Uniqueness and error bounds for eikonal equations with discontinuities

Klaus Deckelnick & Charles M. Elliott

1 Introduction

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a Lipschitz boundary \( \partial \Omega \). We consider the eikonal equation

\[
|\nabla u(x)| = f(x) \quad x \in \Omega \\
u(x) = \phi(x) \quad x \in \partial \Omega,
\]

where \( f \) and \( \phi \) are given functions. The equation arises for example in geometric optics, computer vision or robotic navigation. In certain situations it is desirable to allow \( f \) to be discontinuous, e.g. in geometric optics, when light propagates through a layered medium. The aim of this paper is to study the well-posedness of (1.1), (1.2) for right hand sides \( f \) satisfying a one-sided continuity condition (see (2.2) below), that allows certain types of discontinuities. Furthermore, we shall be concerned with an error analysis for a finite difference scheme to approximate the solution of (1.1), (1.2).

The well-posedness of (1.1), (1.2) in the case of continuous \( f \) follows from the theory of viscosity solutions for Hamilton–Jacobi equations \( H(x, u, \nabla u) = 0 \) developed in [4]. The notion of viscosity solution was generalised by Ishii [5] to allow for discontinuous Hamiltonians \( H \). In [11], Tourin proves a comparison result for Hamiltonians, which are allowed to be discontinuous along a smooth surface. Soravia [10] obtains necessary and sufficient conditions for uniqueness of the solution to the boundary value problem. While the work in [11] and [10] is based on Ishii’s notion of solution, several other approaches have been suggested: in [7], Newcomb & Su consider the Dirichlet problem for \( H(\nabla u) = f \) and introduce a notion of solution which they call Monge solution. They obtain a comparison result as well as uniqueness for the Dirichlet problem provided that \( f \) is lower semicontinuous. Ostrov [8] studies an evolutionary Hamilton–Jacobi equation which occurs in the context of radar satellite tracking and obtains a unique solution as the limit of suitable upper and lower solutions. Recently, Camilli & Siconolfi [3] introduced a new notion of solution for Hamilton–Jacobi equations of the form \( H(x, \nabla u) = 0 \), which allows measurable dependence of \( H \) on \( x \) and involves measure-theoretic limits. They prove representation formulae, comparison principles and uniqueness results.

Our work uses Ishii’s definition of solution which we shall recall in §2. For a class of right hand sides \( f \), which satisfy a suitable one-sided continuity condition we obtain well-posedness of the problem (1.1), (1.2). In §3 we discretize the problem with the help of a finite difference scheme on a regular grid. Under a slightly more restrictive condition on \( f \) we prove that the error between viscosity solution and discrete approximation is of order \( O(\sqrt{h}) \). We have not included all the details of the proofs of existence and of the error analysis. However, a forthcoming paper, which generalises our approach to Hamilton–Jacobi equations of the form \( H(\nabla u) = f \), will provide a detailed convergence analysis for a wide class of finite difference schemes as well as numerical tests.
2 Existence and Uniqueness

In order to allow for discontinuous functions $f$ in (1.1) we shall use the following generalisation of the concept of viscosity solution, which was introduced by Ishii in [5].

Definition 2.1. A function $u \in C^{0}(\bar{\Omega})$ is called a viscosity subsolution (supersolution) of (1.1) if for each $\zeta \in C^{\infty}(\Omega)$: if $u - \zeta$ has a local maximum (minimum) at a point $x_{0} \in \Omega$, then

$$|\nabla \zeta(x_{0})| \leq f^{*}(x_{0}) \quad (\geq f_{*}(x_{0})).$$

Here,

$$f^{*}(x) := \lim_{r \rightarrow 0} \sup \{f(y) \mid y \in B_{r}(x) \cap \Omega\}, \quad f_{*}(x) := \lim_{r \rightarrow 0} \inf \{f(y) \mid y \in B_{r}(x) \cap \Omega\}.$$

A viscosity solution of (1.1), (1.2) then is a function $u \in C^{0}(\bar{\Omega})$ which is both a viscosity sub- and supersolution and which satisfies $u(x) = \phi(x)$ for all $x \in \partial \Omega$.

