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Uniqueness and error bounds for eikonal equations with discontinuities

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We consider the eikonal equation

$$\begin{aligned} |\nabla u(x)| &= f(x) \quad x \in \Omega & (1.1)\\ u(x) &= \phi(x) \quad x \in \partial\Omega, & (1.2) \end{aligned}$$

where f and ϕ are given functions. The equation arises for example in geometric optics, computer vision or robotic navigation. In certain situations it is desirable to allow f to be discontinuous, e.g. in geometric optics, when light propagates through a layered medium. The aim of this paper is to study the well–posedness of (1.1), (1.2) for right hand sides f satisfying a one–sided continuity condition (see (2.2) below), that allows certain types of discontinuities. Furthermore, we shall be concerned with an error analysis for a finite difference scheme to approximate the solution of (1.1), (1.2).
The well–posedness of (1.1), (1.2) in the case of continuous f follows from the theory of viscosity solutions for Hamilton–Jacobi equations $H(x,u,\nabla u) = 0$ developed in [4]. The notion of viscosity solution was generalised by Ishii [5] to allow for discontinuous Hamiltonians H. In [11], Tourin proves a comparison result for Hamiltonians, which are allowed to be discontinuous along a smooth surface. Soravia [10] obtains necessary and sufficient conditions for uniqueness of the solution to the boundary value problem. While the work in [11] and [10] is based on Ishii's notion of solution, several other approaches have been suggested: in [7], Newcomb & Su consider the Dirichlet problem for $H(\nabla u) = f$ and introduce a notion of solution which they call Monge solution. They obtain a comparison result as well as uniqueness for the Dirichlet problem provided that f is lower semicontinuous. Ostrov [8] studies an evolutionary Hamilton–Jacobi equation which occurs in the context of radar satellite tracking and obtains a unique solution as the limit of suitable upper and lower solutions. Recently, Camilli & Siconolfi [3] introduced a new notion of solution for Hamilton–Jacobi equations of the form $H(x,\nabla u) = 0$, which allows measurable dependence of H on x and involves measure–theoretic limits. They prove representation formulae, comparison principles and uniqness results.
Our work uses Ishii's definition of solution which we shall recall in §2. For a class of right hand sides f, which satisfy a suitable one–sided continuity condition we obtain well–posedness of the problem (1.1), (1.2). In §3 we discretize the problem with the help of a finite difference scheme on a regular grid. Under a slightly more restrictive condition on f we prove that the error between viscosity solution and discrete approximation is of order $\mathcal{O}(\sqrt{h})$. We have not included all the details of the proofs of existence and of the error analysis. However, a forthcoming paper, which generalises our approach to Hamilton–Jacobi equations of the form $H(\nabla u) = f$, will provide a detailed convergence analysis for a wide class of finite difference schemes as well as numerical tests.

2 Existence and Uniqueness

In order to allow for discontinuous functions f in (1.1) we shall use the following generalisation of the concept of viscosity solution, which was introduced by Ishii in [5].

Definition 2.1. A function $u \in C^0(\bar{\Omega})$ is called a viscosity subsolution (supersolution) of (1.1) if for each $\zeta \in C^\infty(\Omega)$: if $u - \zeta$ has a local maximum (minimum) at a point $x_0 \in \Omega$, then

$$|\nabla\zeta(x_0)| \le f^*(x_0) \quad (\ge f_*(x_0)).$$

Here,

$$f^*(x) := \lim_{r\to 0} \sup\{f(y) \,|\, y \in B_r(x) \cap \Omega\}, \qquad f_*(x) := \lim_{r\to 0} \inf\{f(y) \,|\, y \in B_r(x) \cap \Omega\}.$$

A viscosity solution of (1.1), (1.2) then is a function $u \in C^0(\bar{\Omega})$ which is both a viscosity sub– and supersolution and which satisfies $u(x) = \phi(x)$ for all $x \in \partial\Omega$.
Let us next formulate our assumptions on the data of the problem. We suppose that $f : \Omega \to \mathbb{R}$ is Borel measurable and that there exist $0 < m \le M < \infty$ such that

$$m \le f(x) \le M \qquad \forall x \in \Omega. \tag{2.1}$$

Furthermore, we assume that for every $x \in \Omega$ there exist $\epsilon_x > 0$ and $n_x \in S^{n-1}$ so that for all $y \in \Omega, r > 0$ and all $d \in S^{n-1}$ with $|d - n_x| < \epsilon_x$ we have

$$f(y + rd) - f(y) \le \omega\big(|y - x| + r\big), \tag{2.2}$$

where $\omega : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing and satisfies $\omega(0) = 0$. A similar type of condition was used in [11]; however, in (2.2) it is sufficient to estimate values of f for vectors whose difference is close to a given direction.

