Uniqueness and error bounds for eikonal equations with discontinuities

Klaus Deckelnick & Charles M. Elliott

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We consider the eikonal equation

\begin{align}
|\nabla u(x)| &= f(x) \quad x \in \Omega \tag{1.1} \\
u(x) &= \phi(x) \quad x \in \partial\Omega, \tag{1.2}
\end{align}

where $f$ and $\phi$ are given functions. The equation arises for example in geometric optics, computer vision or robotic navigation. In certain situations it is desirable to allow $f$ to be discontinuous, e.g. in geometric optics, when light propagates through a layered medium. The aim of this paper is to study the well-posedness of (1.1), (1.2) for right hand sides $f$ satisfying a one-sided continuity condition (see (2.2) below), that allows certain types of discontinuities. Furthermore, we shall be concerned with an error analysis for a finite difference scheme to approximate the solution of (1.1), (1.2).

The well-posedness of (1.1), (1.2) in the case of continuous $f$ follows from the theory of viscosity solutions for Hamilton–Jacobi equations $H(x,u,\nabla u) = 0$ developed in [4]. The notion of viscosity solution was generalised by Ishii [5] to allow for discontinuous Hamiltonians $H$. In [11], Tourin proves a comparison result for Hamiltonians, which are allowed to be discontinuous along a smooth surface. Soravia [10] obtains necessary and sufficient conditions for uniqueness of the solution to the boundary value problem. While the work in [11] and [10] is based on Ishii’s notion of solution, several other approaches have been suggested: in [7], Newcomb & Su consider the Dirichlet problem for $H(\nabla u) = f$ and introduce a notion of solution which they call Monge solution. They obtain a comparison result as well as uniqueness for the Dirichlet problem provided that $f$ is lower semicontinuous. Ostrov [8] studies an evolutionary Hamilton–Jacobi equation which occurs in the context of radar satellite tracking and obtains a unique solution as the limit of suitable upper and lower solutions. Recently, Camilli & Siconolfi [3] introduced a new notion of solution for Hamilton–Jacobi equations of the form $H(x,\nabla u) = 0$, which allows measurable dependence of $H$ on $x$ and involves measure-theoretic limits. They prove representation formulae, comparison principles and uniqueness results.

Our work uses Ishii’s definition of solution which we shall recall in §2. For a class of right hand sides $f$, which satisfy a suitable one-sided continuity condition we obtain well-posedness of the problem (1.1), (1.2). In §3 we discretize the problem with the help of a finite difference scheme on a regular grid. Under a slightly more restrictive condition on $f$ we prove that the error between viscosity solution and discrete approximation is of order $O(\sqrt{h})$. We have not included all the details of the proofs of existence and of the error analysis. However, a forthcoming paper, which generalises our approach to Hamilton–Jacobi equations of the form $H(\nabla u) = f$, will provide a detailed convergence analysis for a wide class of finite difference schemes as well as numerical tests.
2 Existence and Uniqueness

In order to allow for discontinuous functions \( f \) in (1.1) we shall use the following generalisation of the concept of viscosity solution, which was introduced by Ishii in [5].

**Definition 2.1.** A function \( u \in C^{0}(ar{\Omega}) \) is called a viscosity subsolution (supersolution) of (1.1) if for each \( \zeta \in C^{\infty}(\Omega) \): if \( u - \zeta \) has a local maximum (minimum) at a point \( x_0 \in \Omega \), then

\[
|\nabla \zeta(x_0)| \leq f^{*}(x_0) \quad (\geq f_{*}(x_0)).
\]

Here,

\[
f^{*}(x) := \lim_{r \to 0} \sup\{f(y) \mid y \in B_r(x) \cap \Omega\}, \quad f_{*}(x) := \lim_{r \to 0} \inf\{f(y) \mid y \in B_r(x) \cap \Omega\}.
\]

A viscosity solution of (1.1), (1.2) then is a function \( u \in C^{0}(ar{\Omega}) \) which is both a viscosity sub- and supersolution and which satisfies \( u(x) = \phi(x) \) for all \( x \in \partial \Omega \).

