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Derivation of variational problems from microscopic interface model

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1 Introduction

This note reviews recent results on the $\nabla \varphi$ interface model considered over the wall or under the weak effects of additional self-potentials. We are, in particular, interested in the scaling limit which passes from microscopic to macroscopic levels. The results are classified into two types:

1. Static results.
   - Large Deviation Principle
   - Derivation of Variational Problems (VP)
     - Wulff Shape, Winterbottom Shape
     - Alt-Caffarelli or Alt-Caffarelli-Friedman’s VP

2. Dynamic results.
   - Motion by Mean Curvature (MMC) with anisotropy
   - Dynamic Large Deviation Principle
   - MMC with reflection
     - Evolutionary Variational Inequality
   - Fluctuation
     - Stochastic PDE with reflection
2 Static results

2.1. Let us introduce the $\nabla \varphi$ interface model briefly. We are concerned with a surface (interface) in $\mathbb{R}^{d+1}$, which separates two distinct pure phases, described by the height variables $\phi = \{\phi(x) \in \mathbb{R}, x \in \Gamma\}$ measured from a reference hyperplane $\Gamma$ located in $\mathbb{R}^d$. To avoid complications, we assume that the interface has no overhangs nor bubbles. The variables $\phi$ are microscopic objects, and the space $\Gamma$ is discretized and taken as $\Gamma = D_N (\equiv ND \cap \mathbb{Z}^d)$, lattice torus $(\mathbb{Z}/N\mathbb{Z})^d (\equiv \{1, 2, \ldots, N\}^d)$ or $\mathbb{Z}^d$. Here $D$ is a (macroscopic) bounded domain in $\mathbb{R}^d$ and $N$ represents the size of the microscopic system.

An energy is associated with each height variable $\phi : \Gamma \to \mathbb{R}$ as the sum over all bonds $(x, y)$ in $\Gamma$ (or in $\Gamma \cup \partial \Gamma$)

$$H(\phi) \equiv H_\Gamma^\psi(\phi) = \sum_{(x,y) \subset \Gamma (\text{or} \Gamma \cup \partial \Gamma)} V(\phi(x) - \phi(y)),$$

and the equilibrium state (Gibbs measure) is defined by

$$d\mu \equiv d\mu_\Gamma^\psi = Z^{-1} e^{-H(\phi)} \prod_{x \in \Gamma} d\phi(x),$$

where $Z$ is a normalization constant. The potential $V$ is symmetric, smooth and strictly convex ($0 < c_- \leq V'' \leq c_+ < \infty$). Note that the boundary conditions $\psi = \{\psi(x), x \in \partial \Gamma\}$ are required to define $H(\phi)$ and $\mu$ when $\Gamma = D_N$. An infinite volume limit (thermodynamic limit) as $\Gamma \nearrow \mathbb{Z}^d$ exists when $d \geq 3$ and the limit measure $\mu$ has a long correlation. More information on the $\nabla \varphi$ interface model can be found in [7] and [13].

2.2. Our main interest is in the scaling limit, which passes from microscopic to macroscopic levels, defined by

$$h^N(\theta) = \frac{1}{N} \phi([N\theta]), \quad \theta \in D \ (\text{or} \in \mathbb{T}^d \equiv [0,1)^d, \mathbb{R}^d),$$

where $[N\theta]$ stands for the integral part of $N\theta (\in \mathbb{R}^d)$ taken componentwise. The function $h^N$ is the macroscopic height variable associated with the microscopic $\phi : \Gamma \to \mathbb{R}$. The surface tension $\sigma = \sigma(u)$ is the macroscopic energy for a surface with tilt $u \in \mathbb{R}^d$ determined by

$$\mu (\text{tilt of } h^N \sim u) \sim_{N \to \infty} \exp\{-N^d \sigma(u)\}.$$

Theorem 1. (Large Deviation, Deuschel-Giacomin-Ioffe [4]) Consider the Gibbs measure $\mu_{D_{N}}^{0}$ on $\Gamma = D_{N}$ with 0-boundary conditions $\psi(x) = 0, x \in \partial D_{N}$. Then, the probability that $h^{N}$ is close to a given macroscopic surface $h \in H_{0}^{1}(D)$ behaves as

$$\mu_{D_{N}}^{0}(h^{N} \sim h) \sim \exp\{-N^{d}\Sigma_{D}(h)\},$$

where $\Sigma_{D}(h)$ is the total surface tension of $h$ defined by

$$\Sigma_{D}(h) = \int_{D} \sigma(\nabla h(\theta)) d\theta.$$

This result is an analogue of Dobrushin-Kotecky-Shlosman [5] for the Ising model.

