeta_{*2} \, d\zeta_{*3}, \tag{3c}$$

$$\mathrm{Kn} = \ell_r/L, \tag{3d}$$

where $\mathbf{e}$ is a unit vector, ζ_* is the variable of integration corresponding to ζ, $d\Omega(\mathbf{e})$ is the solid angle element in the direction of $\mathbf{e}$, and $\hat{B}^{\beta\alpha}(|\mathbf{e} \cdot \hat{\mathbf{V}}|, |\hat{\mathbf{V}}|)$ are nonnegative functions of $|\mathbf{e} \cdot \hat{\mathbf{V}}|$ and $|\hat{\mathbf{V}}|$ depending on the molecular model. The domain of integration in Eq. (2) is the whole space of ζ_* and all directions of $\mathbf{e}$. In Eq. (3d), ℓ_r is the mean free path of a molecule of the A–component (vapor) when it is in the equilibrium state at rest with molecular number density n_r and temperature T_r, and Kn is the corresponding Knudsen number, which represents the degree of rarefaction of the system. Here and in what follows, the Greek letters α and β are used to represent the labels A and B of the components. Since no confusion is expected, the notes such as $\alpha = A, B$ in Eq. (1) will mostly be omitted below. As in the left-hand side of Eq. (1), the summation convention (i.e., $a_i b_i = \sum_{i=1}^3 a_i b_i$) is used throughout the paper.

It should be noted that the function $\hat{B}^{\beta\alpha}$ also depends on the dimensionless parameter $U_r^{\beta\alpha}/kT_r$, where $U_r^{\beta\alpha}$ is the characteristic size of the intermolecular potential for the interaction of a molecule of the α–component with a molecule of the β–component, though it is not shown explicitly in Eq. (2). This fact was pointed out by Sone for the collision term for a single-component gas (Sone [4, 5], Sone and Aoki [32]). When both of the components are hard-sphere gases, $\hat{B}^{\beta\alpha}$ and ℓ_r are given by

$$\hat{B}^{\beta\alpha} = \frac{1}{4\sqrt{2\pi}} \left(\frac{d^\beta + d^\alpha}{2d^A} \right)^2 |\mathbf{e} \cdot \hat{\mathbf{V}}|, \qquad \ell_r = \frac{1}{\sqrt{2}\pi(d^A)^2 n_r}, \tag{4}$$

where d^α is the diameter of a molecule of the α–component. The $\hat{B}^{\beta\alpha}$ does not depend on $U_r^{\beta\alpha}/kT_r$ in this case.

We now denote the temperature of the boundary by $T_r \hat{T}_w$, its velocity by $(2kT_r/m^A)^{1/2}\hat{v}_{wi}$, and the saturation pressure of the vapor at temperature $T_r\hat{T}_w$ by $p_r\hat{p}_w^A$. Further, since the problem is steady, $\hat{v}_{wi}n_i = 0$ is assumed, where n_i is the unit vector normal to the boundary pointing to the gas region. Then the boundary conditions are written as

$$\hat{F}^\alpha = \sigma_w^\alpha \hat{T}_w^{-3/2} \left(\frac{\hat{m}^\alpha}{\pi} \right)^{3/2} \exp\left(-\frac{\hat{m}^\alpha(\zeta_i - \hat{v}_{wi})^2}{\hat{T}_w} \right), \quad \zeta_i n_i > 0, \tag{5}$$

with

$$\sigma_w^A = \hat{p}_w^A/\hat{T}_w, \tag{6a}$$

$$\sigma_w^B = -2 \left(\frac{\pi \hat{m}^B}{\hat{T}_w} \right)^{1/2} \int_{\zeta_i n_i < 0} \zeta_i n_i \hat{F}^B d^3\zeta, \tag{6b}$$

where $d^3\zeta = d\zeta_1 \, d\zeta_2 \, d\zeta_3$. Equation (5) with $\alpha = A$ means that the vapor molecules leaving the boundary obey the corresponding part of the Maxwellian distribution characterized by $\hat{T}_w$, $\hat{v}_{wi}$, and $\hat{p}_w^A$ (complete condensation condition). On the other hand, Eq. (5) with $\alpha = B$ means that the noncondensable

gas molecules reflected by the boundary obey the corresponding part of the Maxwellian distribution characterized by $\hat{T}_w$ and $\hat{v}_{wi}$ and that there is no net mass flow of this component across the boundary (diffuse reflection).

Next, we introduce macroscopic variables as the moments of the velocity distribution functions. For each component, we define its number density $n_r\hat{n}^\alpha$, density $n_r m^A \hat{\rho}^\alpha$, flow velocity $(2kT_r/m^A)^{1/2}\hat{v}_i^\alpha$, temperature $T_r\hat{T}^\alpha$, and partial pressure $p_r\hat{p}^\alpha$ by

$$\hat{n}^\alpha = \int \hat{F}^\alpha \mathrm{d}^3\zeta, \qquad \hat{\rho}^\alpha = \hat{m}^\alpha \int \hat{F}^\alpha \mathrm{d}^3\zeta \ (= \hat{m}^\alpha \hat{n}^\alpha), \tag{7a}$$

$$\hat{v}_i^\alpha = \frac{1}{\hat{n}^\alpha} \int \zeta_i \hat{F}^\alpha \mathrm{d}^3\zeta, \qquad \hat{T}^\alpha = \frac{2}{3}\frac{\hat{m}^\alpha}{\hat{n}^\alpha} \int (\zeta_i - \hat{v}_i^\alpha)^2 \hat{F}^\alpha \mathrm{d}^3\zeta, \tag{7b}$$

$$\hat{p}^\alpha = \frac{2}{3}\hat{m}^\alpha \int (\zeta_i - \hat{v}_i^\alpha)^2 \hat{F}^\alpha \mathrm{d}^3\zeta \ (= \hat{n}^\alpha \hat{T}^\alpha). \tag{7c}$$

Here and in what follows, the domain of integration with respect to ζ_i is its whole space, unless otherwise stated. For the total mixture, the number density $n_r\hat{n}$, density $n_r m^A \hat{\rho}$, flow velocity $(2kT_r/m^A)^{1/2}\hat{v}_i$, temperature $T_r\hat{T}$, and pressure $p_r\hat{p}$ are defined by

$$\hat{n} = \int \sum_{\beta=A,B} \hat{F}^\beta \mathrm{d}^3\zeta, \qquad \hat{\rho} = \int \sum_{\beta=A,B} \hat{m}^\beta \hat{F}^\beta \mathrm{d}^3\zeta, \tag{8a}$$

$$\hat{v}_i = \frac{1}{\hat{\rho}} \int \zeta_i \sum_{\beta=A,B} \hat{m}^\beta \hat{F}^\beta \mathrm{d}^3\zeta, \qquad \hat{T} = \frac{2}{3\hat{n}} \int (\zeta_i - \hat{v}_i)^2 \sum_{\beta=A,B} \hat{m}^\beta \hat{F}^\beta \mathrm{d}^3\zeta, \tag{8b}$$

$$\hat{p} = \frac{2}{3} \int (\zeta_i - \hat{v}_i)^2 \sum_{\beta=A,B} \hat{m}^\beta \hat{F}^\beta \mathrm{d}^3\zeta \ (= \hat{n}\hat{T}). \tag{8c}$$

Thus the macroscopic variables for the total mixture are expressed in terms of those for individual components as follows:

$$\hat{n} = \sum_{\beta=A,B} \hat{n}^\beta, \qquad \hat{\rho} = \sum_{\beta=A,B} \hat{\rho}^\beta, \qquad \hat{\rho}\hat{v}_i = \sum_{\beta=A,B} \hat{\rho}^\beta \hat{v}_i^\beta, \tag{9a}$$

$$\hat{p} = \sum_{\beta=A,B} \left[\hat{p}^\beta + \frac{2}{3}\hat{\rho}^\beta(\hat{v}_i^\beta - \hat{v}_i)^2 \right]. \tag{9b}$$

In the literature, the temperature $\hat{T}^\alpha$ and partial pressure $\hat{p}^\alpha$ of each component are often defined in a different way, i.e., by the second equation of Eq. (7b) and Eq. (7c) with $\hat{v}_i^\alpha$ being replaced by $\hat{v}_i$ [the first equation of Eq. (8b)] (e.g., Kogan [29] and Chapman and Cowling [30]). If these definitions are adopted, the pressure $\hat{p}$ of the total mixture, defined by Eq. (8c), is expressed by the simple sum of $\hat{p}^\alpha$ instead of Eq. (9b).

3 Asymptotic analysis for small Knudsen numbers

In this section, we carry out a systematic asymptotic analysis of the boundary-value problem (1) and (5) for small Knudsen numbers under the situation described in Sec. 2.1, namely,

$$n_{av}^B/n_r = O(\mathrm{Kn}). \tag{10}$$

The basic guideline of the analysis is the asymptotic theory developed by Sone (Sone [33, 34, 3, 4, 5], Sone and Onishi [1], Onishi and Sone [2], Aoki and Sone [6], Sone and Aoki [32], Sone et al. [22, 35]). In the course of the analysis, we use the small parameter ϵ:

$$\epsilon = (\sqrt{\pi}/2)\mathrm{Kn}, \tag{11}$$

rather than the Knudsen number Kn.