Let us next formulate our assumptions on the data of the problem. We suppose that \( f : \Omega \to \mathbb{R} \) is Borel measurable and that there exist \( 0 < m \leq M < \infty \) such that

\[
m \leq f(x) \leq M \quad \forall x \in \Omega.
\]

(2.1)

Furthermore, we assume that for every \( x \in \Omega \) there exist \( \epsilon_x > 0 \) and \( n_x \in S^{n-1} \) so that for all \( y \in \Omega, r > 0 \) and all \( d \in S^{n-1} \) with \( |d - n_x| < \epsilon_x \) we have

\[
f(y + rd) - f(y) \leq \omega(|y - x| + r),
\]

where \( \omega : [0, \infty) \to [0, \infty) \) is continuous, nondecreasing and satisfies \( \omega(0) = 0 \). A similar type of condition was used in [11]; however, in (2.2) it is sufficient to estimate values of \( f \) for vectors whose difference is close to a given direction.

**Example:** Suppose that a surface \( \Gamma \) splits \( \Omega \) into two subdomains \( \Omega_1 \) and \( \Omega_2 \), that \( f|\Omega_1 \in C^{0}(\Omega_1), f|\Omega_2 \in C^{0}(\Omega_2) \) and that

\[
\lim_{y \to x, y \in \Omega_1} f(y) < \lim_{y \to x, y \in \Omega_2} f(y) \quad \text{for all } x \in \Gamma.
\]

In addition, assume that the following uniform cone property holds: for every \( x \in \Gamma \) there exists a neighborhood \( U_x \) and a cone \( C_x \) (which is congruent to a fixed given cone \( C_0 \)) such that \( y \in U_x \cap \Omega_1 \) implies that \( y + C_x \subset \Omega_1 \). Then (2.2) holds with \( n = n_x \) given by the direction of the cone \( C_x \).

To see this, observe that the cone condition prevents a situation where \( y \in \Omega_1, y + rd \in \Omega_2 \), which would lead to a violation of (2.2) (cf. [11], where \( \Gamma \) is assumed to be smooth).

One can also consider e.g. a two-dimensional domain \( \Omega \), where three curves of discontinuity meet at a triple junction.

It is not difficult to verify that (2.2) implies

\[
f^{*}(y + rd) - f_{*}(y) \leq \omega(|y - x| + r)
\]

for all \( y \in \Omega, r > 0 \) and \( d \in S^{n-1}, |d - n_x| < \epsilon_x \).

Finally, we suppose for simplicity that \( \phi \equiv 0 \).

**Lemma 2.2.** There exists a viscosity solution \( u \in C^{0,1}(\bar{\Omega}) \) of (1.1), (1.2).
Proof. We only sketch the main ideas. Consider the sup-convolution of $f$, i.e.

$$f_{\varepsilon}(x) := \sup_{y \in \Omega} \{ f(y) - \frac{1}{\varepsilon} |x - y|^2 \}, \quad x \in \Omega, \varepsilon > 0.$$  

Clearly, $f_{\varepsilon}$ is continuous and $f^{*}(x) \leq f_{\varepsilon}(x)$ for all $x \in \Omega$. Let

$$L_{\varepsilon}(x, y) := \inf \left\{ \int_{0}^{1} f_{\varepsilon}(\gamma(t)), |\gamma'(t)| \, dt \mid \gamma \in W^{1, \infty}((0, 1); \Omega) \text{ with } \gamma(0) = x, \gamma(1) = y \right\}.$$  

It is well-known that $u_{\varepsilon}(x) := \inf_{y \in \partial \Omega} L_{\varepsilon}(x, y)$ is a solution of

$$|\nabla u_{\varepsilon}| = f_{\varepsilon}(x) \quad x \in \Omega$$  
$$u_{\varepsilon}(x) = 0 \quad x \in \partial \Omega$$  

in the viscosity sense. Furthermore, it can be shown that

$$\|u_{\varepsilon}\|_{C^{0,1}(\overline{\Omega})} \leq C(M, \Omega) \quad \text{uniformly in } \varepsilon > 0.$$  

Thus, there exists a sequence $(\varepsilon_{k})_{k \in \mathbb{N}}$ with $\varepsilon_{k} \searrow 0, k \to \infty$ and $u_{\varepsilon} \in C^{0,1}(\overline{\Omega})$ such that $u_{\varepsilon^{k}} \to u$ uniformly in $\Omega$ as $k \to \infty$. Using well-known arguments from the theory of viscosity solutions one verifies that $u$ is a solution of (1.1), (1.2).