Example: Suppose that a surface Γ splits Ω into two subdomains Ω_1 and Ω_2, that $f_{|\Omega_1} \in C^0(\bar{\Omega}_1)$, $f_{|\Omega_2} \in C^0(\bar{\Omega}_2)$ and that

$$\lim_{y\to x, y\in\Omega_1} f(y) < \lim_{y\to x, y\in\Omega_2} f(y) \qquad \text{for all } x \in \Gamma.$$

In addition, assume that the following uniform cone property holds: for every $x \in \Gamma$ there exists a neighborhood U_x and a cone C_x (which is congruent to a fixed given cone C_0) such that $y \in U_x \cap \bar{\Omega}_1$ implies that $y + C_x \subset \Omega_1$. Then (2.2) holds with $n = n_x$ given by the direction of the cone C_x.
To see this, observe that the cone condition prevents a situation where $y \in \bar{\Omega}_1, y + rd \in \Omega_2$, which would lead to a violation of (2.2) (cf. [11], where Γ is assumed to be smooth).
One can also consider e.g. a two–dimensional domain Ω, where three curves of discontinuity meet at a triple junction.

It is not difficult to verify that (2.2) implies

$$f^*(y + rd) - f_*(y) \le \omega\big(|y - x| + r\big) \tag{2.3}$$

for all $y \in \Omega, r > 0$ and $d \in S^{n-1}, |d - n_x| < \epsilon_x$.
Finally, we suppose for simplicity that $\phi \equiv 0$.

Lemma 2.2. *There exists a viscosity solution $u \in C^{0,1}(\bar{\Omega})$ of (1.1), (1.2).*

Proof. We only sketch the main ideas. Consider the sup–convolution of f, i.e.

$$f_\epsilon(x) := \sup_{y\in\Omega}\{f(y) - \frac{1}{\epsilon}|x-y|^2\}, \qquad x \in \Omega, \epsilon > 0.$$

Clearly, f_ϵ is continuous and $f^*(x) \le f_\epsilon(x)$ for all $x \in \Omega$. Let

$$L_\epsilon(x,y) := \inf\{\int_0^1 f_\epsilon(\gamma(t)), |\gamma'(t)|\,dt \mid \gamma \in W^{1,\infty}((0,1);\bar\Omega) \text{ with } \gamma(0) = x,\ \gamma(1) = y\}.$$

It is well–known that $u_\epsilon(x) := \inf_{y\in\partial\Omega} L_\epsilon(x,y)$ is a solution of

$$\begin{aligned} |\nabla u^\epsilon| &= f_\epsilon(x) \quad x \in \Omega \\ u^\epsilon(x) &= 0 \quad x \in \partial\Omega \end{aligned}$$

in the viscosity sense. Furthermore, it can be shown that

$$\|u^\epsilon\|_{C^{0,1}(\bar\Omega)} \le C(M,\Omega) \qquad \text{uniformly in } \epsilon > 0.$$

Thus, there exists a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ with $\epsilon_k \searrow 0, k \to \infty$ and $u \in C^{0,1}(\bar\Omega)$ such that $u^{\epsilon_k} \to u$ uniformly in $\bar\Omega$ as $k \to \infty$. Using well–known arguments from the theory of viscosity solutions one verifies that u is a solution of (1.1), (1.2). ∎

Uniqueness of the viscosity solution follows from

Theorem 2.3. *Suppose that $u \in C^0(\bar\Omega)$ is a subsolution of (1.1), $v \in C^0(\bar\Omega)$ is a supersolution of (1.1) and that at least one of the functions belongs to $C^{0,1}(\bar\Omega)$. If $u \le v$ on $\partial\Omega$ then $u \le v$ in $\bar\Omega$.*

