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# Neumann problems for singular degenerate parabolic equations on nonsmooth domains

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## 1 Introduction

This is a joint work with F. Da Lio. In this paper we are concerned with the following boundary value problem

(1.1) $$u_t + F(t,x,u,Du,D^2u) = 0 \qquad \text{in } Q = (0,T)\times\Omega,$$

(1.2) $$\frac{\partial u}{\partial\gamma} = 0 \qquad \text{in } S = (0,T)\times\partial\Omega,$$

where $\Omega$ is a bounded domain in $\mathbf{R}^n$ and $T>0$. Here $u_t = \partial u/\partial t$, and $Du$ and $D^2u$ denote, respectively, the gradient and Hessian of $u$. Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ and $\Omega = \bigcap_{i\in I}\Omega_i$ where $I$ is a finite index set and $\Omega_i's$ are domains in $\mathbf{R}^n$ with relatively regular boundary such that $\partial\Omega_i \in C^3$. For $x\in\partial\Omega$ we denote by $I(x)$ the set of those indices $i$ which satisfy $x\in\partial\Omega_i$. Let $\{\gamma_i\}_{i\in I}$ be a set of vector fields on $\mathbf{R}^n$ such that each $\gamma_i$ is oblique to $\Omega_i$ on $\partial\Omega_i$, i.e.,$\langle\gamma_i(x), n_i(x)\rangle > 0$ for $x\in\partial\Omega_i$, whwre $n_i(x)$ denotes the outward unit normal vector of $\Omega_i$ at $x$. We deal with equations (1.1) in a class of singular degenerate parabolic equations which includes the mean curvature flow equation. In the case when $F$ is continuous in its variables, there is already a comparison and existence result for viscosity solutions of second order degenerate parabolic PDE with boundary condition (1.2). We refer for this to [D-I]. In the case of singular PDE like the mean curvature flow equation and $\partial\Omega$ is smooth, Giga and Sato [G-S] have established comparison and existence results for viscosity solutions under the Neumann condition and the author [S], Ishii-Sato [I-S] and Barles [B] treated the case of fully nonlinear boundary condition including capillaly boundary condition. Our aim in this paper is to establish comparison and existence theorems concerning viscosity solutions of (1.1)-(1.2) when $\Omega$ is piecewise smooth.

This paper is organized as follows. In Section 2 we state and prove our comparison result and establish our existence result and we explain how to build test functions which are needed in the proof of the comparison and existence theorems.

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# 2 A comparison and existence theorem

Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ and $\Omega = \bigcap_{i \in I} \Omega_i$ where $I$ is a finite index set and $\Omega_i' s$ are domains in $\mathbf{R}^n$ with relatively regular boundary. For $x \in \partial\Omega$ we denote by $I(x)$ the set of those indices $i$ which satisfy $x \in \partial\Omega_i$. Let $\{\gamma_i\}_{i \in I}$ be a set of vector fields on $\mathbf{R}^n$ such that each $\gamma_i$ is oblique to $\Omega_i$ on $\partial\Omega_i$, i.e.,$\langle \gamma_i(x), n_i(x) \rangle > 0$ for $x \in \partial\Omega_i$, whwre $n_i(x)$ denotes the outward unit normal vector of $\Omega_i$ at $x$.

We start by listing our assumptions. Henceforth, for $p, q \in \mathbf{R}^n \setminus \{0\}$ we write $\overline{p} = \frac{p}{|p|}$ and $\rho(p,q) = [(|p| \wedge |q|)^{-1}|p-q|] \wedge 1$. Here and henceforth we use the notation: $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$.

(F1) $\qquad F \in C([0,T] \times \overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n),$

where $\mathcal{S}^n$ denotes the space of $n \times n$ real matrices equipped with the usual ordering.

(F2) There exists a constant $\gamma \in \mathbf{R}$ such that for each $(t,x,p,X) \in [0,T] \times \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n$ the function $u \mapsto F(t,x,u,p,X) - \gamma u$ is non-decreasing on $\mathbf{R}$.

(F3) For each $R > 0$ there exists a continuous function $\omega_R : [0,\infty) \to [0,\infty)$ satisfying $\omega_R(0) = 0$ such that if $X, Y \in \mathcal{S}^n$ and $\mu_1, \mu_2 \in [0,\infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$\begin{aligned} &F(t,x,u,p,X) - F(t,y,u,q,-Y) \\ &\qquad \ge -\omega_R(\mu_1(|x-y|^2 + \rho(p,q)^2) + \mu_2 + |p-q| + |x-y|(|p| \vee |q| + 1)). \end{aligned}$$

for all $t \in [0,T],\ x,y \in \overline{\Omega},\ u \in \mathbf{R}$, with $|u| \le R$, and $p,q \in \mathbf{R}^n \setminus \{0\}$.