3.1 Hilbert solution

Let us first seek the moderately varying solutions $\hat{F}_H^\alpha$ $[\partial \hat{F}_H^\alpha / \partial x_i = O(\hat{F}_H^\alpha)]$ of the Boltzmann equation (1) in a power series of ϵ:

$$\hat{F}_H^\alpha = \hat{F}_{H0}^\alpha + \hat{F}_{H1}^\alpha \epsilon + \cdots . \tag{12}$$

Correspondingly, the macroscopic variables are expanded as

$$h_H^\alpha = h_{H0}^\alpha + h_{H1}^\alpha \epsilon + \cdots , \tag{13a}$$

$$h_H = h_{H0} + h_{H1}\epsilon + \cdots , \tag{13b}$$

where h denotes $\hat{n}$, $\hat{\rho}$, $\hat{v}_i$, $\hat{T}$, or $\hat{p}$. Here, h_H^α and h_H are defined by Eqs. (7a)–(7c) and (8a)–(8c) with $\hat{F}^\alpha$ replaced by $\hat{F}_H^\alpha$, and the expansion coefficients h_{Hm}^α and h_{Hm} are obtained by substituting the expansions (12)–(13b) into the definitions of h_H^α and h_H and by equating the coefficients of the same power of ϵ. This solution (or expansion) is called the Hilbert solution (or expansion). Some examples of h_{Hm}^α and h_{Hm} are given in Appendix A. If we substitute Eq. (12) into Eq. (1), we obtain the following sequence of integral equations:

$$\sum_{\beta=A,B} \hat{J}^{\beta\alpha}(\hat{F}_{H0}^\beta, \hat{F}_{H0}^\alpha) = 0, \tag{14}$$

$$\sum_{\beta=A,B} [\hat{J}^{\beta\alpha}(\hat{F}_{Hm}^\beta, \hat{F}_{H0}^\alpha) + \hat{J}^{\beta\alpha}(\hat{F}_{H0}^\beta, \hat{F}_{Hm}^\alpha)]$$

$$= \zeta_i \frac{\partial \hat{F}_{Hm-1}^\alpha}{\partial x_i} - \sum_{\beta=A,B} \sum_{n=1}^{m-1} \hat{J}^{\beta\alpha}(\hat{F}_{Hm-n}^\beta, \hat{F}_{Hn}^\alpha), \tag{15}$$

where $m = 1, 2, \ldots$, and $\sum_1^0 = 0$ when $m = 1$ in Eq. (15). Equation (14) is the system of nonlinear integral equations for $\hat{F}_{H0}^\alpha$, while Eq. (15) is that of inhomogeneous linear integral equations for $\hat{F}_{Hm}^\alpha$. The series of equations can, in principle, be solved successively from the lowest order.

The solution of Eq. (14) is given by local equilibrium distributions (Chapman and Cowling [30]), namely, local Maxwellian distributions with common flow velocity and temperature, which can be expressed as

$$\hat{F}_{H0}^\alpha = \hat{n}_{H0}^\alpha \hat{T}_{H0}^{-3/2} \left(\frac{\hat{m}^\alpha}{\pi}\right)^{3/2} \exp\left(-\frac{\hat{m}^\alpha(\zeta_i - \hat{v}_{iH0})^2}{\hat{T}_{H0}}\right), \tag{16}$$

by the use of the leading-order terms $\hat{n}_{H0}^\alpha$, $\hat{v}_{iH0}$, and $\hat{T}_{H0}$ of the expansions (13a) and (13b) [see Eqs. (A1a), (A2b), and (A2c)]. For this distribution, of course, the flow velocity and the temperature of each component are the same as those of the total mixture, i.e.,

$$\hat{v}_{iH0}^A = \hat{v}_{iH0}^B = \hat{v}_{iH0}, \qquad \hat{T}_{H0}^A = \hat{T}_{H0}^B = \hat{T}_{H0}. \tag{17}$$

To be consistent with Eq. (10), we need to assume that

$$\hat{n}_{H0}^B \equiv 0, \qquad \text{(i.e., } \hat{F}_{H0}^B \equiv 0), \tag{18}$$

because otherwise n_{av}^B / n_r becomes of $O(1)$. Then, Eq. (15) with $\alpha = B$ and $m = 1$ reduces to

$$\hat{J}^{AB}(\hat{F}_{H0}^A, \hat{F}_{H1}^B) = 0. \tag{19}$$

The solution to this equation is given by a local Maxwellian distribution with the same flow velocity and temperature as $\hat{F}_{H0}^A$ (Cercignani [36]), i.e.,

$$\hat{F}_{H1}^B = \hat{n}_{H1}^B \hat{T}_{H0}^{-3/2} \left(\frac{\hat{m}^B}{\pi}\right)^{3/2} \exp\left(-\frac{\hat{m}^B(\zeta_i - \hat{v}_{iH0})^2}{\hat{T}_{H0}}\right), \tag{20}$$

where $\hat{n}_{H1}^B$ appears because the first equation of Eq. (A1a) ($\alpha = B$ and $m = 1$) has been used. Since Eq. (20) also satisfies

$$\hat{J}^{BA}(\hat{F}_{H1}^B, \hat{F}_{H0}^A) = 0, \tag{21}$$

Eq. (15) with $\alpha = A$ and $m = 1$ reduces to

$$\hat{J}^{AA}(\hat{F}_{H1}^A, \hat{F}_{H0}^A) + \hat{J}^{AA}(\hat{F}_{H0}^A, \hat{F}_{H1}^A) = \zeta_i \frac{\partial \hat{F}_{H0}^A}{\partial x_i}, \tag{22}$$

which is the linear equation for $\hat{F}_{H1}^A$ and is of the same form as the corresponding equation in the pure vapor case (Aoki and Sone [6]). The homogeneous equation of Eq. (22) has the independent nontrivial solutions $\hat{F}_{H0}^A$, $\zeta_i \hat{F}_{H0}^A$, and $\zeta_j^2 \hat{F}_{H0}^A$. Therefore, the inhomogeneous term of Eq. (22) should satisfy the following solvability condition in order for the equation to have a solution:

$$\int (1, \zeta_i, \zeta_j^2) \zeta_\ell \frac{\partial \hat{F}_{H0}^A}{\partial x_\ell} \, d^3\zeta = 0. \tag{23}$$

If we substitute Eq. (16) with $\alpha = A$ into Eq. (23), we obtain

$$\frac{\partial \hat{\rho}_{H0}^A \hat{v}_{jH0}}{\partial x_j} = 0, \tag{24a}$$

$$\hat{\rho}_{H0}^A \hat{v}_{jH0} \frac{\partial \hat{v}_{iH0}}{\partial x_j} + \frac{1}{2} \frac{\partial \hat{p}_{H0}^A}{\partial x_i} = 0, \tag{24b}$$

$$\hat{v}_{jH0} \frac{\partial}{\partial x_j} \left(\frac{5}{2} \hat{T}_{H0} + \hat{v}_{\ell H0}^2 \right) = 0, \tag{24c}$$

$$\hat{p}_{H0}^A = \hat{\rho}_{H0}^A \hat{T}_{H0}, \tag{24d}$$

where $\hat{\rho}_{H0}^A = \hat{n}_{H0}^A$. The set of Eqs. (24a)–(24d) is the Euler set for an ideal gas. In deriving Eqs. (24b) and (24c) from Eq. (23), we have used Eq. (24a). Our next task is to derive the appropriate boundary condition for the Euler set. This will be discussed in the following subsections.

The nonzero $\hat{n}_{H1}^B$ does not contradict Eq. (10). However, we can consistently assume that

$$\hat{n}_{Hm}^B \equiv 0, \qquad \text{(i.e., } \hat{F}_{Hm}^B \equiv 0 \text{)}, \qquad (m = 1, 2, ...). \tag{25}$$

That is, the analysis can be carried out consistently with Eq. (25). The reasoning of Eq. (25), which is also related to the discussions in Secs. 3.2–3.4, is given in Appendix B.

3.2 Knudsen-layer correction

So far, we have paid no attention to the boundary condition. In order for the Hilbert solution (16) (with $\alpha = A$) to satisfy the boundary condition (5) (with $\alpha = A$), we have to impose the following conditions on the boundary:

$$\hat{n}_{H0}^A = \hat{p}_w^A / \hat{T}_w^A, \qquad \hat{v}_{iH0} = \hat{v}_{wi}, \qquad \hat{T}_{H0} = \hat{T}_w. \tag{26}$$

However, these conditions are too many for the Euler set of equations. In other words, we cannot satisfy the boundary condition (5) only with the Hilbert solution. Therefore, we need to introduce the Knudsen-layer correction. Let us seek the solution in the form

$$\hat{F}^\alpha = \hat{F}_H^\alpha + \hat{F}_K^\alpha, \tag{27}$$

where $\hat{F}_K^\alpha$ is the Knudsen-layer correction, which is a correction term to the Hilbert solution near the boundary. More precisely, $\hat{F}_K^\alpha$ is assumed to have the length scale of variation of the order of ϵ (or the mean free path in the dimensional physical space) in the direction normal to the boundary, i.e., $n_j \partial \hat{F}_K^\alpha / \partial x_j = O(\hat{F}_K^\alpha / \epsilon)$, and to be appreciable only in the thin layer with thickness of the order of

ϵ adjacent to the boundary. In order to analyze the Knudsen-layer correction, we introduce the new coordinate system (η, χ_1, χ_2) defined by

$$x_i = \epsilon \eta n_i(\chi_1, \chi_2) + x_{wi}(\chi_1, \chi_2), \tag{28}$$

where x_{wi} represents the boundary, η is the coordinate normal to the boundary stretched by $1/\epsilon$ ($\eta = 0$ corresponds to the boundary), and χ_1 and χ_2 are the coordinates on the boundary orthogonal to each other. We consider $\hat{F}_K^\alpha$ to be a function of η, χ_1, and χ_2 $[\partial \hat{F}_K^\alpha / \partial \eta = O(\hat{F}_K^\alpha)]$, as well as ζ_i, vanishing rapidly as $\eta \to \infty$. We assume that $\hat{F}_K^\alpha$ is also expanded in a power series of ϵ as

$$\hat{F}_K^\alpha = \hat{F}_{K0}^\alpha + \hat{F}_{K1}^\alpha \epsilon + \cdots . \tag{29}$$

We now substitute Eq. (27) with Eq. (29) into Eq. (1) and take into account the properties of $\hat{F}_K^\alpha$ as well as the fact that $\hat{F}_H^\alpha$ is a solution of Eq. (1). In particular, we use $\hat{F}_{H0}^B = \hat{F}_{H1}^B = 0$ [Eqs. (18) and (25)] and the following rearranged expansion of $\hat{F}_H^A$ in the Knudsen layer:

$$\hat{F}_H^A = (\hat{F}_{H0}^A)_b + \left[(\hat{F}_{H1}^A)_b + \left(\frac{\partial \hat{F}_{H0}^A}{\partial x_i} \right)_b n_i \eta \right] \epsilon + \cdots , \tag{30}$$

where $(\ )_b$ indicates that the quantity in the parentheses is evaluated on the boundary. Then, we obtain the sequence of equations for $\hat{F}_{Km}^\alpha$ ($m = 0, 1, \ldots$). If we introduce the following $\hat{F}_0^A$ and $\hat{F}_0^B$:

$$\hat{F}_0^A = (\hat{F}_{H0}^A)_b + \hat{F}_{K0}^A, \qquad \hat{F}_0^B = \hat{F}_{K0}^B, \tag{31}$$

the equations for $m = 0$ are written as

$$\zeta_i n_i \frac{\partial \hat{F}_0^A}{\partial \eta} = \hat{J}^{AA}(\hat{F}_0^A, \hat{F}_0^A) + \hat{J}^{BA}(\hat{F}_0^B, \hat{F}_0^A), \tag{32a}$$

$$\zeta_i n_i \frac{\partial \hat{F}_0^B}{\partial \eta} = \hat{J}^{AB}(\hat{F}_0^A, \hat{F}_0^B) + \hat{J}^{BB}(\hat{F}_0^B, \hat{F}_0^B). \tag{32b}$$