Proof. Let us assume that $v \in C^{0,1}(\bar\Omega)$. We shall use the approach presented in [6] (see also [11]). Fix $\theta \in (0,1)$ and define $u_\theta(x) := \theta u(x)$. Next, choose $x_0 \in \bar\Omega$ such that

$$u_\theta(x_0) - v(x_0) = \max_{x\in\bar\Omega}\big(u_\theta(x) - v(x)\big) =: \mu, \tag{2.4}$$

and suppose that $\mu > 0$. Upon replacing u, v by $u+k, v+k$, we may assume that $u \ge 0$ in $\bar\Omega$, so that $u_\theta \le u$ in $\bar\Omega$. In particular, $u_\theta \le v$ on $\partial\Omega$, which implies that $x_0 \in \Omega$. Choose $\epsilon = \epsilon_{x_0}$ and $n = n_{x_0} \in S^{n-1}$ according to (2.2) and define for $\lambda > 0, L \ge 1$

$$\Phi(x,y) := u_\theta(x) - v(y) - L\lambda\,|x - y - \frac{1}{\lambda}n\,|^2 - |x - x_0|^2, \qquad (x,y) \in \bar\Omega \times \bar\Omega.$$

Choose $(x_\lambda, y_\lambda) \in \bar\Omega \times \bar\Omega$ such that

$$\Phi(x_\lambda, y_\lambda) = \max_{(x,y)\in\bar\Omega\times\bar\Omega} \Phi(x,y).$$

Since $x_0 \in \Omega$ we also have $x_0 - \frac{1}{\lambda}n \in \Omega$ for large λ; using the relation $\Phi(x_\lambda, y_\lambda) \ge \Phi(x_0, x_0 - \frac{1}{\lambda}n)$ together with (2.4) we infer

$$\begin{aligned} L\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n|^2 + |x_\lambda - x_0|^2 &\le u_\theta(x_\lambda) - v(y_\lambda) - u_\theta(x_0) + v(x_0 - \frac{1}{\lambda}n) \\ &= (u_\theta(x_\lambda) - v(x_\lambda)) - (u_\theta(x_0) - v(x_0)) + v(x_\lambda) - v(y_\lambda) - v(x_0) + v(x_0 - \frac{1}{\lambda}n) \\ &\le \operatorname{lip}(v)\big(|x_\lambda - y_\lambda| + \frac{1}{\lambda}\big) \\ &\le \operatorname{lip}(v)\big(|x_\lambda - y_\lambda - \frac{1}{\lambda}n| + \frac{2}{\lambda}\big). \end{aligned} \tag{2.5}$$

This implies

$$L\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n|^2 + |x_\lambda - x_0|^2 \le \frac{C}{\lambda},$$

where C depends on $\mathrm{lip}(v)$ and as a consequence,

$$x_\lambda,\, y_\lambda \to x_0, \qquad \text{as} \quad \lambda \to \infty \tag{2.6}$$

$$\lambda\,|x_\lambda - y_\lambda - \frac{1}{\lambda}n| \quad \le \quad \frac{C}{\sqrt{L}} < \frac{\epsilon}{2+\epsilon} \tag{2.7}$$

provided that L is large enough. Since u is a subsolution, we may deduce from the relation $\Phi(x_\lambda, y_\lambda) \ge \Phi(x, y_\lambda)$ for $x \in \bar{\Omega}$ that

$$|2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda}n) + 2(x_\lambda - x_0)| \le \theta f^*(x_\lambda)$$

for large λ and similarly,

$$|2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda}n)| \ge f_*(y_\lambda).$$

Combining the above inequalities, we infer

$$(1-\theta)f^*(y_\lambda) \le 2|x_\lambda - x_0| + \theta\big(f^*(x_\lambda) - f_*(y_\lambda)\big). \tag{2.8}$$

In order to apply (2.2) we write $x_\lambda = y_\lambda + r_\lambda d_\lambda$, where

$$d_\lambda = \frac{n + w_\lambda}{|n + w_\lambda|}, \quad r_\lambda = \frac{1}{\lambda}\,|n + w_\lambda|, \quad w_\lambda = \lambda\left(x_\lambda - y_\lambda - \frac{1}{\lambda}n\right). \tag{2.9}$$

Recalling (2.7) we deduce

$$|d_\lambda - n| \le \frac{2|w_\lambda|}{1 - |w_\lambda|} \le \frac{\frac{2\epsilon}{2+\epsilon}}{1 - \frac{\epsilon}{2+\epsilon}} = \epsilon$$

and (2.3) therefore yields

$$f^*(x_\lambda) - f_*(y_\lambda) = f^*(y_\lambda + r_\lambda d_\lambda) - f_*(y_\lambda) \le \omega(|y_\lambda - x_0| + r_\lambda). \tag{2.10}$$