Uniqueness of the viscosity solution follows from

**Theorem 2.3.** Suppose that $u \in C^{0,1}(\overline{\Omega})$ is a subsolution of (1.1), $v \in C^{0}(\Omega)$ is a supersolution of (1.1) and that at least one of the functions belongs to $C^{0,1}(\overline{\Omega})$. If $u \leq v$ on $\partial \Omega$ then $u \leq v$ in $\Omega$.

**Proof.** Let us assume that $v \in C^{0,1}(\overline{\Omega})$. We shall use the approach presented in [6] (see also [11]). Fix $\theta \in (0, 1)$ and define $u_{\theta}(x) := \theta u(x)$. Next, choose $x_{0} \in \Omega$ such that

$$u_{\theta}(x_{0}) - v(x_{0}) = \max_{x \in \Omega}(u_{\theta}(x) - v(x)) =: \mu,$$  

and suppose that $\mu > 0$. Upon replacing $u, v$ by $u + k, v + k$, we may assume that $u \geq 0$ in $\Omega$, so that $u_{\theta} \leq u$ in $\Omega$. In particular, $u_{\theta} \leq v$ on $\partial \Omega$, which implies that $x_{0} \in \Omega$. Choose $\varepsilon = \varepsilon_{x_{0}}$ and $n = n_{x_{0}} \in S^{n-1}$ according to (2.2) and define for $\lambda > 0, L \geq 1$

$$\Phi(x, y) := u_{\theta}(x) - v(y) - L\lambda |x - y - \frac{1}{\lambda} n|^2 - |x - x_{0}|^2, \quad (x, y) \in \Omega \times \Omega.$$  

Choose $(x_{\lambda}, y_{\lambda}) \in \Omega \times \Omega$ such that

$$\Phi(x_{\lambda}, y_{\lambda}) = \max_{(x, y) \in \Omega \times \Omega} \Phi(x, y).$$  

Since $x_{0} \in \Omega$ we also have $x_{0} - \frac{1}{\lambda} n \in \Omega$ for large $\lambda$; using the relation $\Phi(x_{\lambda}, y_{\lambda}) \geq \Phi(x_{0}, x_{0} - \frac{1}{\lambda} n)$ together with (2.4) we infer

$$L\lambda |x_{\lambda} - y_{\lambda} - \frac{1}{\lambda} n|^2 + |x_{\lambda} - x_{0}|^2 \leq u_{\theta}(x_{\lambda}) - v(y_{\lambda}) - u_{\theta}(x_{0}) + v(x_{0} - \frac{1}{\lambda} n)$$

$$= (u_{\theta}(x_{\lambda}) - v(x_{\lambda})) - (u_{\theta}(x_{0}) - v(x_{0})) + v(x_{\lambda}) - v(y_{\lambda}) - v(x_{0}) + v(x_{0} - \frac{1}{\lambda} n)$$