Let us next formulate our assumptions on the data of the problem. We suppose that $f : \Omega \rightarrow \mathbb{R}$ is Borel measurable and that there exist $0 < m \leq M < \infty$ such that

$$m \leq f(x) \leq M \quad \forall x \in \Omega.$$  \hspace{1cm} (2.1)

Furthermore, we assume that for every $x \in \Omega$ there exist $\epsilon_{x} > 0$ and $n_{x} \in S^{n-1}$ so that for all $y \in \Omega$, $r > 0$ and all $d \in S^{n-1}$ with $|d - n_{x}| < \epsilon_{x}$ we have

$$f(y + rd) - f(y) \leq \omega(|y - x| + r), \hspace{1cm} (2.2)$$

where $\omega : [0, \infty) \rightarrow [0, \infty)$ is continuous, nondecreasing and satisfies $\omega(0) = 0$. A similar type of condition was used in [11]; however, in (2.2) it is sufficient to estimate values of $f$ for vectors whose difference is close to a given direction.

Example: Suppose that a surface $\Gamma$ splits $\Omega$ into two subdomains $\Omega_{1}$ and $\Omega_{2}$, that $f_{|\Omega_{1}} \in C^{0}(\bar{\Omega}_{1})$, $f_{|\Omega_{2}} \in C^{0}(\bar{\Omega}_{2})$ and that

$$\lim_{y \rightarrow x, y \in \Omega_{1}} f(y) < \lim_{y \rightarrow x, y \in \Omega_{2}} f(y) \quad \text{for all } x \in \Gamma.$$

In addition, assume that the following uniform cone property holds: for every $x \in \Gamma$ there exists a neighborhood $U_{x}$ and a cone $C_{x}$ (which is congruent to a fixed given cone $C_{0}$) such that $y \in U_{x} \cap \bar{\Omega}_{1}$ implies that $y + C_{x} \subset \Omega_{1}$. Then (2.2) holds with $n = n_{x}$ given by the direction of the cone $C_{x}$.

To see this, observe that the cone condition prevents a situation where $y \in \Omega_{1}, y + rd \in \Omega_{2}$, which would lead to a violation of (2.2) (cf. [11], where $\Gamma$ is assumed to be smooth).

One can also consider e.g. a two–dimensional domain $\Omega$, where three curves of discontinuity meet at a triple junction.

It is not difficult to verify that (2.2) implies

$$f^{*}(y + rd) - f_{*}(y) \leq \omega(|y - x| + r) \hspace{1cm} (2.3)$$

for all $y \in \Omega$, $r > 0$ and $d \in S^{n-1}$, $|d - n_{x}| < \epsilon_{x}$. Finally, we suppose for simplicity that $\phi \equiv 0$.

Lemma 2.2. There exists a viscosity solution $u \in C^{0,1}(\bar{\Omega})$ of (1.1), (1.2).
Proof. We only sketch the main ideas. Consider the sup-convolution of $f$, i.e.

$$f_\varepsilon(x) := \sup_{y \in \Omega} \{ f(y) - \frac{1}{\varepsilon} |x - y|^2 \}, \quad x \in \Omega, \varepsilon > 0.$$ 

Clearly, $f_\varepsilon$ is continuous and $f^*(x) \leq f_\varepsilon(x)$ for all $x \in \Omega$. Let

$$L_\varepsilon(x, y) := \inf \left\{ \int_0^1 f_\varepsilon(\gamma(t)), \frac{1}{2} \lambda'^2(t) \, dt \mid \gamma \in W^{1,\infty}((0,1); \bar{\Omega}) \text{ with } \gamma(0) = x, \gamma(1) = y \right\}.$$ 

It is well-known that $u_\varepsilon(x) := \inf_{y \in \partial \Omega} L_\varepsilon(x, y)$ is a solution of

$$|\nabla u^\varepsilon| = f_\varepsilon(x) \quad x \in \Omega$$

$$u^\varepsilon(x) = 0 \quad x \in \partial \Omega$$

in the viscosity sense. Furthermore, it can be shown that

$$\| u^\varepsilon \|_{C^{0,1}(\bar{\Omega})} \leq C(M, \Omega) \quad \text{uniformly in } \varepsilon > 0.$$ 

Thus, there exists a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ with $\varepsilon_k \searrow 0$, $k \to \infty$ and $u \in C^{0,1}(\bar{\Omega})$ such that $u^{\varepsilon_k} \to u$ uniformly in $\bar{\Omega}$ as $k \to \infty$. Using well-known arguments from the theory of viscosity solutions one verifies that $u$ is a solution of (1.1), (1.2).