The boundary conditions for Eqs. (32a) and (32b) on the boundary are, from Eqs. (5), (27) [with Eqs. (12) and (29)], and (31), given by

$$\hat{F}_0^\alpha = \sigma_{w0}^\alpha \hat{T}_w^{-3/2} \left(\frac{\hat{m}^\alpha}{\pi} \right)^{3/2} \exp \left(- \frac{\hat{m}^\alpha (\zeta_i - \hat{v}_{wi})^2}{\hat{T}_w} \right), \quad \zeta_i n_i > 0, \tag{33}$$

with

$$\sigma_{w0}^A = \hat{p}_w^A / \hat{T}_w, \tag{34a}$$

$$\sigma_{w0}^B = -2 \left(\frac{\pi \hat{m}^B}{\hat{T}_w} \right)^{1/2} \int_{\zeta_i n_i < 0} \zeta_i n_i \hat{F}_0^B d^3\zeta. \tag{34b}$$

On the other hand, $\hat{F}_K^\alpha$ vanishes rapidly when η going to infinity. Therefore, as $\eta \to \infty$,

$$\hat{F}_0^A \to (\hat{F}_{H0}^A)_b$$

$$= (\hat{n}_{H0}^A)_b (\hat{T}_{H0})_b^{-3/2} \left(\frac{1}{\pi} \right)^{3/2} \exp \left(- \frac{[\zeta_i - (\hat{v}_{iH0})_b]^2}{(\hat{T}_{H0})_b} \right), \tag{35a}$$

$$\hat{F}_0^B \to 0. \tag{35b}$$

Equations (32a)–(35b) form a half-space boundary-value problem of the spatially one-dimensional Boltzmann equation, which will be discussed in the next subsection. As we will see, in order for the problem to have a solution, the boundary values $(\hat{n}_{H0}^A)_b$, $(\hat{v}_{iH0})_b$, and $(\hat{T}_{H0})_b$ of the Hilbert part contained in Eq. (35a) and the quantities $\hat{T}_w$, $\hat{v}_{wi}$, and $\hat{p}_w^A$ contained in Eq. (33) must satisfy certain relations. These relations give the essential part of the boundary condition for the Euler set (24a)–(24d).

If we integrate Eq. (32b) with respect to ζ_i over its whole space and take into account Eq. (35b), we have

$$\Phi_{iK0}n_i = 0, \qquad \text{for } 0 \leq \eta, \tag{36}$$

where Φ_{iKm} denotes the particle flux corresponding to $\hat{F}_{Km}^B$ (note that $\hat{F}_0^B = \hat{F}_{K0}^B$), i.e.,

$$\Phi_{iKm} = \int \zeta_i \hat{F}_{Km}^B \mathrm{d}^3\zeta. \tag{37}$$

Since we are interested in the behavior in the continuum limit, Eqs. (32a)–(35b) play the main role. However, we need a piece of information from the first-order Knudsen-layer correction. The first-order equation corresponding to Eq. (32b) is given by

$$\begin{aligned}
\zeta_i n_i \frac{\partial \hat{F}_{K1}^B}{\partial \eta} = {} & \hat{J}^{AB}\left((\hat{F}_{H1}^A)_b + \left(\frac{\partial \hat{F}_{H0}^A}{\partial x_i}\right)_b n_i \eta + \hat{F}_{K1}^A, \hat{F}_{K0}^B\right) \\
& + \hat{J}^{AB}\left((\hat{F}_{H0}^A)_b + \hat{F}_{K0}^A, \hat{F}_{K1}^B\right) \\
& + \hat{J}^{BB}(\hat{F}_{K1}^B, \hat{F}_{K0}^B) + \hat{J}^{BB}(\hat{F}_{K0}^B, \hat{F}_{K1}^B) \\
& - \zeta_j\left[\left(\frac{\partial \chi_1}{\partial x_j}\right)_b \frac{\partial \hat{F}_{K0}^B}{\partial \chi_1} + \left(\frac{\partial \chi_2}{\partial x_j}\right)_b \frac{\partial \hat{F}_{K0}^B}{\partial \chi_2}\right].
\end{aligned} \tag{38}$$

If we integrate this equation with respect to ζ_i over its whole space and use Eq. (36), we have

$$\frac{\partial}{\partial \eta}[\,\Phi_{iK1}\,n_i] + \chi_{1,1}\frac{\partial}{\partial \chi_1}[\Phi_{iK0}\,t_i^{(1)}] + \chi_{2,2}\frac{\partial}{\partial \chi_2}[\Phi_{iK0}\,t_i^{(2)}]$$
$$+ g_2\,\Phi_{iK0}\,t_i^{(1)} - g_1\,\Phi_{iK0}\,t_i^{(2)} = 0. \tag{39}$$

Here, $t_i^{(1)}$ and $t_i^{(2)}$ are, respectively, unit tangential vectors to the boundary in the direction of increasing χ_1 and χ_2 taken in such a way that $t_i^{(1)}$, $t_i^{(2)}$, and n_i form a right-hand system, $\chi_{1,1}$ and $\chi_{2,2}$ are defined by

$$\chi_{1,1} = \left(\frac{\partial \chi_1}{\partial x_j}\right)_b t_j^{(1)}, \qquad \chi_{2,2} = \left(\frac{\partial \chi_2}{\partial x_j}\right)_b t_j^{(2)}, \tag{40}$$

and g_1 and g_2 are, respectively, the geodesic curvatures (Kreyszig [37]) (in the dimensionless x_i space) of the χ_1 and χ_2 coordinate lines on the boundary (see Sone et al. [35] for the details). Equation (39) is the continuity equation for the B-component in the Knudsen layer. Because of the diffuse reflection condition (5) (with $\alpha = B$) and (6b), the condition $\int \zeta_i n_i \hat{F}^B \mathrm{d}^3\zeta = 0$, i.e., $\int \zeta_i n_i(\hat{F}_{Hm}^B + \hat{F}_{Km}^B)\,\mathrm{d}^3\zeta = 0$, is always satisfied on the boundary. Therefore, it follows from Eq. (25) that $\Phi_{iKm}n_i = 0$ at $\eta = 0$. The same relations also hold at $\eta = \infty$ because $\hat{F}_{Km} \to 0$ as $\eta \to \infty$. If we make use of $\Phi_{iK1}n_i = 0$ at $\eta = 0$ and ∞, the integration of Eq. (39) with respect to η from $\eta = 0$ to ∞ yields

$$\chi_{1,1}\frac{\partial}{\partial \chi_1}(\hat{\mathcal{N}}_{i0}t_i^{(1)}) + \chi_{2,2}\frac{\partial}{\partial \chi_2}(\hat{\mathcal{N}}_{i0}t_i^{(2)}) + g_2\hat{\mathcal{N}}_{i0}t_i^{(1)} - g_1\hat{\mathcal{N}}_{i0}t_i^{(2)} = 0, \tag{41}$$

where

$$\hat{\mathcal{N}}_{i0} = \int_0^\infty \Phi_{iK0}\,\mathrm{d}\eta = \int_0^\infty \left(\int \zeta_i \hat{F}_0^B \mathrm{d}^3\zeta\right)\mathrm{d}\eta. \tag{42}$$

(Note that $\hat{F}_{K0}^B = \hat{F}_0^B$.) Incidentally, $\hat{\mathcal{N}}_{i0}n_i = 0$ because of Eq. (36).

In the two-dimensional case, we may assume the physical quantities to be independent of χ_2. For simplicity, let us denote χ_1 by χ and $t_i^{(1)}$ by t_i. Then, because $g_1 = g_2 = 0$ in this case, it follows from Eq. (41) that

$$\hat{\mathcal{N}}_{i0}\,t_i = \text{const.} \tag{43}$$

As we will see, Eq. (41) or (43) (in the two-dimensional case) is a part of the boundary condition for the Euler set of equations (24a)–(24d) on the condensing surface.

For convenience of the later use, we introduce the following $\tilde{\Gamma}$ and $\hat{\mathcal{N}}_{i0}^*$:

$$\tilde{\Gamma} = \int_0^\infty \left(\int \hat{F}_0^B d^3\zeta \right) d\eta = \int_0^\infty \hat{n}_0^B d\eta, \tag{44}$$

$$\hat{\mathcal{N}}_{i0}^* = \int_0^\infty \left[\int (\zeta_i - \hat{v}_{wi}) \hat{F}_0^B d^3\zeta \right] d\eta = \int_0^\infty [\Phi_{iK0} - \hat{n}_0^B \hat{v}_{wi}] d\eta$$

$$= \hat{\mathcal{N}}_{i0} - \hat{v}_{wi} \tilde{\Gamma}, \tag{45}$$

where

$$\hat{n}_0^B = \int \hat{F}_0^B d^3\zeta, \tag{46}$$

and $n_r \hat{n}_0^B$ is the molecular number density of the noncondensable gas in the Knudsen layer. The $(\sqrt{\pi}/2) n_r \ell_r \tilde{\Gamma}$, $(\sqrt{\pi}/2) m^B n_r (2kT_r/m^A)^{1/2} \ell_r \hat{\mathcal{N}}_{i0}$, and $(\sqrt{\pi}/2) m^B n_r (2kT_r/m^A)^{1/2} \ell_r \hat{\mathcal{N}}_{i0}^*$ are, respectively, the total number, total momentum, and total momentum based on the velocity relative to the boundary of the B-molecules, contained in the Knudsen layer per unit area on the boundary.