$$\leq \text{lip}(v)(|x_{\lambda} - y_{\lambda}| + \frac{1}{\lambda})$$

$$\leq \text{lip}(v)(|x_{\lambda} - y_{\lambda} - \frac{1}{\lambda} n| + \frac{2}{\lambda}).$$
This implies
\[ L\lambda |x_\lambda - y_\lambda - \frac{1}{\lambda} n|^2 + |x_\lambda - x_0|^2 \leq \frac{C}{\lambda}, \]
where $C$ depends on lip($v$) and as a consequence,
\[ x_\lambda, y_\lambda \to x_0, \quad \text{as} \quad \lambda \to \infty \]
\[ \lambda |x_\lambda - y_\lambda - \frac{1}{\lambda} n| \leq \frac{C}{\sqrt{L}} < \frac{\epsilon}{2+\epsilon} \]
(2.7)
provided that $L$ is large enough. Since $u$ is a subsolution, we may deduce from the relation $\Phi(x_\lambda, y_\lambda) \geq \Phi(x, y_\lambda)$ for $x \in \overline{\Omega}$ that
\[ |2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda} n) + 2(x_\lambda - x_0)| \leq \theta f^*(x_\lambda) \]
for large $\lambda$ and similarly,
\[ |2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda} n)| \geq f_*(y_\lambda). \]
Combining the above inequalities, we infer
\[ (1 - \theta)f^*(y_\lambda) \leq 2|x_\lambda - x_0| + \theta(f^*(x_\lambda) - f_*(y_\lambda)) \]
(2.8)
In order to apply (2.2) we write $x_\lambda = y_\lambda + r_\lambda d_\lambda$, where
\[ d_\lambda = \frac{n + w_\lambda}{|n + w_\lambda|}, \quad r_\lambda = \frac{1}{\lambda} |n + w_\lambda|, \quad w_\lambda = \lambda (x_\lambda - y_\lambda - \frac{1}{\lambda} n). \]
(2.9)
Recalling (2.7) we deduce
\[ |d_\lambda - n| \leq \frac{2|w_\lambda|}{1 - |w_\lambda|} \leq \frac{\frac{2\epsilon}{2+\epsilon}}{1 - \frac{\epsilon}{2+\epsilon}} = \epsilon \]
and (2.3) therefore yields
\[ f^*(x_\lambda) - f_*(y_\lambda) = f^*(y_\lambda + r_\lambda d_\lambda) - f_*(y_\lambda) \leq \omega(|y_\lambda - x_0| + r_\lambda). \]
(2.10)
If we send $\lambda \to \infty$ in (2.8) we finally obtain from (2.1), (2.10) and (2.6) that $(1 - \theta)m \leq 0$, a contradiction. Thus, $u_\theta \leq v$ in $\overline{\Omega}$ and sending $\theta \searrow 1$ gives the desired result.

3 Numerical scheme and error analysis

Let us assume that $\Omega = \Pi_{i=1}^{n}(0, b_i)$ and that the grid size $h > 0$ is chosen in such a way that $b_i = N_i h$ for some $N_i \in \mathbb{N}$, $i = 1, \ldots, n$. We then define
\[ \Omega_h := Z_h^n \cap \Omega, \quad \partial\Omega_h := Z_h^n \cap \partial\Omega, \quad \bar{\Omega}_h := \Omega_h \cup \partial\Omega_h, \]
where $Z_h^n = \{x_\alpha = (h\alpha_1, \ldots, h\alpha_n) | \alpha_i \in \mathbb{Z}, i = 1, \ldots, n\}$. Our aim is to approximate the viscosity solution $u$ of (1.1), (1.2) by a grid function $U : \bar{\Omega}_h \to \mathbb{R}$ and to prove an estimate for $\max_{x_\alpha \in \Omega_h} |u(x_\alpha) - U(x_\alpha)|$. Let us abbreviate $U_\alpha = U(x_\alpha)$ and recall the usual backward and forward difference quotients,
\[ D_k^- U_\alpha := \frac{U_\alpha - U_{\alpha-e_k}}{h}, \quad D_k^+ U_\alpha := \frac{U_{\alpha+e_k} - U_\alpha}{h}, \quad x_\alpha \in \Omega_h, \quad k = 1, \ldots, n. \]
In order to define the numerical method we introduce the function $G : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$G(p_1, q_1, \ldots, p_n, q_n) := \left( \sum_{k=1}^{n} \max(p_k^+, -q_k^-)^2 \right)^{\frac{1}{2}},$$

where $x^+ = \max(x, 0), x^- = \min(x, 0)$. The discrete problem now reads: find $U : \tilde{\Omega}_h \to \mathbb{R}$ such that

$$G(D_1^- U_\alpha, D_1^+ U_\alpha, \ldots, D_n^- U_\alpha, D_n^+ U_\alpha) = f(x_\alpha) \quad x_\alpha \in \Omega_h$$
$$U_\alpha = 0 \quad x_\alpha \in \partial \Omega_h.$$ 

(3.1)

(3.2)

The above scheme was examined for continuous $f$ in [9] in the context of shape-from-shading and convergence to the viscosity solution was obtained as a consequence of a result of Barles and Souganidis [2]. In the case of a constant right hand side $f \equiv 1$, Zhao [12] recently obtained an $O(h)$ error bound. The scheme can be derived by interpreting the viscosity solution $u$ as the value function of an optimal control problem. For further information and a corresponding list of references we refer to Appendix A (written by M. Falcone) in [1].