Uniqueness of the viscosity solution follows from

**Theorem 2.3.** Suppose that $u \in C^0(\bar{\Omega})$ is a subsolution of (1.1), $v \in C^0(\bar{\Omega})$ is a supersolution of (1.1) and that at least one of the functions belongs to $C^{0,1}(\bar{\Omega})$. If $u \leq v$ on $\partial \Omega$ then $u \leq v$ in $\Omega$.

*Proof.* Let us assume that $v \in C^{0,1}(\bar{\Omega})$. We shall use the approach presented in [6] (see also [11]). Fix $\theta \in (0,1)$ and define $u_\theta(x) := \theta u(x)$. Next, choose $x_0 \in \Omega$ such that

$$u_\theta(x_0) - v(x_0) = \max_{x \in \Omega} (u_\theta(x) - v(x)) =: \mu,$$

(2.4)

and suppose that $\mu > 0$. Upon replacing $u, v$ by $u + k, v + k$, we may assume that $u \geq 0$ in $\bar{\Omega}$, so that $u_\theta \leq u$ in $\bar{\Omega}$. In particular, $u_\theta \leq v$ on $\partial \Omega$, which implies that $x_0 \in \Omega$. Choose $\varepsilon = \varepsilon_{x_0}$ and $n = n_{x_0} \in S^{n-1}$ according to (2.2) and define for $\lambda > 0, L \geq 1$

$$\Phi(x, y) := u_\theta(x) - v(y) - L \lambda |x - y - \frac{1}{\lambda} n|^2 - |x - x_0|^2, \quad (x, y) \in \bar{\Omega} \times \bar{\Omega}.$$ 

Choose $(x_\lambda, y_\lambda) \in \bar{\Omega} \times \bar{\Omega}$ such that

$$\Phi(x_\lambda, y_\lambda) = \max_{(x, y) \in \bar{\Omega} \times \bar{\Omega}} \Phi(x, y).$$ 

Since $x_0 \in \Omega$ we also have $x_0 - \frac{1}{\lambda} n \in \Omega$ for large $\lambda$; using the relation $\Phi(x_\lambda, y_\lambda) \geq \Phi(x_0, x_0 - \frac{1}{\lambda} n)$ together with (2.4) we infer

$$L \lambda |x_\lambda - y_\lambda - \frac{1}{\lambda} n|^2 + |x_\lambda - x_0|^2 \leq u_\theta(x_\lambda) - v(y_\lambda) - u_\theta(x_0) + v(x_0 - \frac{1}{\lambda} n)$$

$$= (u_\theta(x_\lambda) - u_\theta(x_0)) - (u_\theta(x_0) - v(x_0)) + v(x_\lambda) - v(y_\lambda) - v(x_0) + v(x_0 - \frac{1}{\lambda} n)$$

$$\leq \text{lip}(v)(|x_\lambda - y_\lambda| + \frac{1}{\lambda})$$

$$\leq \text{lip}(v)(|x_\lambda - y_\lambda - \frac{1}{\lambda} n| + 2 \lambda).$$