3.3 Half-space problem of evaporation or condensation

The boundary-value problem for the Knudsen-layer correction, consisting of Eqs. (32a) and (32b) and boundary conditions (33)–(35b), is equivalent to the following half-space problem of (strong) evaporation or condensation.

Consider a half space $X_1 > 0$ filled with the vapor (A-component), bounded by an infinite plane condensed phase of the vapor located at $X_1 = 0$ and at rest, where (X_1, X_2, X_3) is a system of rectangular space coordinates. The condensed phase is kept at temperature T_s, and the saturation vapor pressure at temperature T_s is denoted by p_s^A. At infinity, there is a uniform equilibrium flow of the vapor with pressure p_∞^A, temperature T_∞, and flow velocity $(v_{1\infty}, v_{2\infty}, 0)$ $(v_{2\infty} \geq 0)$. On the condensed phase, steady evaporation $(v_{1\infty} > 0)$ or condensation $(v_{1\infty} < 0)$ is taking place. The noncondensable gas (B-component) may be present near the condensed phase. Investigate the steady behavior of the vapor and the noncondensable gas under the complete condensation condition for the vapor and the diffuse reflection condition for the noncondensable gas on the condensed phase.

In fact, the basic equation and the boundary condition for this problem are given by Eqs. (32a)–(35b) if we take into account the following correspondence between the parameters and variables of the problem and those in Eqs. (32a)–(35b): T_s, p_s^A, T_∞, p_∞^A, $v_{1\infty}$, $v_{2\infty}$, and X_1 correspond to $T_r \hat{T}_w$, $p_r \hat{p}_w^A$, $T_r(\hat{T}_{H0})_b$, $p_r(\hat{p}_{H0}^A)_b$, $(2kT_r/m^A)^{1/2}(\hat{v}_{jH0})_b n_j$, $(2kT_r/m^A)^{1/2}|(\hat{v}_{iH0})_b - \hat{v}_{wi} - (\hat{v}_{jH0})_b n_j n_i|$, and $(\sqrt{\pi}/2)\ell_r \eta$, respectively, and the direction of the positive X_1 axis corresponds to that of n_i.

First we consider the case where evaporation is taking place $(v_{1\infty} > 0)$. In this case, being blown away by the evaporating vapor flow, the noncondensable gas cannot stay near the condensed phase or in the Knudsen layer, namely $\hat{F}_0^B = 0$. This fact, which is intuitively obvious and is supported by some numerical results (the transition process in which the noncondensable gas initially occupying the half space is swept away by the evaporating vapor is investigated numerically in Doi et al. [38]), can be shown rigorously (Taguchi et al. [39]) for the Boltzmann equation for the Maxwellian molecules, as well as for the model equations based on the Maxwellian molecules, such as the model proposed by Garzó et al. [40]. Therefore, the problem is reduced to that of an evaporating flow of the pure vapor, which has been investigated by many authors (Kogan and Makashev [41], Murakami and Oshima [42], Ytrehus [43], Sone [44, 45], Sone and Sugimoto [46], Aoki et al. [47], Sone et al. [48], Bobylev et al. [49]). There is a steady solution to the half-space problem only when the parameters T_s, p_s^A, T_∞, p_∞^A, $v_{1\infty}$, and $v_{2\infty}$ satisfy the following relations (Sone and Sugimoto [46]).

$$M_t = 0, \qquad M_n \leq 1,$$

$$\frac{p_\infty^A}{p_s^A} = h_1(M_n), \qquad \frac{T_\infty}{T_s} = h_2(M_n), \tag{47}$$

where

$$M_t = v_{2\infty}(5kT_\infty/3m^A)^{-1/2}, \qquad M_n = v_{1\infty}(5kT_\infty/3m^A)^{-1/2}. \tag{48}$$

The M_t and M_n are, respectively, the Mach number at infinity based on the tangential flow speed and that based on the normal flow speed. The functions $h_1(M_n)$ and $h_2(M_n)$ are obtained accurately by means of a numerical analysis of the BGK model (Bhatnagar et al. [50], Welander [51], and Kogan [52]) of the Boltzmann equation in Sone and Sugimoto [46]. The numerical values of these functions are given in Table I. The analytical form of these functions for $M_n \ll 1$ was obtained by Sone [44] (see also Sone [45, 5] and Sone and Aoki [32]).

The case where the condensation is taking place ($v_{1\infty} < 0$) is studied in Sone et al. [53] and Aoki and Doi [54]. In Sone et al. [53], by considering the case where the molecule of the noncondensable gas is mechanically identical with that of the vapor, the problem is successfully decomposed into two problems, one for the total mixture and the other for the noncondensable gas. The former problem is the same as the half-space problem of condensation for a pure vapor, which has extensively been investigated in the literature (Kogan and Makashev [41], Sone [44, 45], Sone et al. [55], Aoki et al. [56], Aoki et al. [47], Kogan and Abramov [57], Kryukov [58], Sone et al. [59], Sone et al. [48], Bobylev et al. [49]). For example, the condition that allows a steady solution has been clarified in a series of analytical and numerical studies (Sone [44, 45], Sone et al. [55], Aoki et al. [56], Aoki et al. [47], Sone et al. [59]) (see also Sone [4, 5]). Therefore, the above decomposition enables us to exploit the comprehensive results for the pure-vapor case obtained so far. Furthermore, this approach not only reduces the necessary amount of computation drastically, but also gives the clear understanding of the basic structure of the solution. In Sone et al. [53] and Aoki and Doi [54], the study is concentrated on the case where the vapor is condensing perpendicularly ($v_{2\infty} = 0$). Recently, the same analysis was extended to the general case where the vapor is condensing at incidence ($v_{2\infty} \neq 0$) (Taguchi et al. [39, 60]). According to this result, under the above condition that the molecules of the two components are identical, the solution to the half-space problem exists only when the parameters T_s, p_s^A, T_∞, p_∞^A, $v_{1\infty}$, and $v_{2\infty}$ satisfy the following relation.

$$\frac{p_\infty^A}{p_s^A} = \mathcal{F}_s\left(|M_n|, M_t, \frac{T_\infty}{T_w}, \Gamma\right), \qquad (|M_n| < 1), \tag{49a}$$

$$\frac{p_\infty^A}{p_s^A} > \mathcal{F}_b\left(|M_n|, M_t, \frac{T_\infty}{T_w}, \Gamma\right), \qquad (|M_n| > 1), \tag{49b}$$

where

$$\Gamma = (2/\sqrt{\pi})(N^B/n_\infty^A \ell_\infty), \qquad N^B = \int_0^\infty n^B \mathrm{d}X_1. \tag{50}$$

Here, M_t and M_n are defined by Eq. (48) ($|M_n|$ is the Mach number based on the normal flow speed at infinity); $n_\infty^A = p_\infty^A/kT_\infty$ is the number density of the vapor molecules at infinity; ℓ_∞ is the mean free path of the vapor molecules in the equilibrium state at rest with number density n_∞^A and temperature T_∞; n^B is the molecular number density of the noncondensable gas; and N^B is the total number of the noncondensable-gas molecules contained in the semi-infinite column ($X_1 > 0$) with the base of unit area standing perpendicularly on the condensed phase. The Γ is a parameter to be specified and is a measure of the amount of the noncondensable gas contained in the half space.

The functions $\mathcal{F}_s$ and $\mathcal{F}_b$ are, respectively, monotonically increasing and decreasing functions in $|M_n|$. For $0 \leq \Gamma < \Gamma_{cr}$, where Γ_{cr} is a critical value depending on T_∞/T_w and M_t, the $\mathcal{F}_s$ and $\mathcal{F}_b$ meet at $|M_n| = 1$, i.e., $\mathcal{F}_s(1_-, M_t, T_\infty/T_w, \Gamma) = \mathcal{F}_b(1_+, M_t, T_\infty/T_w, \Gamma)$. In this case, Eqs. (49a) and (49b) should be supplemented by

$$\frac{p_\infty^A}{p_s^A} \geq \mathcal{F}_s\left(1_-, M_t, \frac{T_\infty}{T_w}, \Gamma\right), \qquad (|M_n| = 1). \tag{51}$$

For $\Gamma \geq \Gamma_{cr}$, the $\mathcal{F}_s$ and $\mathcal{F}_b$ increase infinitely as $|M_n| \to c_s$ (≤ 1) and $|M_n| \to c_b$ (≥ 1), respectively, where c_s and c_b depend on M_t, T_∞/T_w, and Γ, and $c_s = c_b = 1$ when $\Gamma = \Gamma_{cr}$. The functions $\mathcal{F}_s$ and $\mathcal{F}_b$ were constructed numerically in Sone et al. [53], Aoki and Doi [54], Taguchi et al. [39, 60], where the numerical data of the corresponding functions for the pure-vapor case (Sone et al. [55], Aoki et al. [56], Aoki et al. [47]), obtained by using the BGK model, were exploited, and additional computations were carried out by the use of the model Boltzmann equation for a mixture proposed by Garzó et al. [40]. (It should be noted that the Γ-dependence of $\mathcal{F}_s$ and $\mathcal{F}_b$ is obtained explicitly.) As an example, $\mathcal{F}_s$ and $\mathcal{F}_b$ are shown in Figs. 1 and 2, respectively, as the functions of $|M_n|$ for several Γ and for $M_t = 1$ and 3 in

the case of $T_\infty/T_w = 1$ (see Taguchi et al. [39, 60] for the details). The numerical results in Taguchi et al. [39, 60] show that the dependence of $\mathcal{F}_s$ and $\mathcal{F}_b$ on M_t and T_∞/T_w is weak, as in the pure vapor case ($\Gamma = 0$) (Aoki et al. [47]). Some data obtained by a DSMC computation for hard-sphere gases indicate that a relation of the same form as Eq. (49a) holds in the general case where the molecules of the two components are different (i.e., different masses and diameters) (see Taguchi et al. [39]).

