Anisotropic convexified Gauss curvature flow of bounded open sets: stochastic approximation, weak solution and viscosity solution (Viscosity Solutions of Differential Equations and Related Topics)

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Anisotropic convexified Gauss curvature flow of bounded open sets:
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1 Introduction

Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach (see [2]).

In [3] we proposed and studied a two dimensional random crystalline algorithm for the curvature flow of smooth simple closed convex curves.

In [4] we studied a convexified Gauss curvature flow of compact sets by the level set approach in the theory of viscosity solutions.

In this talk we discuss a random crystalline algorithm of and PDE on an anisotropic convexified Gauss curvature flow of bounded open sets in $\mathbb{R}^N$ for any $N \geq 2$ (see [5]).

We introduce an assumption and a notation before we describe the PDE under consideration.

(A.1). $R \in L^1(\mathbb{S}^{N-1} : [0, \infty), d\mathcal{H}^{N-1})$, and $||R||_{L^1(\mathbb{S}^{N-1})} = 1$. 
For $p \in \mathbb{R}^N$ and a $N \times N$-symmetric real matrix $X$, put $G(o, X) := 0$ and

$$G(p, X) := |p| \det \left( -\left( I - \frac{p}{|p|} \otimes \frac{p}{|p|} \right) \frac{X}{|p|} \left( I - \frac{p}{|p|} \otimes \frac{p}{|p|} \right) + \frac{p}{|p|} \otimes \frac{p}{|p|} \right)$$

if $p \neq o$.

We discuss a weak solution and a viscosity solution of the following PDE in this talk:

$$0 = \partial_t u(t, x) + R\left( \frac{Du(t,x)}{|Du(t,x)|} \right) \sigma^+(u, Du(t,x), t, x) G(Du(t,x), D^2 u(t,x))$$

(1.1)

$((t, x) \in (0, \infty) \times \mathbb{R}^N)$. Here

$$\sigma^+(u, p, t, x) := \begin{cases} 1 & \text{if } u(t, \cdot) \leq u(t, x) \text{ on } H(p, x) \text{ and } p \in \mathbb{R}^N \setminus \{o\}, \\ 0 & \text{otherwise}, \end{cases}$$

$$H(p, x) := \{y \in \mathbb{R}^N \setminus \{x\} | <y-x, p> \leq 0\}.$$

To introduce the notion of a weak solution to (1.1), we give several notations.

Let $F$ be a closed convex subset of $\mathbb{R}^N$. For $x \in \partial F$, put

$$N_F(x) := \{p \in S^{N-1} | F \subset \{y| < y - x, p > \leq 0\}\}.$$

**Definition 1** Suppose that (A.1) holds. Let $u : D(u)(\subset \mathbb{R}^N) \rightarrow \mathbb{R}$ be bounded and $r \in \mathbb{R}$. For any $B \in B(\mathbb{R}^N)$, put
\[ \omega_r(R, u, B) := \int_{N(\cap u^{-1}([r, \infty)))-(B \cap \partial(u^{-1}([r, \infty))))} R(p) dH^{N-1}(p), \]

\[ w(R, u, B) := \int_{R} dr \omega_r(R, u, B), \]

provided the right hand side is well defined.

**Definition 2 (Weak Solutions)** Suppose that (A.1) holds.

(i) A family of bounded open sets \( \{D(t)\}_{t \geq 0} \) in \( \mathbb{R}^N \) is called an anisotropic convexified Gauss curvature flow if

\[ D(t) = \begin{cases} (\cap D(t)) \cap D(0) & \text{for } t \in [0, \text{Vol}(D)), \\ \emptyset & \text{for } t \geq \text{Vol}(D) \end{cases} \]

(1.2)

and for any \( \varphi \in C_0(\mathbb{R}^N) \) and any \( t \geq 0 \),

\[ \int_{\mathbb{R}^N} \varphi(x) (I_{D(t)}(x) - I_{D(0)}(x)) dx = \int_0^t ds \int_{\mathbb{R}^N} \varphi(x) \omega_1(I_{D(s)}(\cdot), dx). \]

(1.3)