(2.5)
This implies
\[ L\lambda |x_\lambda - y_\lambda - \frac{1}{\lambda} n|^2 + |x_\lambda - x_0|^2 \leq \frac{C}{\lambda}, \]
where \( C \) depends on \( \text{lip}(v) \) and as a consequence,
\[ x_\lambda, y_\lambda \to x_0, \quad \text{as} \quad \lambda \to \infty \quad (2.6) \]
\[ \lambda |x_\lambda - y_\lambda - \frac{1}{\lambda} n| \leq \frac{C}{\sqrt{L}} \leq \frac{\epsilon}{2 + \epsilon} \quad (2.7) \]
provided that \( L \) is large enough. Since \( u \) is a subsolution, we may deduce from the relation \( \Phi(x_\lambda, y_\lambda) \geq \Phi(x, y_\lambda) \) for \( x \in \overline{\Omega} \) that
\[ 2L\lambda (x_\lambda - y_\lambda - \frac{1}{\lambda} n) + 2(x_\lambda - x_0) \leq \theta f^*(x_\lambda) \]
for large \( \lambda \) and similarly,
\[ 2L\lambda (x_\lambda - y_\lambda - \frac{1}{\lambda} n) \geq f_*(y_\lambda). \]
Combining the above inequalities, we infer
\[ (1 - \theta)f^*(y_\lambda) \leq \omega(|y_\lambda - x_0| + r_\lambda) \quad (2.8) \]
In order to apply (2.2) we write \( x_\lambda = y_\lambda + r_\lambda d_\lambda \), where
\[ d_\lambda = \frac{n + w_\lambda}{|n + w_\lambda|}, \quad r_\lambda = \frac{1}{\lambda} |n + w_\lambda|, \quad w_\lambda = \lambda \left( x_\lambda - y_\lambda - \frac{1}{\lambda} n \right). \quad (2.9) \]
Recalling (2.7) we deduce
\[ |d_\lambda - n| \leq \frac{2|w_\lambda|}{1 - |w_\lambda|} \leq \frac{2\epsilon}{2 + \epsilon} = \epsilon \]
and (2.3) therefore yields
\[ f^*(x_\lambda) - f_*(y_\lambda) = f^*(y_\lambda + r_\lambda d_\lambda) - f_*(y_\lambda) \leq \omega(|y_\lambda - x_0| + r_\lambda). \quad (2.10) \]
If we send \( \lambda \to \infty \) in (2.8) we finally obtain from (2.1), (2.10) and (2.6) that \( (1 - \theta)m \leq 0 \), a contradiction. Thus, \( u_\theta \leq v \) in \( \overline{\Omega} \) and sending \( \theta \nearrow 1 \) gives the desired result.

3 Numerical scheme and error analysis

Let us assume that \( \Omega = \Pi_{i=1}^n (0, b_i) \) and that the grid size \( h > 0 \) is chosen in such a way that \( b_i = N_i h \) for some \( N_i \in \mathbb{N}, i = 1, \ldots, n \). We then define
\[ \Omega_h := Z^n_h \cap \Omega, \quad \partial \Omega_h := Z^n_h \cap \partial \Omega, \quad \tilde{\Omega}_h := \Omega_h \cup \partial \Omega_h, \]
where \( Z^n_h = \{ x_\alpha = (h\alpha_1, \ldots, h\alpha_n) \mid \alpha_i \in \mathbb{Z}, i = 1, \ldots, n \} \). Our aim is to approximate the viscosity solution \( u \) of (1.1), (1.2) by a grid function \( U : \tilde{\Omega}_h \to \mathbb{R} \) and to prove an estimate for \( \max_{x_\alpha \in \Omega_h} |u(x_\alpha) - U(x_\alpha)| \). Let us abbreviate \( U_\alpha = U(x_\alpha) \) and recall the usual backward and forward difference quotients,
\[ D^-_k U_\alpha := \frac{U_{\alpha} - U_{\alpha-e_k}}{h}, \quad D^+_k U_\alpha := \frac{U_{\alpha+e_k} - U_\alpha}{h}, \quad x_\alpha \in \Omega_h, \quad k = 1, \ldots, n. \]
In order to define the numerical method we introduce the function \( G : \mathbb{R}^{2n} \to \mathbb{R} \) as
\[
G(p_1, q_1, ..., p_n, q_n) := \left( \sum_{k=1}^{n} \max(p_k^+, -q_k^-)^2 \right)^{\frac{1}{2}},
\]
where \( x^+ = \max(x, 0), x^- = \min(x, 0) \). The discrete problem now reads: find \( U : \bar{\Omega}_h \to \mathbb{R} \) such that
\[
G(D^-_1 U_\alpha, D^+_1 U_\alpha, ..., D^-_n U_\alpha, D^+_n U_\alpha) = f(x_\alpha) \quad x_\alpha \in \Omega_h \quad (3.1)
\]
\[
U_\alpha = 0 \quad x_\alpha \in \partial \Omega_h. \quad (3.2)
\]