The numerical results by Taguchi et al. [39, 60] show that in this half-space problem of condensation, a macroscopic flow of the noncondensable gas is caused along the condensed phase in the positive X_2 direction when $v_{2\infty} > 0$, but it is not when $v_{2\infty} = 0$. That is, the direction of the flow of the noncondensable gas along the condensed phase is the same as the direction of the component of the flow velocity of the vapor parallel to the condensed phase at infinity. Let $\mathcal{N}_f$ be the particle flow of the noncondensable gas in the positive X_2 direction per unit time and unit width in X_3 and $\hat{\mathcal{N}}_f$ be its dimensionless counterpart defined by

$$\hat{\mathcal{N}}_f = (2/\sqrt{\pi})[n_\infty^A \ell_\infty (2kT_\infty/m^A)^{1/2}]^{-1} \mathcal{N}_f. \tag{52}$$

Note that $m^B \mathcal{N}_f$ is the total momentum of the B-molecules contained in the column used in the definition of N^B [see the sentence below Eq. (50)]. Since $\hat{\mathcal{N}}_f$ is determined by a solution of the half-space problem, its dependence of the parameters is given as follows.

$$\hat{\mathcal{N}}_f = G_s\left(|M_n|, M_t, \frac{T_\infty}{T_w}, \Gamma\right), \qquad (|M_n| < 1), \tag{53a}$$

$$\hat{\mathcal{N}}_f = G_b\left(|M_n|, M_t, \frac{T_\infty}{T_w}, \frac{p_\infty^A}{p_s^A}, \Gamma\right), \qquad (|M_n| \geq 1), \tag{53b}$$

where $G_s = G_b = 0$ at $M_t = 0$ because $\hat{\mathcal{N}}_f$ vanishes when $v_{2\infty} = 0$. The functions G_s and G_b obtained numerically are given in Taguchi et al. [39, 60]. To be more precise, G_b has been obtained in a restricted manner because it is a function of five variables. On the other hand, G_s turns out to be of the form

$$G_s = \Gamma \, \mathcal{G}(|M_n|, M_t, T_\infty/T_w), \tag{54}$$

in the case where the molecules of the two components are mechanically the same. The function $\mathcal{G}$ based on the model Boltzmann equation by Garzó et al., which was found to be almost independent of $|M_n|$ and T_∞/T_w, is shown for $|M_n| = 0.1$ and $T_\infty/T_w = 1$ in Fig. 3.

3.4 Boundary condition for the Euler equations

The relations (47), (49a), and (49b) [and (51)] given in Sec. 3.3 essentially provide the boundary condition for the Euler set of equations (24a)–(24d). One only needs to rewrite them by the use of the variables in Sec. 3.1. First, noting that n^B in Eq. (50) corresponds to $n_r \hat{n}_0^B$, we obtain the following relation between Γ and $\widetilde{\Gamma}$ [Eq. (44)]:

$$\Gamma = \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b} \int_0^\infty \left(\int \hat{F}_0^B d^3\zeta \right) d\eta = \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b} \widetilde{\Gamma}, \tag{55}$$

where, ℓ_b is the mean free path of the vapor molecules in the equilibrium state at rest with number density $n_r(\hat{n}_{H0}^A)_b$ and temperature $T_r(\hat{T}_{H0})_b$. Since $\widetilde{\Gamma}$ does not depend on the local state of the vapor [such as $(\hat{n}_{H0}^A)_b$ and ℓ_b], it is preferable to use $\widetilde{\Gamma}$ rather than Γ as a quantity related to the amount of the noncondensable gas in the Knudsen layer. Then, the boundary conditions can be summarized in the following form: the conditions on the evaporating surface, where $M_n > 0$, are

$$\begin{aligned} M_t &= 0, & M_n &\leq 1, \\ (\hat{p}_{H0}^A)_b/\hat{p}_w^A &= h_1(M_n), & (\hat{T}_{H0})_b/\hat{T}_w &= h_2(M_n), \end{aligned} \tag{56}$$

and those on the condensing surface, where $M_n < 0$, are

$$\frac{(\hat{p}_{H0}^A)_b}{\hat{p}_w^A} = \mathcal{F}_s\left(|M_n|, M_t, \frac{(\hat{T}_{H0})_b}{\hat{T}_w}, \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b} \widetilde{\Gamma}\right), \qquad (|M_n| < 1), \tag{57a}$$

$$\frac{(\hat{p}_{H0}^A)_b}{\hat{p}_w^A} > \mathcal{F}_b\left(|M_n|, M_t, \frac{(\hat{T}_{H0})_b}{\hat{T}_w}, \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b} \widetilde{\Gamma}\right), \qquad (|M_n| > 1). \tag{57b}$$

Here,

$$M_n = \sqrt{6/5}(\hat{T}_{H0})_b^{-1/2}(\hat{v}_{jH0})_b n_j, \tag{58a}$$

$$M_t = \sqrt{6/5}(\hat{T}_{H0})_b^{-1/2}|(\hat{v}_{iH0})_b - \hat{v}_{wi} - (\hat{v}_{jH0})_b n_j n_i|. \tag{58b}$$

Equations (57a) and (57b) are supplemented by

$$\frac{(\hat{p}_{H0}^A)_b}{\hat{p}_w^A} \geq \mathcal{F}_s\left(1_-, M_t, \frac{(\hat{T}_{H0})_b}{\hat{T}_w}, \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b} \widetilde{\Gamma}\right), \qquad (|M_n| = 1), \tag{59}$$

when $\widetilde{\Gamma} < (\hat{n}_{H0}^A)_b \ell_b \ell_r^{-1} \Gamma_{cr}$.

In the case of a pure vapor (or in the case of $\widetilde{\Gamma} = 0$), Eqs. (56)–(57b), and (59) are known to be consistent boundary conditions (Aoki and Sone [6], Sone [4, 5]). In the spatially one-dimensional case, such as the two-surface problem of evaporation and condensation, $\widetilde{\Gamma}$ is a constant that can be specified (see Aoki et al. [18] and Aoki [21]), and therefore the situation is essentially the same as that in the pure vapor case. However, in the general geometry, $\widetilde{\Gamma}$ is not a constant but a function of χ_1 and χ_2, as is seen from Eq. (44). In other words, $\widetilde{\Gamma}$ varies along the boundary. Therefore, an additional relation is required as the boundary condition on the condensing surface. This relation is provided by Eq. (41) [or Eq. (43) for the two-dimensional case], i.e.,

$$\chi_{1,1}\frac{\partial}{\partial\chi_1}(\hat{\mathcal{N}}_{i0}t_i^{(1)}) + \chi_{2,2}\frac{\partial}{\partial\chi_2}(\hat{\mathcal{N}}_{i0}t_i^{(2)}) + g_2\hat{\mathcal{N}}_{i0}t_i^{(1)} - g_1\hat{\mathcal{N}}_{i0}t_i^{(2)} = 0, \tag{60}$$

if we have the relation between $\hat{\mathcal{N}}_{i0}$ and $\widetilde{\Gamma}$.

Here, let us note the following correspondence between $\hat{\mathcal{N}}_{i0}^*$ [Eq. (45)] and $\hat{\mathcal{N}}_f$ [Eq. (52)]:

$$(\sqrt{\pi}/2)n_r(2kT_r/m^A)^{1/2}\ell_r|\hat{\mathcal{N}}_{i0}^*| \iff (\sqrt{\pi}/2)n_\infty^A(2kT_\infty/m^A)^{1/2}\ell_\infty\hat{\mathcal{N}}_f. \tag{61}$$

Using Eqs. (53a) and (53b) and taking into account the statement above Eq. (52), we can write

$$\hat{\mathcal{N}}_{i0} = \hat{v}_{wi}\widetilde{\Gamma} + \frac{\ell_b}{\ell_r}(\hat{n}_{H0}^A)_b(\hat{T}_{H0})_b^{1/2} G_s\left(|M_n|, M_t, \frac{(\hat{T}_{H0})_b}{\hat{T}_w}, \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b}\widetilde{\Gamma}\right)a_i, \quad (|M_n| < 1), \tag{62a}$$

$$\hat{\mathcal{N}}_{i0} = \hat{v}_{wi}\widetilde{\Gamma} + \frac{\ell_b}{\ell_r}(\hat{n}_{H0}^A)_b(\hat{T}_{H0})_b^{1/2} G_b\left(|M_n|, M_t, \frac{(\hat{T}_{H0})_b}{\hat{T}_w}, \frac{(\hat{p}_{H0}^A)_b}{\hat{p}_w^A}, \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b}\widetilde{\Gamma}\right)a_i, \quad (|M_n| \geq 1), \tag{62b}$$

where a_i is a unit vector defined by

$$a_i = \frac{(\hat{v}_{iH0})_b - \hat{v}_{wi} - (\hat{v}_{jH0})_b n_j n_i}{|(\hat{v}_{iH0})_b - \hat{v}_{wi} - (\hat{v}_{jH0})_b n_j n_i|}. \tag{63}$$

Equations (62a) and (62b) give the relation between $\hat{\mathcal{N}}_{i0}$ and $\widetilde{\Gamma}$ mentioned above and thus complete the boundary condition.

To summarize, the boundary conditions for the Euler set of equations (24a)–(24d) are given by Eq. (56) on the evaporating surface and Eqs. (57a) and (57b) [and (59)] with Eqs. (60), (62a), and (62b) on the condensing surface. It should be noted that this system determines $\hat{v}_{iH0}$, $\hat{T}_{H0}$, and $\hat{p}_{H0}^A$ together with $\widetilde{\Gamma}$, which is a function of the position on the boundary and specifies in what manner the noncondensable gas is distributed along the boundary. Of course, we need to specify a quantity, such as n_{av}^B/n_r [$= O(\text{Kn})$], related to the amount of the noncondensable gas in order to have a solution of the Euler system. This imposes a constraint on $\widetilde{\Gamma}$. However, a specified value of n_{av}^B/n_r does not necessarily guarantee a unique solution, as is obvious from the physical point of view. That is, we may obtain a different solution in which the same amount of the noncondensable gas is present in a different part of the boundary. The uniqueness and nonuniqueness of the solution highly depend on the geometry of the problem under consideration. Therefore, the Euler system should be applied with careful consideration based on physical intuition.

Apart from the problem of nonuniqueness mentioned above, the application of the boundary conditions is straightforward if we know on which part of the boundary evaporation will take place and on which part condensation will take place. Such an example will be given in the next section. But, when we do not know it beforehand, a sort of iteration is necessary in the process of analysis or numerical analysis.