The function $G$ has the following crucial properties:

a) **Consistency:**

$$G(p_1, p_1, \ldots, p_n, p_n) = |p| \quad \text{for all } p = (p_1, \ldots, p_n) \in \mathbb{R}^n.$$ 

(3.3)

b) **Monotonicity:**

let $a = (a_1, a_2, \ldots, a_{2n-1}, a_{2n}), b = (b_1, b_2, \ldots, b_{2n-1}, b_{2n}) \in \mathbb{R}^{2n}$ with $a_k \geq b_k$ for $k = 1, \ldots, 2n$. Then

$$G(t - a_1, a_2 - t, \ldots, t - a_{2n-1}, a_{2n} - t) \leq G(t - b_1, b_2 - t, \ldots, t - b_{2n-1}, b_{2n} - t) \quad \forall t \in \mathbb{R}.$$ 

(3.4)

Note that the above properties imply in particular that the solution of (3.1), (3.2) cannot have a local minimum in $\Omega_h$ and therefore $U_\alpha \geq 0$ in $\tilde{\Omega}_h$. In order to carry out our error analysis we need to strengthen (2.2) in that we assume that there exist $\epsilon > 0, K \geq 0$ such that for all $x \in \Omega$ there is a direction $n = n_x \in S^{n-1}$ with

$$f(y + rd) - f(y) \leq Kr \quad \forall y \in \Omega, \ |y - x| < \epsilon \ \forall d \in S^{n-1}, \ |d - n| < \epsilon \ \forall r > 0.$$ 

(3.5)

**Theorem 3.1.** Let $u$ be the viscosity solution of (1.1), (1.2) and $U$ a solution of (3.1), (3.2). Then there exists a constant $C$, which is independent of $h$ such that

$$\max_{x_\alpha \in \Omega_h} |u(x_\alpha) - U(x_\alpha)| \leq C\sqrt{h}.$$ 

**Proof.** We again only sketch the main ideas. As it seems difficult to use the argument from the uniqueness proof in order to control the maximum of $u - U$, we shall resort to the Kružkov transform. Thus, let $\tilde{u} : \tilde{\Omega} \rightarrow \mathbb{R}, \bar{U} : \tilde{\Omega}_h \rightarrow \mathbb{R}$ be defined by

$$\tilde{u}(x) := -e^{-u(x)}, \ x \in \tilde{\Omega}, \ \bar{U}_\alpha := -e^{-U_\alpha}, \ x_\alpha \in \tilde{\Omega}_h.$$ 

One verifies (cf. [4]) that $\tilde{u}$ is a viscosity solution of

$$f(x)\tilde{u} + |\nabla \tilde{u}| = 0 \quad x \in \Omega$$
$$\tilde{u}(x) = -1 \quad x \in \partial \Omega,$$

(3.6)

(3.7)
and that $\tilde{U}$ satisfies
\begin{equation}
 f(x_{\alpha})\tilde{U}_{\alpha} + G(D_{i}^{-}\tilde{U}_{\alpha}, D_{i}^{+}\tilde{U}_{\alpha}, \ldots, D_{n}^{-}\tilde{U}_{\alpha}, D_{n}^{+}\tilde{U}_{\alpha}) = F_{\alpha}^{h} \quad x_{\alpha} \in \Omega_{h} \tag{3.8}
\end{equation}

where
\begin{equation}
 \max_{x_{\alpha} \in \partial \Omega_{h}} |F_{\alpha}^{h}| \leq Ch. \tag{3.10}
\end{equation}