If we send $\lambda \to \infty$ in (2.8) we finally obtain from (2.1), (2.10) and (2.6) that $(1-\theta)m \le 0$, a contradiction. Thus, $u_\theta \le v$ in $\bar{\Omega}$ and sending $\theta \nearrow 1$ gives the desired result. ∎

3 Numerical scheme and error analysis

Let us assume that $\Omega = \Pi_{i=1}^n (0, b_i)$ and that the grid size $h > 0$ is chosen in such a way that $b_i = N_i h$ for some $N_i \in \mathbb{N}$, $i = 1, \dots, n$. We then define

$$\Omega_h := \mathbb{Z}_h^n \cap \Omega, \quad \partial\Omega_h := \mathbb{Z}_h^n \cap \partial\Omega, \quad \bar{\Omega}_h := \Omega_h \cup \partial\Omega_h,$$

where $\mathbb{Z}_h^n = \{x_\alpha = (h\alpha_1, \dots, h\alpha_n) \mid \alpha_i \in \mathbb{Z}, i = 1, \dots, n\}$. Our aim is to approximate the viscosity solution u of (1.1), (1.2) by a grid function $U : \bar{\Omega}_h \to \mathbb{R}$ and to prove an estimate for $\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)|$. Let us abbreviate $U_\alpha = U(x_\alpha)$ and recall the usual backward and forward difference quotients,

$$D_k^- U_\alpha := \frac{U_\alpha - U_{\alpha - e_k}}{h}, \quad D_k^+ U_\alpha := \frac{U_{\alpha + e_k} - U_\alpha}{h}, \quad x_\alpha \in \Omega_h, \quad k = 1, \dots, n.$$

In order to define the numerical method we introduce the function $G : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$G(p_1, q_1, \dots, p_n, q_n) := \Big(\sum_{k=1}^{n} \max(p_k^+, -q_k^-)^2\Big)^{\frac{1}{2}},$$

where $x^+ = \max(x, 0), x^- = \min(x, 0)$. The discrete problem now reads: find $U : \bar{\Omega}_h \to \mathbb{R}$ such that

$$\begin{aligned} G(D_1^- U_\alpha, D_1^+ U_\alpha, \dots, D_n^- U_\alpha, D_n^+ U_\alpha) &= f(x_\alpha) \quad x_\alpha \in \Omega_h && (3.1)\\ U_\alpha &= 0 \qquad\;\; x_\alpha \in \partial\Omega_h. && (3.2)\end{aligned}$$

The above scheme was examined for continuous f in [9] in the context of shape–from–shading and convergence to the viscosity solution was obtained as a consequence of a result of Barles and Souganidis [2]. In the case of a constant right hand side $f \equiv 1$, Zhao [12] recently obtained an $\mathcal{O}(h)$ error bound. The scheme can be derived by interpreting the viscosity solution u as the value function of an optimal control problem. For further information and a corresponding list of references we refer to Appendix A (written by M. Falcone) in [1].
The function G has the following crucial properties:
a) *Consistency:*

$$G(p_1, p_1, \dots, p_n, p_n) = |p| \qquad \text{for all } p = (p_1, \dots, p_n) \in \mathbb{R}^n. \tag{3.3}$$

b) *Monotonicity:*
let $a = (a_1, a_2, \dots, a_{2n-1}, a_{2n}), b = (b_1, b_2, \dots, b_{2n-1}, b_{2n}) \in \mathbb{R}^{2n}$ with $a_k \geq b_k$ for $k = 1, \dots, 2n$. Then

$$G(t - a_1, a_2 - t, \dots, t - a_{2n-1}, a_{2n} - t) \leq G(t - b_1, b_2 - t, \dots, t - b_{2n-1}, b_{2n} - t) \quad \forall t \in \mathbb{R}. \tag{3.4}$$