3.5 Continuum limit

Now let us discuss the continuum limit where ϵ (or Kn) goes to zero. In this limit, the thickness of the Knudsen layer vanishes, and the velocity distribution function of each component reduces to the leading-order term of the Hilbert expansion except on the boundary, i.e., $\hat{F}^A \rightarrow \hat{F}^A_{H0}$ and $\hat{F}^B \rightarrow 0$. The $\hat{n}^A_{H0}$ $(= \hat{\rho}^A_{H0})$, $\hat{v}_{iH0}$, and $\hat{T}_{H0}$ occurring in $\hat{F}^A_{H0}$ [Eq. (16)] are determined by the Euler set, Eqs. (24a)–(24d), and the boundary conditions, Eqs. (56), (57a), (57b), [(59)], (60), (62a), and (62b). On the other hand, it follows from Eq. (10) that $n^B_{av}/n_r \rightarrow 0$ in this limit. In other words, the average concentration of the noncondensable gas over the whole domain becomes infinitely small. (This is consistent with $\hat{F}^B \rightarrow 0$.) Therefore, it would seem that the effect of the noncondensable gas disappears in this limit. However, if we look at the boundary condition on the condensing surface, Eqs. (57a), (57b), and (59), then we notice that it depends on $\tilde{\Gamma}$, which is related to the noncondensable gas and is of the order of unity, independent of ϵ. This means that in the continuum limit, the overall vapor flow is still affected by the noncondensable gas through the boundary condition on the condensing surface, in spite of the fact that the average concentration of the noncondensable gas is infinitesimal.

The physical picture of this situation is as follows. Being blown by the overwhelming vapor flow, the noncondensable gas of an infinitesimal concentration accumulates in a thickless Knudsen layer on the condensing surface, where its local number density becomes comparable to that of the vapor, and has a significant effect on the condensing vapor flow. This fact was pointed out by Aoki et al. [18] for the simple two-surface problem, i.e., a vapor flow caused by evaporation and condensation in the gap between two parallel plane condensed phases. In connection with the two-surface problem, it should be mentioned that an interesting experiment to measure the temperature distribution between the two condensed phases was performed by Shankar and Deshpande [61, 62]. Their aim was to perform the measurement in a pure vapor flow, but the effect of the impurity, i.e., the presence of a small amount of the noncondensable gas, is also discussed in Shankar and Deshpande [62].

3.6 Case of an infinite domain

In this subsection, we give a short comment on the case of an infinite domain. As is seen from the course of analysis, the results obtained in Secs. 3.1-3.5 are also valid in the case of an infinite domain, more precisely, in the case where there is a flow of the vapor in an infinite domain, and the noncondensable gas is confined in the Knudsen layer on (a part of) the boundary where condensation is taking place. That is, the vapor flow in this situation is described by the Euler set (24a)–(24d) and the boundary conditions (56) on the evaporating surface and (57a) and (57b) [and (59)] with (60), (62a), and (62b) on the condensing surface, supplemented by the boundary condition at infinity. Only the difference is that n^B_{av} has no more sense, so that the amount of the noncondensable gas should be specified in a different way. For example, let us suppose that the noncondensable gas is present only in the part S with a finite area of the boundary. Then, we can obtain such a solution of the Euler system by specifying the total number of the noncondensable gas molecules, say M^B. In fact, it is related to $\tilde{\Gamma}$ as $M^B = (\sqrt{\pi}/2)n_r \ell_r L^2 \int_S \tilde{\Gamma} da$, where da is a surface element on the boundary in the dimensionless x_i space, and this gives a constraint on $\tilde{\Gamma}$. However, as in the case of a closed domain, a specified value of M^B does not guarantee a unique solution. In the continuum limit, the average concentration of the noncondensable gas becomes infinitesimal in a subdomain with the extent of L that contains the part S of the boundary.

4 Application

In this section we give an application of the Euler set of equations and their boundary conditions derived in the previous section. The problem that we are going to investigate is as follows.

Consider a vapor (A-component) in a gap between two condensed phases at rest, one is of sinusoidal shape located at $X_1 = A\cos(\pi X_2/L)$, and the other is a plane located at $X_1 = L$, where X_i is a (dimensional) coordinate system (Fig. 4). Let the temperature of the sinusoidal condensed phase be T_I and that of the plane one be T_{II}, and let the saturation vapor pressure at temperature T_I be p_I and that at temperature T_{II} be p_{II}. A small amount of a noncondensable gas (B-component) is contained in the gap, as specified below. Let n^B_{av} denote the average molecular number density of the noncondensable gas over the gap. We investigate the steady flow of the vapor caused by evaporation and condensation on the

condensed phases and the effect of the noncondensable gas on the vapor flow when the Knudsen number Kn (with respect to the vapor) going to zero (continuum limit).

Here, we take L as the reference length and T_I, p_I, and $n_I = p_I/kT_I$ as the reference quantities $(T_r = T_I, p_r = p_I$, and $n_r = n_I)$. The problem is characterized by the following dimensionless parameters:

$$\frac{T_{II}}{T_I}, \quad \frac{p_{II}}{p_I}, \quad \frac{A}{L}, \quad \text{Kn} = \frac{\ell_r}{L}, \quad \frac{n_{av}^B}{n_I}, \tag{64}$$

where ℓ_r is the mean free path of the molecules of the vapor when it is in the equilibrium state at rest with temperature T_I and pressure p_I. To be consistent with Eq. (10), we put

$$n_{av}^B/n_I = \Delta \text{Kn}, \tag{65}$$

and specify Δ rather than n_{av}^B/n_I.

We assume that the flow field is periodic (with period $2L$) in the X_2 direction and is symmetric with respect to the X_1 axis. Therefore, we may consider the problem in the closed domain

$$A\cos(\pi X_2/L) \le X_1 \le L, \qquad 0 \le X_2 \le L, \tag{66}$$

imposing the specular reflection condition on $X_2 = 0$ and L.

Now let us apply the Euler set of equations and its boundary conditions to the present problem. We consider the case where evaporation of the vapor is taking place on the plane condensed phase and condensation on the sinusoidal condensed phase. In this two-dimensional problem, Eq. (43) holds along the latter condensed phase. But $\mathcal{N}_{i0} t_i = 0$ at $X_2 = 0$ and L (or $x_2 = 0$ and 1) because of the specularly reflecting boundary. Therefore, $\mathcal{N}_{i0} t_i = 0$ holds identically. It follows from Eqs. (62a) and (62b) (with $\hat{v}_{wi} = 0$) and the property of G_s and G_b that $M_t = 0$ on the condensing surface if $\tilde{\Gamma} \ne 0$ there. Let us restrict ourselves to the case where $|M_n| < 1$. Then the boundary condition for the Euler set (24a)–(24d) is Eq. (56), i.e.,

$$M_t = 0, \qquad (\hat{p}_{H0}^A)_b = \frac{p_{II}}{p_I} h_1(M_n), \qquad (\hat{T}_{H0})_b = \frac{T_{II}}{T_I} h_2(M_n), \tag{67}$$

on the plane condensed phase ($x_1 = 1$; note that $\hat{p}_w^A = p_{II}/p_I$ and $\hat{T}_w = T_{II}/T_I$ there) and Eq. (57a) with $M_t = 0$, i.e.,

$$(\hat{p}_{H0}^A)_b = \mathcal{F}_s\left(|M_n|, 0, (\hat{T}_{H0})_b, \frac{\ell_r}{(\hat{n}_{H0}^A)_b \ell_b}\tilde{\Gamma}\right), \tag{68}$$

on the sinusoidal condensed phase $[x_1 = (A/L)\cos(\pi x_2)$; note that $\hat{p}_w^A = \hat{T}_w = 1$ there]. It should be noted that $\tilde{\Gamma}$ can be zero in a certain part of the sinusoidal condensed phase, where we should use

$$(\hat{p}_{H0}^A)_b = \mathcal{F}_s\left(|M_n|, M_t, (\hat{T}_{H0})_b, 0\right), \tag{69}$$

no restriction being imposed on M_t. These boundary conditions are supplemented by the condition $\hat{v}_{2H0} = 0$, which corresponds to the specular reflection condition, on $x_2 = 0$ and 1. Finally we need the relation between Δ and $\tilde{\Gamma}$, which is, as described below, given by

$$\Delta = \frac{\sqrt{\pi}}{2} \int \tilde{\Gamma}\, ds, \tag{70}$$

where ds is the line element along the sinusoidal boundary in the dimensionless $x_1 x_2$ plane, and the range of integration is from $x_2 = 0$ to 1. This relation is obtained by noting that $(\sqrt{\pi}/2)n_r\ell_r\tilde{\Gamma}$ is the total number of the noncondensable gas in the Knudsen layer per unit area of the boundary [see Eq. (44) and the sentence following Eq. (46)] and that its total number in the entire domain per unit width in X_3, i.e., $n_{av}^B L^2$, is therefore given by $(\sqrt{\pi}/2)n_r\ell_r L \int \tilde{\Gamma} ds$. The ratio ℓ_r/ℓ_b occurring in Eq. (68) depends on the molecular model; for example, $\ell_r/\ell_b = (\hat{n}_{H0}^A)_b$ for hard-sphere molecules [cf. Eq. (4)], and $\ell_r/\ell_b = (\hat{n}_{H0}^A)_b/(\hat{T}_{H0})_b^{1/2}$ for the BGK model.

We solve this boundary-value problem for the Euler set of equations numerically. In Fig. 5, the stream lines of $\hat{v}_{iH0}$ for $\Delta = 0$ (pure vapor case) and 2 are shown in the case where $A/L = 0.2$, $T_{II}/T_I = 1$, and

$p_{II}/p_I = 2$. Figure 6, where the result for $\Delta = 4$ is also included, shows the distribution of $\tilde{\Gamma}$ along the sinusoidal condensed phase in the same case as Fig. 5. Here, we have assumed that the vapor molecules are mechanically identical with the noncondensable-gas molecules and used the numerical data of the functions h_1 and h_2 based on the BGK model (Table I) and those of the function $\mathcal{F}_s$ based on the model Boltzmann equation by Garzó et al., which is consistent with the BGK model for a single component case (see Fig. 1 for some examples of $\mathcal{F}_s$). Therefore, we have used the relation $\ell_r/\ell_b = (\hat{n}^A_{H0})_b/(\hat{T}_{H0})_b^{1/2}$ in Eq. (68). If the explicit form of the boundary is used, Eq. (70) becomes

$$\Delta = \frac{\sqrt{\pi}}{2} \int_0^1 \tilde{\Gamma}\sqrt{1 + (\pi A/L)^2 \sin^2(\pi x_2)} \, dx_2, \tag{71}$$

where $\tilde{\Gamma}$ is considered to be a function of x_2.