Next, choose $x_{\beta} \in \tilde{\Omega}_{h}$ such that
\begin{equation}
 |\tilde{u}(x_{\alpha}) - \tilde{U}_{\alpha}| = \max_{x_{a} \in \partial \Omega_{h}} |\tilde{u}(x_{\alpha}) - \tilde{U}_{\alpha}| \quad \text{and assume that} \quad \tilde{u}(x_{\beta}) \geq \tilde{U}_{\beta}. \tag{3.9}
\end{equation}

The opposite case can be treated similarly. If $\text{dist}(x_{\beta}, \partial \Omega) \leq \sqrt{h}$, it follows from (3.7), (3.9) and the Lipschitz continuity of $\tilde{u}$ that
\begin{equation}
 \max_{x_{\alpha} \in \partial \Omega_{h}} |\tilde{u}(x_{\alpha}) - \tilde{U}_{\alpha}| = \tilde{u}(x_{\beta}) - \tilde{U}_{\beta} \leq C\sqrt{h}. \tag{3.11}
\end{equation}

Now suppose that $\text{dist}(x_{\beta}, \partial \Omega) > \sqrt{h}$ and define
\begin{equation}
 \Phi(x, x_{\alpha}) := \tilde{u}(x) - \tilde{U}_{\alpha} - \frac{L_{1}}{\sqrt{h}} |x - x_{\alpha} - \sqrt{h}n|^2 - L_{2}\sqrt{h} |x_{\alpha} - x_{\beta}|^2, \quad (x, x_{\alpha}) \in \tilde{\Omega} \times \tilde{\Omega}_{h}. \tag{3.12}
\end{equation}

Here, $n = n_{x_{\beta}}$ and $L_{1}, L_{2} \geq 0$ are constants that do not depend on $h$ and which will be chosen later. There exists $(x_{h}, x_{\alpha_{h}}) \in \Omega \times \tilde{\Omega}_{h}$ such that
\begin{equation}
 \Phi(x_{h}, x_{\alpha_{h}}) = \max_{(x, x_{\alpha}) \in \Omega \times \tilde{\Omega}_{h}} \Phi(x, x_{\alpha}). \tag{3.13}
\end{equation}

Since $\text{dist}(x_{\beta}, \partial \Omega) > \sqrt{h}$, we have $x_{\beta} + \sqrt{h}n \in \tilde{\Omega}$ and therefore
\begin{equation}
 \Phi(x_{h}, x_{\alpha_{h}}) \geq \Phi(x_{\beta} + \sqrt{h}n, x_{\beta}). \tag{3.14}
\end{equation}

From this we infer in a similar way as in (2.5) that
\begin{equation}
 |x_{\alpha_{h}} - x_{\beta}| < \epsilon, \tag{3.15}
\end{equation}
\begin{equation}
 \frac{1}{\sqrt{h}} |x_{h} - x_{\alpha_{h}} - \sqrt{h}n| < \frac{\epsilon}{2 + \epsilon} \tag{3.16}
\end{equation}

provided that $L_{i} = L_{i}(\text{lip}(\tilde{u}), \epsilon), i = 1, 2$ are sufficiently large ($\epsilon$ from (3.5)).

Suppose first that $(x_{h}, x_{\alpha_{h}}) \in \Omega \times \tilde{\Omega}_{h}$. Since $\tilde{u}$ is a subsolution of (3.6) we infer
\begin{equation}
 f^{*}(x_{h})\tilde{u}(x_{h}) + \frac{2L_{1}}{\sqrt{h}} (x_{h} - x_{\alpha_{h}} - \sqrt{h}n) \leq 0. \tag{3.17}
\end{equation}