The above scheme was examined for continuous \( f \) in [9] in the context of shape-from-shading and convergence to the viscosity solution was obtained as a consequence of a result of Barles and Souganidis [2]. In the case of a constant right hand side \( f \equiv 1 \), Zhao [12] recently obtained an \( O(h) \) error bound. The scheme can be derived by interpreting the viscosity solution \( u \) as the value function of an optimal control problem. For further information and a corresponding list of references we refer to Appendix A (written by M. Falcone) in [1].

The function \( G \) has the following crucial properties:

a) **Consistency**:
\[
G(p_1, p_1, ..., p_n, p_n) = |p| \quad \text{for all} \quad p = (p_1, ..., p_n) \in \mathbb{R}^n. \quad (3.3)
\]

b) **Monotonicity**:

Let \( a = (a_1, a_2, ..., a_{2n-1}, a_{2n}) \), \( b = (b_1, b_2, ..., b_{2n-1}, b_{2n}) \) \( \in \mathbb{R}^{2n} \) with \( a_k \geq b_k \) for \( k = 1, ..., 2n \). Then
\[
G(t-a_1, a_2-t, ..., t-a_{2n-1}, a_{2n}-t) \leq G(t-b_1, b_2-t, ..., t-b_{2n-1}, b_{2n}-t) \quad \forall t \in \mathbb{R}. \quad (3.4)
\]

Note that the above properties imply in particular that the solution of (3.1), (3.2) cannot have a local minimum in \( \Omega_h \) and therefore \( U_\alpha \geq 0 \) in \( \Omega_h \). In order to carry out our error analysis we need to strengthen (2.2) in that we assume that there exist \( \epsilon > 0 \), \( K \geq 0 \) such that for all \( x \in \Omega \) there is a direction \( n = n_x \in S^{n-1} \) with
\[
f(y + rd) - f(y) \leq K r \quad \forall y \in \Omega, \quad |y - x| < \epsilon \quad \forall d \in S^{n-1}, \quad |d - n| < \epsilon \quad \forall r > 0. \quad (3.5)
\]

**Theorem 3.1.** Let \( u \) be the viscosity solution of (1.1), (1.2) and \( U \) a solution of (3.1), (3.2). Then there exists a constant \( C \), which is independent of \( h \) such that
\[
\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)| \leq C \sqrt{h}.
\]

**Proof.** We again only sketch the main ideas. As it seems difficult to use the argument from the uniqueness proof in order to control the maximum of \( u - U \), we shall resort to the Kružkov transform. Thus, let \( \tilde{u} : \bar{\Omega} \to \mathbb{R}, \tilde{U} : \bar{\Omega}_h \to \mathbb{R} \) be defined by
\[
\tilde{u}(x) := -e^{-u(x)}, \quad x \in \bar{\Omega}, \quad \tilde{U}_\alpha := -e^{-U_\alpha}, \quad x_\alpha \in \bar{\Omega}_h.
\]

One verifies (cf. [4]) that \( \tilde{u} \) is a viscosity solution of
\[
f(x)\tilde{u} + |\nabla \tilde{u}| = 0 \quad x \in \Omega \quad \tilde{u}(x) = -1 \quad x \in \partial \Omega, \quad (3.6)
\]
\[
\tilde{u}(x) = -1 \quad x \in \partial \Omega. \quad (3.7)
\]
and that \( \bar{U} \) satisfies
\[
f(x_{\alpha})\bar{U}_{\alpha} + G(D_{1}^{-}\bar{U}_{\alpha}, D_{1}^{+}\bar{U}_{\alpha}, \ldots, D_{n}^{-}\bar{U}_{\alpha}, D_{n}^{+}\bar{U}_{\alpha}) = F_{\alpha}^{h} \quad x_{\alpha} \in \Omega_{h} \tag{3.8}
\]
where
\[
\max_{x_{\alpha} \in \Omega_{h}} |F_{\alpha}^{h}| \leq Ch. \tag{3.10}
\]