Note that the above properties imply in particular that the solution of (3.1), (3.2) cannot have a local minimum in Ω_h and therefore $U_\alpha \geq 0$ in $\bar{\Omega}_h$. In order to carry out our error analysis we need to strengthen (2.2) in that we assume that there exist $\epsilon > 0$, $K \geq 0$ such that for all $x \in \Omega$ there is a direction $n = n_x \in S^{n-1}$ with

$$f(y + rd) - f(y) \leq Kr \qquad \forall y \in \Omega, |y - x| < \epsilon \quad \forall d \in S^{n-1}, |d - n| < \epsilon \quad \forall r > 0. \tag{3.5}$$

Theorem 3.1. *Let u be the viscosity solution of (1.1), (1.2) and U a solution of (3.1), (3.2). Then there exists a constant C, which is independent of h such that*

$$\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)| \leq C\sqrt{h}.$$

Proof. We again only sketch the main ideas. As it seems difficult to use the argument from the uniqueness proof in order to control the maximum of $u - U$, we shall resort to the Kružkov transform. Thus, let $\tilde{u} : \bar{\Omega} \to \mathbb{R}, \tilde{U} : \bar{\Omega}_h \to \mathbb{R}$ be defined by

$$\tilde{u}(x) := -e^{-u(x)}, \, x \in \bar{\Omega}, \quad \tilde{U}_\alpha := -e^{-U_\alpha}, \, x_\alpha \in \bar{\Omega}_h.$$

One verifies (cf. [4]) that $\tilde{u}$ is a viscosity solution of

$$\begin{aligned} f(x)\tilde{u} + |\nabla \tilde{u}| &= 0 \quad\;\; x \in \Omega && (3.6)\\ \tilde{u}(x) &= -1 \quad x \in \partial\Omega, && (3.7)\end{aligned}$$

and that $\tilde{U}$ satisfies

$$\begin{aligned} f(x_\alpha)\tilde{U}_\alpha + G(D_1^-\tilde{U}_\alpha, D_1^+\tilde{U}_\alpha, ..., D_n^-\tilde{U}_\alpha, D_n^+\tilde{U}_\alpha) &= F_\alpha^h \quad x_\alpha \in \Omega_h &\quad (3.8)\\ \tilde{U}_\alpha &= -1 \quad x_\alpha \in \partial\Omega_h, &\quad (3.9)\end{aligned}$$

where

$$\max_{x_\alpha\in\Omega_h} |F_\alpha^h| \leq Ch. \tag{3.10}$$

Next, choose $x_\beta \in \bar{\Omega}_h$ such that

$$|\tilde{u}(x_\beta) - \tilde{U}_\beta| = \max_{x_\alpha\in\bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha|$$

and assume that $\tilde{u}(x_\beta) \geq \tilde{U}_\beta$. The opposite case can be treated similarly. If $\operatorname{dist}(x_\beta, \partial\Omega) \leq \sqrt{h}$, it follows from (3.7), (3.9) and the Lipschitz continuity of $\tilde{u}$ that

$$\max_{x_\alpha\in\bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha| = \tilde{u}(x_\beta) - \tilde{U}_\beta \leq C\sqrt{h}.$$

Now suppose that $\operatorname{dist}(x_\beta, \partial\Omega) > \sqrt{h}$ and define

$$\Phi(x, x_\alpha) := \tilde{u}(x) - \tilde{U}_\alpha - \frac{L_1}{\sqrt{h}}|x - x_\alpha - \sqrt{h}\,n|^2 - L_2\sqrt{h}\,|x_\alpha - x_\beta|^2, \quad (x, x_\alpha) \in \bar{\Omega} \times \bar{\Omega}_h.$$

Here, $n = n_{x_\beta}$ and $L_1, L_2 \geq 0$ are constants that do not depend on h and which will be chosen later. There exists $(x_h, x_{\alpha_h}) \in \bar{\Omega} \times \bar{\Omega}_h$ such that

$$\Phi(x_h, x_{\alpha_h}) = \max_{(x,x_\alpha)\in\bar{\Omega}\times\bar{\Omega}_h} \Phi(x, x_\alpha).$$