(ii) \( u \in C_b([0, \infty) \times \mathbb{R}^N) \) is called a weak solution to (1.1) if the following holds: for any \( \varphi \in C_0(\mathbb{R}^N) \) and any \( t \geq 0 \),

\[ \int_{\mathbb{R}^N} \varphi(x) (u(0, x) - u(t, x)) dx = \int_0^t ds \int_{\mathbb{R}^N} \varphi(x) w(u(s, \cdot), dx). \]

(1.4)

Let \( M \) be a smooth oriented hypersurface in \( \mathbb{R}^N \) and \( K(x) \) denote Gauss curvature of \( M \) at \( x \). Define \( \sigma : M \mapsto \{0, 1\} \) by

\[ \sigma(x) = \begin{cases} 1 & \text{if } x \in M \cap \partial(\cap M), \\ 0 & \text{otherwise}, \end{cases} \]

and call \( \sigma(x)K(x) \) the convexified Gauss curvature of \( M \) at \( x \).
Remark 1 If $\partial D(t)$ is a smooth hypersurface for all $t \in [0, \text{Vol}(D(0))]$, then $t \mapsto \partial D(t)$ is the curvature flow:

$$v = -R(\nu)\sigma K\nu$$  \hspace{1cm} (1.5)

on $[0, \text{Vol}(D(0))]$, where $\nu$ denotes the unit outward normal vector on the surface and $v$ denotes the velocity of the surface.

Before we introduce the notion of a viscosity solution to (1.1), we introduce notations.

$f \in F$ if and only if $f \in C^2([0, \infty))$, $f''(r) > 0$ on $(0, \infty)$, and $f(r)/r^N \rightarrow 0$ as $r \rightarrow 0$.

Let $\Omega$ be an open subset of $(0, \infty) \times \mathbb{R}^N$. $f \in A(\Omega)$ if and only if $\varphi \in C^2(\Omega)$, and for any $(\hat{t}, \hat{x}) \in \Omega$ for which $D\varphi$ vanishes, there exists $f \in F$ such that

$$|\varphi(t, x) - \varphi(\hat{t}, \hat{x}) - \partial_t \varphi(\hat{t}, \hat{x})(t-\hat{t})| \leq f(|x-\hat{x}|) + o(|t-\hat{t}|) \quad \text{as} \quad (t, x) \rightarrow (\hat{t}, \hat{x})$$

Definition 3 (Viscosity solution) (see [7]).

Let $0 < T \leq \infty$ and set $\Omega := (0, T) \times \mathbb{R}^N$.

(i). A function $u \in USC(\Omega)$ is called a viscosity subsolution of (1.1) in $\Omega$ if whenever $\varphi \in A(\Omega)$, $(s, y) \in \Omega$, and $u - \varphi$ attains a local maximum at $(s, y)$, then

$$\partial_t \varphi(s, y) + \sigma^-(u, D\varphi(s, y), s, y)R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right)G(D\varphi(s, y), D^2\varphi(s, y)) \leq 0,$$
where

\[ \sigma^-(u, p, s, y) := \begin{cases} 1 & \text{if } u(s, \cdot) < u(s, y) \text{ on } H(p, y) \text{ and } p \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \]

(ii). A function \( u \in LSC(\Omega) \) is called a viscosity supersolution of (1.1) in \( \Omega \) if whenever \( \varphi \in \mathcal{A}(\Omega), (s, y) \in \Omega, \) and \( u - \varphi \) attains a local minimum at \((s, y),\) then

\[ \partial_t \varphi(s, y) + \sigma^+(u, D\varphi(s, y), s, y)R\left(\frac{D\varphi(s, y)}{|D\varphi(s, y)|}\right)G(D\varphi(s, y), D^2\varphi(s, y)) \geq 0. \]

(1.7)

(iii). A function \( u \in C(\Omega) \) is called a viscosity solution of (1.1) in \( \Omega \) if it is both a viscosity subsolution and a viscosity supersolution of (1.1) in \( \Omega.\)

Next we introduce a class of stochastic processes of which continuum limit becomes an anisotropic convexified Gauss curvature flow.

The following is an assumption on the initial set.