Figure 5 shows the leading-order flow field of the vapor for small Kn, which is at the same time the flow field in the continuum limit, Kn $\rightarrow$ 0. In this limit, the average concentration of the noncondensable gas becomes infinitesimal because of Eq. (65). However, the flow properties still depend on $\tilde{\Gamma}$, which is the trace of the noncondensable gas. In fact, the stream lines for $\Delta = 2$ are distinct from those for the pure vapor flow ($\Delta = 0$). In the former case, the stream lines enter the sinusoidal condensed phase perpendicularly because of the condition (68), i.e., $M_t = 0$, whereas in the latter case, they enter the same condensed phase obliquely because of Eq. (69), which is the condition there in the pure vapor case. The pattern of the stream lines for $\Delta = 4$, which are not shown in Fig. 5, is quite similar to that for $\Delta = 2$, but the flow speeds at the corresponding points are different. In this way, an infinitesimal average concentration (or the trace) of the noncondensable gas has a dramatic effect on the overall vapor flow.

A direct numerical simulation, based on the original Boltzmann system, of the same problem has been carried out for relatively small Kn by Aoki et al. [28], where both components are assumed to be hard-sphere gases, and the direct simulation Monte Carlo (DSMC) method (Bird [63, 64]) is employed as the solution technique. Although it is hard to draw a definite conclusion about the behavior in the continuum limit from the DSMC computation, the result is consistent with the present result; in fact, it gave a proper guideline for the asymptotic analysis in Sec. 3. The difficulty in the simulation for small Kn in the problem arises from the fact that, as Kn becomes small, the noncondensable gas is localized inside the Knudsen layer, the thickness of which becomes small as well. Since the overall flow field is affected by the localized noncondensable gas, its accurate description is required. For this purpose, we need a very fine cell near the condensing surface with a sufficient number of simulation particles for the noncondensable gas. This makes the size of the simulation system very large. This fact confirms the usefulness of the fluid-dynamic description based on the Euler set and its boundary condition.

5 Concluding remarks

In the present paper, we have investigated, on the basis of kinetic theory, steady flows of a vapor with evaporation and condensation on the boundary, consisting of the condensed phase of the vapor, in the presence of a noncondensable gas under the condition that the Knudsen number with respect to the vapor is small and that the average concentration of the noncondensable gas is also small and is of the order of the Knudsen number (see Sec. 2.1). The conventional boundary condition (complete condensation condition) for the vapor and the diffuse reflection condition for the noncondensable gas were employed as the boundary condition on the boundary. After the formulation of the problem in Sec. 2, a systematic asymptotic analysis for small Knudsen numbers was carried out in Sec. 3, where the fluid-dynamic equations (the Euler set of equations) for the vapor and their appropriate boundary conditions on the boundary were derived for the leading-order (or zeroth-order) terms in the Knudsen number. In deriving the boundary conditions, the previously known results for half-space problems have been exploited. On the basis of this system, we discussed the behavior of the continuum limit, i.e., the limit where the Knudsen number vanishes, in Sec. 3.5. There, it was shown that an infinitesimal average concentration of the noncondensable gas may have a significant effect on the overall vapor flow. An example demonstrating such an effect was given in Sec. 4.

The Euler set of equations and its boundary condition presented here give a clear understanding of the behavior of the vapor and the noncondensable gas in the near continuum regime as well as in the continuum limit. For example, the distribution of the noncondensable gas (or its trace), which is confined in the very thin (or infinitely thin) Knudsen layer, along the condensing surface is determined through

the boundary condition there. The Euler system is also useful in practical applications because the direct numerical computation based on the kinetic system for this type of problem is rather hard in general. For the moment, the available numerical boundary condition on the condensing surface has some limitations. More specifically, the numerical values of the functions $\mathcal{F}_s$, $\mathcal{F}_b$, G_s, and G_b occurring in the boundary condition are available only for a model Boltzmann equation and in a special case where the molecule of the vapor and that of the noncondensable gas are mechanically identical. But the qualitative structure of the boundary condition is most likely to be the same in the general case. The extension and enrichment of the numerical data for these functions would upgrade the applicability of the fluid-dynamic system.

Acknowledgements

Appendix A   Hilbert expansion of the macroscopic quantities

In this appendix, we give some of the expressions of the coefficients h_{Hm}^{α} and h_{Hm} in terms of $\hat{F}_{Hm}^{\alpha}$.

$$\hat{n}_{Hm}^{\alpha} = \int \hat{F}_{Hm}^{\alpha}\, d^3\zeta, \qquad \hat{\rho}_{Hm}^{\alpha} = \hat{m}^{\alpha}\hat{n}_{Hm}^{\alpha}, \qquad (m = 0, 1, ...), \tag{A1a}$$

$$\hat{v}_{iH0}^{\alpha} = (1/\hat{n}_{H0}^{\alpha})\int \zeta_i \hat{F}_{H0}^{\alpha}\, d^3\zeta, \tag{A1b}$$

$$\hat{p}_{H0}^{\alpha} = \hat{n}_{H0}^{\alpha}\hat{T}_{H0}^{\alpha} = \frac{2}{3}\hat{m}^{\alpha}\int (\zeta_i - \hat{v}_{iH0}^{\alpha})^2 \hat{F}_{H0}^{\alpha}\, d^3\zeta, \tag{A1c}$$

$$\hat{v}_{iH1}^{\alpha} = (1/\hat{n}_{H0}^{\alpha})\int \zeta_i \hat{F}_{H1}^{\alpha}\, d^3\zeta - (\hat{n}_{H1}^{\alpha}/\hat{n}_{H0}^{\alpha})\hat{v}_{iH0}^{\alpha}, \tag{A1d}$$

$$\hat{p}_{H1}^{\alpha} = \hat{n}_{H0}^{\alpha}\hat{T}_{H1}^{\alpha} + \hat{n}_{H1}^{\alpha}\hat{T}_{H0}^{\alpha} = \frac{2}{3}\hat{m}^{\alpha}\int (\zeta_i - \hat{v}_{iH0}^{\alpha})^2 \hat{F}_{H1}^{\alpha}\, d^3\zeta, \tag{A1e}$$

$$\cdots ,$$

$$\hat{n}_{Hm} = \sum_{\beta=A,B} \hat{n}_{Hm}^{\beta}, \qquad \hat{\rho}_{Hm} = \sum_{\beta=A,B} \hat{\rho}_{Hm}^{\beta}, \qquad (m = 0, 1, ...), \tag{A2a}$$

$$\hat{v}_{iH0} = (1/\hat{\rho}_{H0}) \sum_{\beta=A,B} \hat{\rho}_{H0}^{\beta}\hat{v}_{iH0}^{\beta}, \tag{A2b}$$

$$\hat{p}_{H0} = \hat{n}_{H0}\hat{T}_{H0} = \sum_{\beta=A,B} \left[\hat{p}_{H0}^{\beta} + \frac{2}{3}\hat{\rho}_{H0}^{\beta}(\hat{v}_{iH0}^{\beta} - \hat{v}_{iH0})^2\right], \tag{A2c}$$

$$\hat{v}_{iH1} = (1/\hat{\rho}_{H0}) \sum_{\beta=A,B} (\hat{\rho}_{H0}^{\beta}\hat{v}_{iH1}^{\beta} + \hat{\rho}_{H1}^{\beta}\hat{v}_{iH0}^{\beta}) - (\hat{\rho}_{H1}/\hat{\rho}_{H0})\hat{v}_{iH0}, \tag{A2d}$$

$$\hat{p}_{H1} = \hat{n}_{H0}\hat{T}_{H1} + \hat{n}_{H1}\hat{T}_{H0}$$
$$= \sum_{\beta=A,B} \left\{\hat{p}_{H1}^{\beta} + \frac{2}{3}\left[\hat{\rho}_{H1}^{\beta}(\hat{v}_{iH0}^{\beta} - \hat{v}_{iH0})^2 + 2\hat{\rho}_{H0}^{\beta}(\hat{v}_{iH0}^{\beta} - \hat{v}_{iH0})(\hat{v}_{iH1}^{\beta} - \hat{v}_{iH1})\right]\right\}, \tag{A2e}$$

$$\cdots .$$

Here, $\hat{T}_{H0}^{\alpha}$, $\hat{T}_{H1}^{\alpha}$, $\hat{T}_{H0}$, and $\hat{T}_{H1}$ are defined by Eqs. (A1c), (A1e), (A2c), and (A2e), respectively.

Appendix B Reasoning of assumption (25)

Suppose that the leading-order vapor flow $\hat{v}_{iH0}$ has been established. From Eqs. (18) and (A1d) ($\alpha = B$) multiplied by $\hat{n}_{H0}^B$, the first-order particle flux of the noncondensable gas is $\hat{n}_{H1}^B \hat{v}_{iH0}^B = \int \zeta_i \hat{F}_{H1}^B \mathrm{d}^3\zeta$, which turns out to be equal to $\hat{n}_{H1}^B \hat{v}_{iH0}$ if Eq. (20) is used. Let us suppose that $\hat{n}_{H1}^B$ is not identically zero. Then, we have $\hat{v}_{iH0}^B = \hat{v}_{iH0}$ in the region where $\hat{n}_{H1}^B \neq 0$ (note that the leading-order flow velocity $\hat{v}_{iH0}^B$ is determined through the first-order velocity distribution function $\hat{F}_{H1}^B$). This corresponds to the physical situation that a small amount of the noncondensable gas ($\hat{n}_{H1}^B \epsilon$) is carried by the leading-order vapor flow ($\hat{v}_{iH0}$). If we integrate Eq. (15) with $\alpha = B$ and $m = 2$ over the whole space of ζ, we obtain

$$\int \zeta_j \frac{\partial \hat{F}_{H1}^B}{\partial x_j} \, \mathrm{d}^3\zeta = 0, \tag{B1}$$

because $\hat{J}^{\beta\alpha}$ does not contribute to this integration. [Equation (B1) is a part of the solvability condition for the equations for $\hat{F}_{H2}^\alpha$.] Equations (20) and (B1) give the continuity equation

$$\frac{\partial \hat{n}_{H1}^B \hat{v}_{jH0}}{\partial x_j} = 0. \tag{B2}$$

Because of Eq. (24a), we can transform Eq. (B2) into

$$\hat{v}_{jH0} \frac{\partial}{\partial x_j} \left(\frac{\hat{n}_{H1}^B}{\hat{n}_{H0}^A} \right) = 0, \tag{B3}$$

which means that $\hat{n}_{H1}^B / \hat{n}_{H0}^A = const.$ along a stream line. Therefore, if $\hat{n}_{H1}^B$ vanishes at a point on a stream line, then $\hat{n}_{H1}^B = 0$ holds on the entire stream line.