Keeping the first component of $\Phi$ fixed we obtain on the other hand for all $x_{\alpha} \in \tilde{\Omega}_{h}$
\begin{equation}
 \tilde{U}_{\alpha} = \tilde{U}_{\alpha_{h}} + \frac{L_{1}}{\sqrt{h}} (|x_{h} - x_{\alpha_{h}} - \sqrt{h}n|^2 - |x_{h} - x_{\alpha} - \sqrt{h}n|^2) + L_{2}\sqrt{h} (|x_{\alpha_{h}} - x_{\beta}|^2 - |x_{\alpha} - x_{\beta}|^2) = \tilde{V}_{\alpha}. \tag{3.18}
\end{equation}
Since $\tilde{U}_{\alpha_{h}} = \tilde{V}_{\alpha_{h}}$, (3.4) and (3.3) imply
\[
G(D_{1}^{-}\tilde{U}_{\alpha_{h}}, D_{1}^{+}\tilde{U}_{\alpha_{h}}, \ldots, D_{n}^{-}\tilde{U}_{\alpha_{h}}, D_{n}^{+}\tilde{U}_{\alpha_{h}}) \leq G(D_{1}^{-}\tilde{V}_{\alpha_{h}}, D_{1}^{+}\tilde{V}_{\alpha_{h}}, \ldots, D_{n}^{-}\tilde{V}_{\alpha_{h}}, D_{n}^{+}\tilde{V}_{\alpha_{h}})
\]
\[
\leq \frac{2L_{1}}{\sqrt{\beta}}(x_{h} - x_{\alpha_{h}} - \sqrt{\beta} n) - 2L_{2}\sqrt{\beta}(x_{\alpha_{h} \beta} - x_{\beta}) + C\sqrt{\beta}.
\]
Combining this inequality with (3.8) and (3.10) then yields
\[
f(x_{\alpha_{h}}) \tilde{U}_{\alpha_{h}} + \frac{2L_{1}}{\sqrt{\beta}}(x_{h} - x_{\alpha_{h}} - \sqrt{\beta} n) \geq -|F_{\alpha_{h}}^{h}| - C\sqrt{\beta} \geq -C\sqrt{\beta} \tag{3.14}
\]
As a result of (3.13), (3.14)
\[
f(x_{\alpha_{h}})(\tilde{u}(x_{h}) - \tilde{U}_{\alpha_{h}}) \leq C\sqrt{\beta} + e^{-u(x_{h})}(f^{*}(x_{h}) - f(x_{\alpha_{h}})) = C\sqrt{\beta} + e^{-u(x_{h})}(f^{*}(x_{\alpha_{h}} + r_{h}d_{h}) - f(x_{\alpha_{h}})) \tag{3.15}
\]
where similar to (2.9),
\[
d_{h} = \frac{n + w_{h}}{|n + w_{h}|}, r_{h} = \sqrt{\beta} |n + w_{h}|, w_{h} = \frac{1}{\sqrt{\beta}}(x_{h} - x_{\alpha_{h}} - \sqrt{\beta} n).
\]
Since
\[
\tilde{u}(x_{h}) - \tilde{U}_{\alpha_{h}} = \Phi(x_{h}, x_{\alpha_{h}}) + \frac{L_{1}}{\sqrt{\beta}}|x_{h} - x_{\alpha_{h}} - \sqrt{\beta} n|^{2} + L_{2}\sqrt{\beta}|x_{\alpha_{h} \beta} - x_{\beta}|^{2}
\]
\[
\geq \Phi(x_{\beta}, x_{\beta}) = \tilde{U}_{\beta} - \tilde{u}(x_{\beta}) - L_{1}\sqrt{\beta},
\]
we finally deduce from (2.1), (3.15) and (3.5) that
\[
m(\tilde{u}(x_{\beta}) - \tilde{U}_{\beta}) \leq C\sqrt{\beta} + Kr_{h} \leq C\sqrt{\beta}.
\]
The cases $x_{h} \in \partial \Omega$ or $x_{\alpha_{h}} \in \partial \Omega_{h}$ can be examined with the help of the boundary conditions (3.7), (3.9). Transforming back to $u$ and $U$ implies the desired error bound.

References


Klaus Deckelnick  
Institut für Analysis und Numerik  
Otto–von–Guericke-Universität Magdeburg  
Universitätsplatz 2  
39106 Magdeburg  
Germany  

Charles M. Elliott  
Centre for Mathematical Analysis and Its Applications  
School of Mathematical Sciences  
University of Sussex  
Falmer Brighton BN1 9QH  
United Kingdom