(A.2). \( D \) is a bounded open set in \( \mathbb{R}^N \) such that \( \text{Vol}(\partial D) = 0.\)

Take \( K > 0 \) so that \( \text{co} D \subset [-K + 1, K - 1]^N.\) Put

\[ S_n := \{I_A : [-K, K]^N \cap (\mathbb{Z}^N/n) \mapsto \{0, 1\}|A \subset \mathbb{Z}^N/n\}. \]

For \( x, z \in \mathbb{Z}^N/n \) and \( v \in S_n,\) put

\[ v_{n,z}(x) := \begin{cases} v(x) & \text{if } x \neq z, \\ 0 & \text{if } x = z \end{cases} \]

; and for a bounded \( f : S_n \mapsto \mathbb{R},\) put
$A_n f(v) := n^N \sum_{x \in [-K,K]^N \cap (\mathbb{Z}^N/n)} \omega_1(R, v, \{z\}) \{f(v_{n,z}) - f(v)\}.$

Let $\{Y_n(t, \cdot)\}_{t \geq 0}$ be a Markov process on $S_n$ ($n \geq 1$), with the generator $A_n$, such that $Y_n(0, z) = I_{D \cap (\mathbb{Z}^N/n)}(z)$.

For $(t, x) \in [0, \infty) \times [-K, K]^N$, put also

$$D_n(t) := (co Y_n(t, \cdot)^{-1}(1))^o \cap D.$$  \hspace{1cm} (1.8)

$$X_n(t, x) := I_{D_n(t)}(x).$$  \hspace{1cm} (1.9)

Then $\{X_n(t, \cdot)\}_{t \geq 0}$ is a stochastic process on

$$S := \{f \in L^2([-K, K]^N) : ||f||_{L^2([-K, K]^N)} \leq (2K)^N\}$$

which is a complete separable metric space by the metric

$$d(f, g) := \sum_{k=1}^{\infty} \frac{\max(|<f - g, e_k>_{L^2([-K, K]^N)}|, 1)}{2^k}.$$

Here $\{e_k\}_{k \geq 1}$ denotes a complete orthonormal basis of $L^2([-K, K]^N)$.

By definition, the following holds.

(1) $D_n(0) \to D$ in Hausdorff metric as $n \to \infty$.

(2) $\sum_{x \in (\mathbb{Z}^N/n) \cap [-K,K]^N} |I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 0$ or 1 for all $t \geq 0$.

(3) If $|I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 1$, then $z \in \partial (co D_n(t-)).$

(4) $\sum_{x \in (\mathbb{Z}^N/n) \cap [-K,K]^N} |I_{D_n(t)}(z) - I_{D_n(t-)}(z)| = 1$ if and only if $t = \sigma_{n,i}$ for some $i$, where $0 < \sigma_{n,1} < \sigma_{n,2} < \cdots$ are random variables such that $\{\sigma_{n,i+1} - \sigma_{n,i}\}_{i > 0}$ are independent and that
\[ P(\sigma_{n,i+1} - \sigma_{n,i} \in dt) = n^N \exp(-n^N t) dt. \]

(5) \( P(I_{D_n(\sigma_{n,i})}(z) - I_{D_n(\sigma_{n,i-1})}(z) = 1) = E[\omega_1(R, I_{D_n(\sigma_{n,i-1})}, \{z\})]. \)

**Remark 2** In this paper we try to minimize the number of references because of the page limitation. One can find extensive references in [1]-[7].

## 2 Main Result

In this section we give our main result from [5].

The following theorem implies that \( D_n \) is a random crystalline approximation of an anisotropic convexified Gauss curvature flow.

**Theorem 1** Suppose that (A.1)-(A.2) hold. Then there exists a unique anisotropic convexified Gauss curvature flow \( \{D(t)\}_{t \geq 0} \) with \( D(0) = D \), and for any \( \gamma > 0 \),

\[
\lim_{n \to \infty} P(\sup_{0 \leq t} ||X_n(t, \cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \geq \gamma) = 0. \quad (2.1)
\]

Suppose in addition that \( D \) is convex. Then for any \( T \in [0, \text{Vol}(D)) \) and \( \gamma > 0 \),

\[
\lim_{n \to \infty} P(\sup_{0 \leq t \leq T} d_H(D_n(t), D(t)) \geq \gamma) = 0, \quad (2.2)
\]

where \( d_H \) denotes Hausdorff metric.