Since $\operatorname{dist}(x_\beta, \partial\Omega) > \sqrt{h}$, we have $x_\beta + \sqrt{h}\,n \in \bar{\Omega}$ and therefore

$$\Phi(x_h, x_{\alpha_h}) \geq \Phi(x_\beta + \sqrt{h}\,n, x_\beta).$$

From this we infer in a similar way as in (2.5) that

$$\begin{aligned} |x_{\alpha_h} - x_\beta| &< \epsilon, &\quad (3.11)\\ \frac{1}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}\,n| &< \frac{\epsilon}{2+\epsilon} &\quad (3.12)\end{aligned}$$

provided that $L_i = L_i(\operatorname{lip}(\tilde{u}), \epsilon), i = 1, 2$ are sufficiently large (ϵ from (3.5)).
Suppose first that $(x_h, x_{\alpha_h}) \in \Omega \times \Omega_h$. Since $\tilde{u}$ is a subsolution of (3.6) we infer

$$f^*(x_h)\tilde{u}(x_h) + |\frac{2L_1}{\sqrt{h}}(x_h - x_{\alpha_h} - \sqrt{h}\,n)| \leq 0. \tag{3.13}$$

Keeping the first component of Φ fixed we obtain on the other hand for all $x_\alpha \in \bar{\Omega}_h$

$$\begin{aligned} \tilde{U}_\alpha &\geq \tilde{U}_{\alpha_h} + \frac{L_1}{\sqrt{h}}\Big(|x_h - x_{\alpha_h} - \sqrt{h}\,n|^2 - |x_h - x_\alpha - \sqrt{h}\,n|^2\Big)\\ &\quad + L_2\sqrt{h}\big(|x_{\alpha_h} - x_\beta|^2 - |x_\alpha - x_\beta|^2\big)\\ &=: \tilde{V}_\alpha.\end{aligned}$$

Since $\tilde{U}_{\alpha_h} = \tilde{V}_{\alpha_h}$, (3.4) and (3.3) imply

$$\begin{aligned} G(D_1^-\tilde{U}_{\alpha_h}, D_1^+\tilde{U}_{\alpha_h}, ..., D_n^-\tilde{U}_{\alpha_h}, D_n^+\tilde{U}_{\alpha_h}) &\le G(D_1^-\tilde{V}_{\alpha_h}, D_1^+\tilde{V}_{\alpha_h}, ..., D_n^-\tilde{V}_{\alpha_h}, D_n^+\tilde{V}_{\alpha_h}) \\ &\le |\frac{2L_1}{\sqrt{h}}(x_h - x_{\alpha_h} - \sqrt{h}\,n) - 2L_2\sqrt{h}(x_{\alpha_h} - x_\beta)| + C\sqrt{h}. \end{aligned}$$

Combining this inequality with (3.8) and (3.10) then yields

$$f(x_{\alpha_h})\tilde{U}_{\alpha_h} + |\frac{2L_1}{\sqrt{h}}(x_h - x_{\alpha_h} - \sqrt{h}\,n)| \ge -|F^h_{\alpha_h}| - C\sqrt{h} \ge -C\sqrt{h}. \tag{3.14}$$

As a result of (3.13), (3.14)

$$\begin{aligned} f(x_{\alpha_h})(\tilde{u}(x_h) - \tilde{U}_{\alpha_h}) &\le C\sqrt{h} + e^{-u(x_h)}(f^*(x_h) - f(x_{\alpha_h})) \\ &= C\sqrt{h} + e^{-u(x_h)}(f^*(x_{\alpha_h} + r_h d_h) - f(x_{\alpha_h})) \end{aligned} \tag{3.15}$$

where similar to (2.9), $d_h = \dfrac{n + w_h}{|n + w_h|}$, $r_h = \sqrt{h}\,|n + w_h|$, $w_h = \frac{1}{\sqrt{h}}(x_h - x_{\alpha_h} - \sqrt{h}\,n)$. Since

$$\begin{aligned} \tilde{u}(x_h) - \tilde{U}_{\alpha_h} &= \Phi(x_h, x_{\alpha_h}) + \frac{L_1}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}\,n|^2 + L_2\sqrt{h}|x_{\alpha_h} - x_\beta|^2 \\ &\ge \Phi(x_\beta, x_\beta) = \tilde{U}_\beta - \tilde{u}(x_\beta) - L_1\sqrt{h}, \end{aligned}$$

we finally deduce from (2.1), (3.15) and (3.5) that

$$m(\tilde{u}(x_\beta) - \tilde{U}_\beta) \le C\sqrt{h} + K r_h \le C\sqrt{h}.$$

The cases $x_h \in \partial\Omega$ or $x_{\alpha_h} \in \partial\Omega_h$ can be examined with the help of the boundary conditions (3.7), (3.9). Transforming back to u and U implies the desired error bound. ∎

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