Corollary 2. (Wall and constant volume conditions) For every $v \geq 0$, under the conditional probability $\mu_{D_{N}}^{0}(\cdot | h^{N} \geq 0, \int_{D} h^{N}(\theta) d\theta \geq v)$, the law of large numbers $h^{N} \rightarrow \bar{h}_{v}$ (as $N \rightarrow \infty$) holds, where $\bar{h}_{v}$ is the minimizer (called Wulff shape) of the variational problem

$$\min \left\{ \Sigma_{D}(h); h \in H_{0}^{1}(D), h \geq 0, \int_{D} h(\theta) d\theta = v \right\}.$$

We add a weak self-potential term to the energy $H_{D_{N}}^{\psi}(\phi)$:

$$H_{D_{N}}^{\psi,Q,W}(\phi) = H_{D_{N}}^{\psi}(\phi) + \sum_{x \in D_{N}} Q\left(\frac{x}{N}\right) W(\phi(x)),$$

having the boundary conditions $\psi(x) = Nf(x/N), x \in \partial D_{N}$ determined from macroscopic function $f : \partial D \rightarrow \mathbb{R}$, where $Q : D \rightarrow [0, \infty)$ and $W : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\alpha_{\pm} = \lim_{r \rightarrow \pm \infty} W(r)$ such that $\alpha_{+} \wedge \alpha_{-} \leq W \leq \alpha_{+} \vee \alpha_{-}$. The Gibbs measure is associated and defined by

$$d\mu_{D_{N}}^{\psi,Q,W} = Z_{\psi,Q,W}^{-1} e^{-H_{D_{N}}^{\psi,Q,W}(\phi)} \prod_{x \in D_{N}} d\phi(x).$$

Theorem 3. (Large Deviation, Funaki-Sakagawa [11]) Assume $A := \alpha_{+} - \alpha_{-} \geq 0$. Then,

$$\mu_{D_{N}}^{\psi,Q,W}(h^{N} \sim h) \sim \exp\{-N^{d}I_{D}^{A}(h)\},$$

where

$$I_{D}^{A}(h) = \Sigma_{D}^{A}(h) - \inf_{h' \in H_{1}^{f}(D)} \Sigma_{D}^{A}(h'),$$

$$\Sigma_{D}^{A}(h) = \Sigma_{D}(h) - A \int_{D} Q(\theta) 1_{\{h(\theta) \leq 0\}} d\theta,$$

and $H_{1}^{f}(D)$ is the space of all $h \in H^{1}(D)$ having boundary conditions $f$. \qed
Theorem 5. (Hydrodynamic Limit, Funaki-Spohn [12] on the torus $\mathbb{T}^d$, Nishikawa [14] on $D$ with boundary conditions) As $N \to \infty$, $h^N(t, \theta) \to h(t, \theta)$. The limit $h(t, \theta)$ is a unique weak solution of the nonlinear PDE (MMC with anisotropy): 

$$
\frac{\partial h}{\partial t}(t) = \text{div} \{\nabla \sigma(\nabla h(t))\} 
= \sum_{i=1}^{d} \frac{\partial}{\partial \theta_i} \left\{ \frac{\partial \sigma}{\partial u_i}(\nabla h(t)) \right\} . 
$$
The surface tension has the following properties: \( \sigma \in C^1(\mathbb{R}^d) \), \( \nabla \sigma \) is Lipschitz continuous and \( \sigma \) is strictly convex. The PDE (2) can be regarded as the gradient flow for \( \Sigma = \Sigma_{T^d} \) or \( \Sigma_D \), the total surface tension (1) on \( \mathbb{T}^d \) or \( D \):

\[
\frac{\partial h}{\partial t}(t) = -\frac{\delta}{\delta h(\theta)} \Sigma(h(t)).
\]

**Theorem 6. (Dynamic Large Deviation, Funaki-Nishikawa [9] on \( \mathbb{T}^d \))**

\[
P(h^N(t) \sim h(t), t \leq T) \sim \exp\{-N^d I_T(h)\},
\]

where \( h(t) = h(t, \theta) \) is a given motion of surface and

\[
I_T(h) = \frac{1}{4} \int_0^T dt \int_{\mathbb{T}^d} \left( \frac{\partial h}{\partial t} - \text{div} \{ \nabla \sigma(\nabla h(t)) \} \right)^2 d\theta.
\]

The relation to the static large deviation (Theorem 1) is given by

\[
\lim_{T \to \infty} S_T(\tilde{h}) = \Sigma_{T^d}(\tilde{h}), \quad \tilde{h} = \tilde{h}(\theta),
\]

where

\[
S_T(\tilde{h}) = \inf \{ I_T(h); h(T, \theta) = \tilde{h}(\theta) \}.
\]