Next, choose \( x_{\beta} \in \bar{\Omega}_{h} \) such that
\[
|\bar{u}(x_{\beta}) - \bar{U}_{\beta}| = \max_{x_{\alpha} \in \bar{\Omega}_{h}} |\bar{u}(x_{\alpha}) - \bar{U}_{\alpha}|
\]
and assume that \( \bar{u}(x_{\beta}) \geq \bar{U}_{\beta} \). The opposite case can be treated similarly. If \( \text{dist}(x_{\beta}, \partial \Omega) \leq \sqrt{h} \), it follows from (3.7), (3.9) and the Lipschitz continuity of \( \bar{u} \) that
\[
\max_{x_{\alpha} \in \Omega_{h}} |\bar{u}(x_{\alpha}) - \bar{U}_{\alpha}| = \bar{u}(x_{\beta}) - \bar{U}_{\beta} \leq C\sqrt{h}.
\]

Now suppose that \( \text{dist}(x_{\beta}, \partial \Omega) > \sqrt{h} \) and define
\[
\Phi(x, x_{\alpha}) := \bar{u}(x) - \bar{U}_{\alpha} - \frac{L_{1}}{\sqrt{h}} |x - x_{\alpha} - \sqrt{h}n|^{2} - L_{2}\sqrt{h} |x_{\alpha} - x_{\beta}|^{2}, \quad (x, x_{\alpha}) \in \bar{\Omega} \times \bar{\Omega}_{h}.
\]
Here, \( n = n_{x_{\beta}} \) and \( L_{1}, L_{2} \geq 0 \) are constants that do not depend on \( h \) and which will be chosen later. There exists \( (x_{h}, x_{\alpha_{h}}) \in \Omega \times \bar{\Omega}_{h} \) such that
\[
\Phi(x_{h}, x_{\alpha_{h}}) = \max_{(x, x_{\alpha}) \in \bar{\Omega} \times \bar{\Omega}_{h}} \Phi(x, x_{\alpha}).
\]
Since \( \text{dist}(x_{\beta}, \partial \Omega) > \sqrt{h} \), we have \( x_{\beta} + \sqrt{h}n \in \Omega \) and therefore
\[
\Phi(x_{h}, x_{\alpha_{h}}) \geq \Phi(x_{\beta} + \sqrt{h}n, x_{\beta}).
\]

From this we infer in a similar way as in (2.5) that
\[
|\alpha_{h} - x_{\beta}| < \epsilon, \tag{3.11}
\]
\[
\frac{1}{\sqrt{h}} |x_{h} - x_{\alpha_{h}} - \sqrt{h}n| < \frac{\epsilon}{2 + \epsilon} \tag{3.12}
\]
provided that \( L_{i} = L_{i}(\text{lip}(\bar{u}), \epsilon), i = 1, 2 \) are sufficiently large (\( \epsilon \) from (3.5)).

Suppose first that \( (x_{h}, x_{\alpha_{h}}) \in \Omega \times \bar{\Omega}_{h} \). Since \( \bar{u} \) is a subsolution of (3.6) we infer
\[
f^{*}(x_{h})\bar{u}(x_{h}) + \frac{2L_{1}}{\sqrt{h}} (x_{h} - x_{\alpha_{h}} - \sqrt{h}n) \leq 0. \tag{3.13}
\]