We introduce an additional assumption.
(A.3). $h \in C_b(\mathbb{R}^N)$ and for any $r \in \mathbb{R}$, the set $h^{-1}((r, \infty))$ is bounded or $\mathbb{R}^N$.

The following corollary implies that a level set of a continuous weak solution to (1.1) is determined by that at $t = 0$.

**Corollary 1** Suppose that (A.1) and (A.3) hold. Then there exists a unique bounded continuous weak solution $\{u(t, \cdot)\}_{t \geq 0}$ to (1.1) and for any $r \in \mathbb{R}$, $\{u(t, \cdot)^{-1}((r, \infty))\}_{t \geq 0}$ is a unique anisotropic convexified Gauss curvature flow with initial data $u(0, \cdot)^{-1}((r, \infty))$.

We state properties of anisotropic convexified Gauss curvature flows.

**Theorem 2** Suppose that (A.1)-(A.2) hold. Let $\{D(t)\}_{t \geq 0}$ be a unique anisotropic convexified Gauss curvature flow $\{D(t)\}_{t \geq 0}$ with $D(0) = D$. Then

(a) $t \mapsto D(t)$ is nonincreasing on $[0, \infty)$.

(b) For any $t \in [0, \text{Vol}(D(0)))$,

$$\text{Vol}(D(0) \setminus D(t)) = t. \quad (2.3)$$

(c) Let $\{D_1(t)\}_{t \geq 0}$ be an anisotropic convexified Gauss curvature flow such that $D_1(0)$ is a bounded, convex, open set which contains $D$. Then

$$D(t) \subset D_1(t) \quad \text{for all } t \geq 0, \quad (2.4)$$

where the equality holds if and only if $D(0) = D_1(0)$.

We give an additional assumption and state the result on viscosity solutions to (1.1).

(A.4). $R \in C(S^{N-1} : [0, \infty))$. 

Theorem 3 Suppose that (A.2) and (A.4) hold. Let \( \{D(t)\}_{t \geq 0} \) be a unique anisotropic convexified Gauss curvature flow \( \{D(t)\}_{t \geq 0} \) with \( D(0) = D \). Then \( I_{D(t)}(x) \) and \( I_{D(t)^{-}}(x) \) are a viscosity supersolution and a viscosity subsolution to (1.1), respectively.

The following results imply that \( u \in C_{b}(\mathbb{R}^{N}) \) is a weak solution to (1.1) if and only if it is a viscosity solution to (1.1).

Corollary 2 Suppose that (A.3)-(A.4) hold. Then a unique weak solution \( u \in C_{b}(\mathbb{R}^{N}) \) to (1.1) is a viscosity solution to it.

Corollary 3 (see [6]) Suppose that (A.3)-(A.4) hold. Then a continuous viscosity solution to (1.1) is unique and is a weak solution to it.

3 Sketch of Proof

In this section we explain the main idea of proof.

(Idea of Proof of Theorem 1). We first show that \( \{X_{n}(t, \cdot)\}_{t \geq 0} \) is tight in \( D(\omega_{1}) \). By the weak convergence result on \( \omega_{1} \) by Bakelman [1], we show that any weak limit point of \( \{X_{n}(t, \cdot)\}_{t \geq 0} \) is a weak solution to (1.3).

The following lemma implies the uniqueness of a weak solution to (1.3), and hence completes the proof of (2.1).

Lemma 1 Suppose that (A.1) hold. If \( \{I_{D_{i}(t)}\}_{t \geq 0} \) \( i = 1, 2 \) are weak solutions to (1.3) for which \( D_{1}(0) \subset D_{2}(0) \), then \( D_{1}(t) \subset D_{2}(t) \) for all \( t \geq 0 \). In particular,
$d(D_1(t), D_2(t)^c) \geq d(D_1(0), D_2(0)^c), \quad (3.1)$

for $t \leq Vol(D_1(0))$.