A stream line of the leading-order vapor flow $\hat{v}_{iH0}$ either (i) starts from the evaporating surface or (ii) forms a closed loop in the case of a closed domain. Let us consider the case (i). Since we have not assumed that $\hat{n}_{H1}^B \equiv 0$ (or $\hat{F}_{H1}^B \equiv 0$) in the present discussion, the $\hat{F}_{K1}^B$ in the right-hand side of Eq. (38) should be replaced by $(\hat{F}_{H1}^B)_b + \hat{F}_{K1}^B$. However, this replacement does not affect Eq. (39). Let us recall that, on the evaporating surface where $\hat{v}_{iH0} n_i > 0$, the Knudsen layer in the leading order does not contain the noncondensable gas, namely $\hat{F}_{K0}^B = 0$ (see the fourth paragraph in Sec. 3.3). Therefore, Eq. (39) reduces to $\partial(\Phi_{iK1} n_i)/\partial\eta = 0$, which leads to $\Phi_{iK1} n_i = \int \zeta_i n_i \hat{F}_{K1}^B \mathrm{d}^3\zeta = 0$ because $\hat{F}_{Km}^B \to 0$ as $\eta \to \infty$. Further, $\int \zeta_i n_i (\hat{F}_{Hm}^B + \hat{F}_{Km}^B) \, \mathrm{d}^3\zeta = 0$ holds on the boundary because of the diffuse reflection condition (5) (with $\alpha = B$) and (6b). Therefore, we have

$$\int \zeta_i n_i \hat{F}_{H1}^B \mathrm{d}^3\zeta = \hat{n}_{H1}^B \hat{v}_{iH0} n_i = 0, \tag{B4}$$

on the evaporating surface. But, since $\hat{v}_{iH0} n_i > 0$ on the evaporating surface, $\hat{n}_{H1}^B$ should vanish there. In consequence, $\hat{n}_{H1}^B = 0$ holds along a stream line in case (i). If we consider the problems in which there is no closed stream lines [type (ii)] of the vapor flow or those in which closed stream lines of the vapor flow, if any, do not carry any noncondensable gas, we can put

$$\hat{n}_{H1}^B \equiv 0, \qquad (\text{i.e., } \hat{F}_{H1}^B \equiv 0). \tag{B5}$$

By repeating the same argument successively for $m = 2, 3, \ldots$, we can show that Eq. (25) is a consistent assumption.

In the case of an infinite domain (Sec. 3.6), stream lines starting from infinity may also exist. But, it is obvious that $\hat{n}_{H1}^B = 0$ holds along such a stream line because there is no noncondensable gas at infinity. Therefore, Eq. (25) is consistent also in this case.

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Table I: The functions $h_1(M_n)$ and $h_2(M_n)$.

| M_n	h_1	h_2	M_n	h_1	h_2	M_n	h_1	h_2
0.0000	1.0000	1.0000	0.4000	0.4900	0.8470	0.8000	0.2695	0.7088
0.04999	0.9083	0.9798	0.4400	0.4593	0.8326	0.8400	0.2553	0.6956
0.07998	0.8582	0.9679	0.4800	0.4310	0.8184	0.8800	0.2420	0.6824
0.1200	0.7966	0.9521	0.5200	0.4050	0.8043	0.9200	0.2297	0.6693
0.1600	0.7404	0.9365	0.5600	0.3809	0.7904	0.9600	0.2182	0.6563
0.2000	0.6891	0.9212	0.6000	0.3586	0.7765	0.9700	0.2155	0.6530
0.2400	0.6421	0.9060	0.6400	0.3380	0.7628	0.9800	0.2128	0.6498
0.2800	0.5991	0.8910	0.6800	0.3189	0.7492	0.9900	0.2101	0.6466
0.3200	0.5596	0.8761	0.7200	0.3012	0.7356	1.0000	0.2075	0.6434
0.3600	0.5233	0.8615	0.7600	0.2848	0.7222			
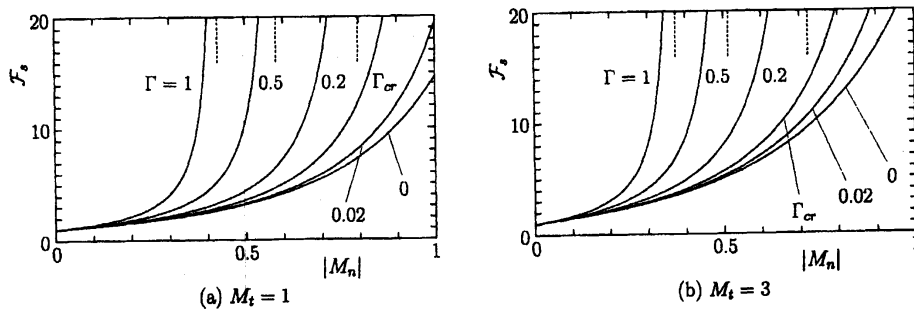
(a) $M_t = 1$

(b) $M_t = 3$

Figure 1: $\mathcal{F}_s(|M_n|, M_t, T_\infty/T_w, \Gamma)$ versus $|M_n|$ for various values of Γ in the case $T_\infty/T_w = 1$. (a) $M_t = 1$, (b) $M_t = 3$. The dashed line indicates the asymptote ($|M_n| = c_s$) of the curve for $\Gamma > \Gamma_{cr}$. The values of Γ_{cr} are 0.080878 ($M_t = 1$) and 0.058957 ($M_t = 3$).
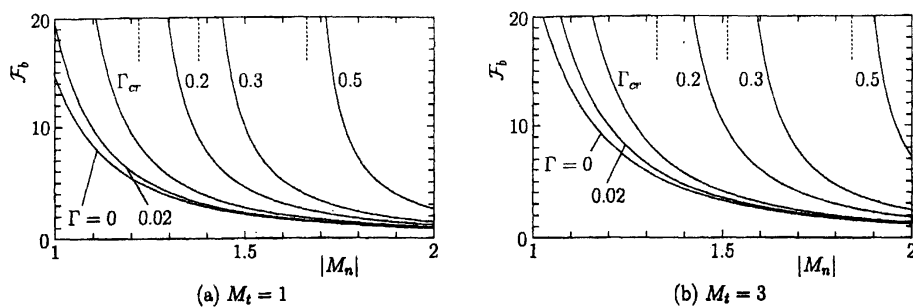
(a) $M_t = 1$ (b) $M_t = 3$

Figure 2: $\mathcal{F}_b(|M_n|, M_t, T_\infty/T_w, \Gamma)$ versus $|M_n|$ for various values of Γ in the case $T_\infty/T_w = 1$. (a) $M_t = 1$, (b) $M_t = 3$. The dashed line indicates the asymptote ($|M_n| = c_b$) of the curve $\Gamma > \Gamma_{cr}$. The values of Γ_{cr} are the same as those for the corresponding M_t in Fig. 1.
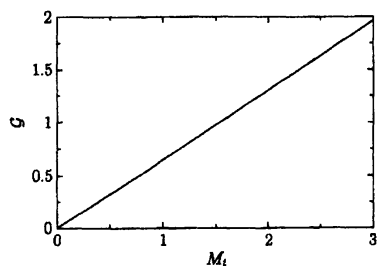

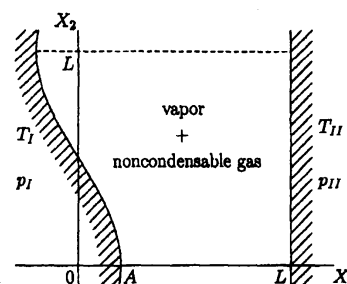Figure 3: $\mathcal{G}(|M_n|, M_t, T_\infty/T_w)$ versus M_t for $|M_n| = 0.1$ and $T_\infty/T_w = 1$.

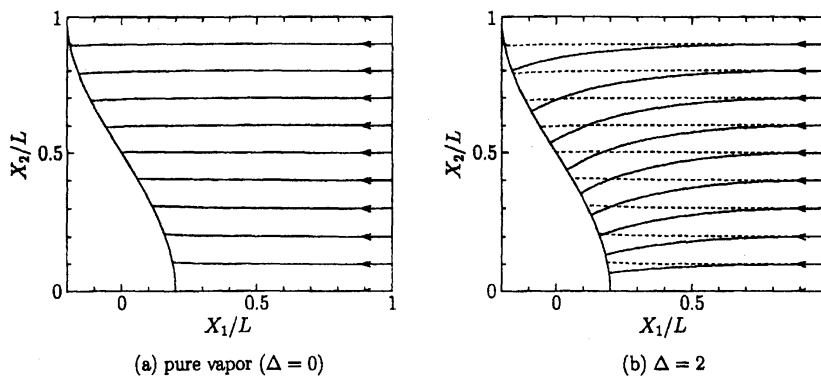Figure 4: Vapor between a plane and a sinusoidal condensed phase in the presence of a noncondensable gas.

(a) pure vapor ($\Delta = 0$)   (b) $\Delta = 2$

Figure 5: Stream lines. (a) pure vapor ($\Delta = 0$), (b) $\Delta = 2$. The stream lines in the pure vapor case [(a)] are also shown by dashed lines in (b) for comparison.
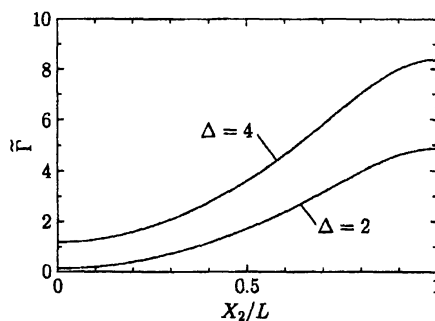

Figure 6: Distribution of $\tilde{\Gamma}$ along the sinusoidal condensed phase.