Keeping the first component of \( \Phi \) fixed we obtain on the other hand for all \( x_{\alpha} \in \bar{\Omega}_{h} \)
\[
\bar{U}_{\alpha} \geq \bar{U}_{\alpha_{h}} + \frac{L_{1}}{\sqrt{h}} (|x_{h} - x_{\alpha_{h}} - \sqrt{h}n|^{2} - |x_{h} - x_{\alpha} - \sqrt{h}n|^{2})
\]
\[
+ L_{2}\sqrt{h} (|x_{\alpha_{h}} - x_{\beta}|^{2} - |x_{\alpha} - x_{\beta}|^{2})
\]
\[
= : \tilde{V}_{\alpha}.
\]
Since $\tilde{U}_{\alpha_{h}} = \bar{V}_{\alpha_{h}}$, (3.4) and (3.3) imply
\[
G(D_{1}^{-}\tilde{U}_{\alpha_{h}}, D_{1}^{+}\tilde{U}_{\alpha_{h}}, \ldots, D_{n}^{-}\tilde{U}_{\alpha_{h}}, D_{n}^{+}\tilde{U}_{\alpha_{h}}) \leq G(D_{1}^{-}\bar{V}_{\alpha_{h}}, D_{1}^{+}\bar{V}_{\alpha_{h}}, \ldots, D_{n}^{-}\bar{V}_{\alpha_{h}}, D_{n}^{+}\bar{V}_{\alpha_{h}})
\]
\[
\leq \left| \frac{2L_{1}}{\sqrt{h}}(x_{h} - x_{\alpha_{h}} - \sqrt{h}n) - 2L_{2}\sqrt{h}(x_{\alpha_{h}} - x_{\beta}) \right| + C\sqrt{h}.
\]
Combining this inequality with (3.8) and (3.10) then yields
\[
f(x_{\alpha_{h}})\tilde{U}_{\alpha_{h}} + \frac{2L_{1}}{\sqrt{h}}(x_{h} - x_{\alpha_{h}} - \sqrt{h}n) \geq -|F_{\alpha_{h}}^{h}| - C\sqrt{h} \geq -C\sqrt{h}.
\] (3.14)
As a result of (3.13), (3.14)
\[
f(x_{\alpha_{h}})(\bar{u}(x_{h}) - \tilde{U}_{\alpha_{h}}) \leq C\sqrt{h} + e^{-u(x_{h})}(f^{*}(x_{h}) - f(x_{\alpha_{h}}))
\]
\[
= C\sqrt{h} + e^{-u(x_{h})}(f^{*}(x_{\alpha_{h}} + r_{h}d_{h}) - f(x_{\alpha_{h}}))
\] (3.15)
where similar to (2.9), $d_{h} = \frac{n + w_{h}}{|n + w_{h}|}$, $r_{h} = \sqrt{h}|n + w_{h}|$, $w_{h} = \frac{1}{\sqrt{h}}(x_{h} - x_{\alpha_{h}} - \sqrt{h}n)$. Since
\[
\bar{u}(x_{h}) - \tilde{U}_{\alpha_{h}} = \Phi(x_{h}, x_{\alpha_{h}}) + \frac{L_{1}}{\sqrt{h}}|x_{h} - x_{\alpha_{h}} - \sqrt{h}n|^{2} + L_{2}\sqrt{h}|x_{\alpha_{h}} - x_{\beta}|^{2}
\]
\[
\geq \Phi(x_{\beta}, x_{\beta}) = \tilde{U}_{\beta} - \tilde{u}(x_{\beta}) - L_{1}\sqrt{h},
\]
we finally deduce from (2.1), (3.15) and (3.5) that
\[
m(\bar{u}(x_{\beta}) - \tilde{U}_{\beta}) \leq C\sqrt{h} + Kr_{h} \leq C\sqrt{h}.
\]
The cases $x_{h} \in \partial\Omega$ or $x_{\alpha_{h}} \in \partial\Omega_{h}$ can be examined with the help of the boundary conditions (3.7), (3.9). Transforming back to $u$ and $U$ implies the desired error bound.

References


Klaus Deckelnick
Institut für Analysis und Numerik
Otto–von–Guericke–Universität Magdeburg
Universitätsplatz 2
39106 Magdeburg
Germany

Charles M. Elliott
Centre for Mathematical Analysis and Its Applications
School of Mathematical Sciences
University of Sussex
Falmer Brighton BN1 9QH
United Kingdom