(2.2) can be shown easily. \hfill \Box

(Sketch of Proof of Corollary 1). For $r \in \mathbb{R}$, let $\{I_{D_r(t)}\}_{t \geq 0}$ denote a unique weak solution of (1.3) with $D_r(0) = h^{-1}((r, \infty))$.

Put

$$u(t, x) := \sup \{r \in \mathbb{R} | x \in D_r(t)\}.$$ Then $u$ is continuous. In particular, for all $t \geq 0$ and $r \in \mathbb{R}$,

$$u(t, \cdot)^{-1}((r, \infty)) = D_r(t).$$

For $n \geq 1$, put $k_{n,1} := [n \sup \{h(y) | y \in \mathbb{R}^N\}]$ and $k_{n,0} := [n \inf \{h(y) | y \in \mathbb{R}^N\}]$. Then for any $\varphi \in C_0(\mathbb{R}^N)$ and any $t \geq 0$,

$$\int_{\mathbb{R}^N} \varphi(x)[\sum_{k_{n,0} \leq k \leq k_{n,1}} \frac{k}{n}(I_{D_r(t)^c}(x) - I_{D_{k_{n,1}}(t)^c}(x))]$$

$$- \sum_{k_{n,0} \leq k \leq k_{n,1}} \frac{k}{n}(I_{D_r(0)^c}(x) - I_{D_{k_{n,1}}(0)^c}(x))]dx$$

$$= \int_0^t ds[\sum_{k_{n,0} \leq k \leq k_{n,1}} \frac{1}{n} \int_{\mathbb{R}^N} \varphi(x)\omega_0(R, I_{D_r(s)^c}(\cdot), dx)].$$

Letting $n \to \infty$, $u$ is shown to be a weak solution to (1.1).

The uniqueness of $u$ follows from that of $D_r(\cdot)$ for all $r$. In fact, we can show that for a continuous weak solution $v$ to (1.1), $\{v(t, \cdot)^{-1}((r, \infty))\}_{t \geq 0}$ is an anisotropic convexified Gauss curvature flow. \hfill \Box
We omit the proof of Theorems 2 and 3. Corollary 3 is an easy consequence of Corollary 2 and [6] where we give the uniqueness of a viscosity solution to (1.1).

(Idea of Proof of Corollary 2) Let $u$ be a weak solution to (1.1).

We first show that $u$ is a viscosity supersolution to (1.1). Suppose that $u$ is smooth in $\Omega$ and that $\varphi \in A(\Omega)$, $(s, y) \in \Omega$, and $u - \varphi$ attains a local maximum at $(s, y)$. Then, putting $\varphi^\varepsilon := \varphi - \varepsilon$ ($\varepsilon > 0$),

$$\partial_t (u - \varphi^\varepsilon)(s, y) \geq 0.$$  

Hence formally, we have, in some neighborhood of $(s, y)$,

$$\partial_t \varphi^\varepsilon(t, x) \leq \partial u(t, x) = -w(u(t, \cdot), dx)/dx \leq -w(\varphi^\varepsilon(t, \cdot), dx)/dx = -R\left(\frac{D\varphi(t, x)}{|D\varphi(t, x)|}\right)G(D\varphi(t, x), D^2 \varphi(t, x)).$$

In the last equality, we use the following lemma.

**Lemma 2** For $\varphi \in C^2(\mathbb{R}^N : \mathbb{R})$ for which $D\varphi(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^N$ and for which all eigenvalues of $-D(D\varphi(x_0)/|D\varphi(x_0)|)$ are nonnegative,

$$\frac{\partial_i \varphi(x_0)}{|D\varphi(x_0)|}G(D\varphi(x_0), D^2 \varphi(x_0)) = \det(Dy_i(x_0)) \quad (i = 1, \cdots, N), \quad (3.2)$$

where

$$y_i(x) := \left(-\delta_{ij} \frac{\partial_j \varphi(x)}{|D\varphi(x)|} + \delta_{ij} \varphi(x)\right)_{j=1}^N.$$  

Similarly one can show that $u$ is a viscosity subsolution to (1.1).  \qed
References