3.2. **Dynamics on the wall** are introduced by SDEs of Skorohod type:

\[
d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t) \, dt + \sqrt{2} dw_t(x) + \frac{1}{N} f\left( \frac{t}{N^2}, \frac{x}{N}, \frac{1}{N} \phi_t(x) \right) dt + d\ell_t(x),
\]

subject to the conditions

\[
\phi_t(x) \geq 0, \quad \ell_t(x) \nearrow \quad \text{and} \quad \int_0^{\infty} \phi_t(x) d\ell_t(x) = 0,
\]

where \( f = f(t, \theta, h) \) is a given macroscopic external field. Note that \( \ell_t(x) \) increases only when \( \phi_t(x) = 0 \). The unique invariant (stationary) measure (when \( f = 0, \Gamma = D_N \) with 0-boundary conditions) is given by \( \mu^0_{D_N}(\cdot | \phi \geq 0) \), which is reversible under the dynamics.

**Theorem 7. (Hydrodynamic Limit, Funaki [8] on \( \mathbb{T}^d \))** As \( N \to \infty \), \( h^N(t, \theta) \longrightarrow h(t, \theta) \). The limit \( h(t, \theta) \) is a unique solution of the evolutionary variational in-
equality (MMC with reflection (obstacle)):

(a) \( h \in L^2(0, T; V), \frac{\partial h}{\partial t} \in L^2(0, T; V'), \forall T > 0, \)

(b) \( \left( \frac{\partial h}{\partial t}(t), h(t) - v \right) + \left( \nabla \sigma(\nabla h(t)), \nabla h(t) - \nabla v \right) \leq (f(t, h(t)), h(t) - v), \quad \text{a.e. } t, \forall v \in V : v \geq 0, \)

(c) \( h(t, \theta) \geq 0, \quad \text{a.e.,} \)

(d) \( h(0, \theta) = h_0(\theta), \)

where \( V = H^1(\mathbb{T}^d), H = L^2(\mathbb{T}^d), V' = H^{-1}(\mathbb{T}^d) \) and \( (\cdot, \cdot) \) denotes the inner product of \( H \) (or \( H^d \)) or the duality between \( V' \) and \( V. \) \( \square \)

Remark 2. Rezakhanlou [15], [16] derived a Hamilton-Jacobi equation under hyperbolic scaling from growing SOS dynamics \( (\phi(x) \in \mathbb{Z}) \) with constraints on the gradients (e.g. \( \nabla \phi(x) \leq v) \). Related results were obtained by Evans-Rezakhanlou [6] and Seppäläinen [17]. \( \square \)

Let us consider the equilibrium dynamics \( \phi_t \) on the wall in one dimension, i.e., \( \phi_t \) is a solution of the SDE (3) with \( d = 1, \Gamma = \{1, 2, \ldots, N-1\}, f = 0 \) and with 0-boundary conditions \( \phi_t(0) = \phi_t(N) = 0 \) and an initial distribution \( \mu_{\Gamma}^0(\cdot | \phi \geq 0) \). Macroscopic fluctuation field (around the hydrodynamic limit \( h(t, \theta) = 0 \)) is defined by

\[ \Phi^N(t, \theta) = \sqrt{N} h^N(t, \theta) (\geq 0), \quad \theta \in [0, 1]. \]

Theorem 8. (Equilibrium Fluctuation, Funaki-Olla [10]) As \( N \to \infty, \Phi^N(t, \theta) \Rightarrow \Phi(t, \theta) \). The limit \( \Phi(t, \theta) \) is a unique weak stationary solution of the stochastic PDE with reflection of Nualart-Pardoux type:

\[ \frac{\partial \Phi}{\partial t}(t, \theta) = q \frac{\partial^2 \Phi}{\partial \theta^2}(t, \theta) + \sqrt{2} \dot{B}(t, \theta) + \xi(t, \theta), \quad \theta \in [0, 1], \]

\[ \Phi(t, \theta) \geq 0, \quad \int_0^\infty \int_0^1 \Phi(t, \theta) \xi(dtd\theta) = 0, \]

\[ \Phi(t, 0) = \Phi(t, 1) = 0, \quad \xi: \text{random measure}, \]

where \( \dot{B}(t, \theta) \) is a space-time white noise and

\[ q = 1/\langle \eta^2 \rangle_{\nu}, \quad \nu(d\eta) = e^{-V(\eta)} d\eta / \int_{\mathbb{R}} e^{-V(\eta')} d\eta'. \]
References