(B1) For each $i \in I$ the boundary $\partial\Omega_i$ is of class $C^3$.

(B2) For each $x \in \partial\Omega$ there is a neighorhood $V$ of $x$ in $\partial\Omega$ such that $I(y) \subset I(x)$ for $y \in V$.

(B3) For each $x \in \partial\Omega$ the convex hull of the vectors $\gamma_i(x)$, with $i \in I(x)$, does not contain the origin.

(B4) For each $z \in \partial\Omega$ there is a family $\{B(x) : x \in W\}$ of compact convex subsets of $\mathbf{R}^n$ with $0 \in W$ for all $x \in W$, where $W$ is an open neighborhood of $z$, such that the family is of class $C^{2,+}$ and such that for all $x \in W \cap \partial\Omega$, $p \in \partial B(x)$, $i \in I(x)$ and $n \in N_p(B(x))$,

$$\langle \gamma_i(x), n \rangle \begin{cases} \geq 0 & \text{if } \langle p, n_i(x) \rangle \geq -1, \\ \leq 0 & \text{if } \langle p, n_i(x) \rangle \leq -1. \end{cases} \tag{2.1}$$

**Theorem 2.1.** *Suppose that (F1)–(F3) and (B1)–(B4) hold. Let $u \in \mathrm{USC}([0,T) \times \overline{\Omega})$ and $v \in \mathrm{LSC}([0,T) \times \overline{\Omega})$ be, respectively, viscosity sub- and supersolutions of (1.1)–(1.2). If $u(0,x) \leq v(0,x)$ for $x \in \overline{\Omega}$, then $u \leq v$ on $(0,T) \times \overline{\Omega}$.*

Let $Q_0 = (0,T) \times \overline{\Omega}$. A function $u : Q_0 \to \mathbf{R}$ is called a viscosity subsolution of (1.1)-(1.2) if it satisfies the following properties:

$$\begin{array}{ll} \text{(i)} & u^* < +\infty \\ \text{(ii)} & \tau + F_*(x,r,p,X) \leq 0 \qquad \text{for } x \in \Omega \qquad (\tau,p,X) \in \rho_{Q_0}^{2,+} u^*(t,x) \\ & \tau + F_*(x,r,p,X) \wedge \min\{\langle \gamma_i(x), p \rangle : i \in I(x)\} \leq 0 \\ & \text{for} \quad x \in \partial\Omega \qquad (\tau,p,X) \in \rho_{Q_0}^{2,+} u^*(t,x) \end{array}$$

Similarly a function $u : Q_0 \to \mathbf{R}$ is called a viscosity subsolution of (1.1)-(1.2) if it satisfies the following properties:

$$\begin{array}{ll} \text{(i)} & u_* > -\infty \\ \text{(ii)} & \tau + F^*(x,r,p,X) \geq 0 \qquad \text{for} \quad x \in \Omega \qquad (\tau,p,X) \in \rho_{Q_0}^{2,-} u_*(t,x) \\ & \tau + F^*(x,r,p,X) \wedge \min\{\langle \gamma_i(x), p \rangle : i \in I(x)\} \geq 0 \\ & \text{for} \quad x \in \partial\Omega \qquad (\tau,p,X) \in \rho_{Q_0}^{2,-} u_*(t,x) \end{array}$$

Here $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$ and $\rho_{Q_0}^{2,+} u^*(t,x)$ (resp.$\rho_{Q_0}^{2,-} u_*(t,x)$) denotes the parabolic super 2-jet in $Q_0$. (see [CIL]) Any function $u$

**Remark 2.2.** Assumptions (F1) and (F3) imply that

$$-\infty < F_*(t,x,u,0,0) = F^*(t,x,u,0,0) < \infty$$

In what follows we use the notation: for any $p, q \in \mathbf{R}^n$,

$$\rho^*(p,q) = \begin{cases} \rho(p,q) & \text{if } p, q \neq 0, \\ 1 & \text{if either } p = 0 \text{ or } q = 0. \end{cases} \tag{2.2}$$

Note that the function $\rho^*$ is upper semi-continuous on $\mathbf{R}^n \times \mathbf{R}^n$.

**Remark 2.3.** We state typical examples of $F$ satisfying (F1)-(F3). Let $A : \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \to M^{n\times m}$, where $M^{n\times m}$ denotes the space of real $n \times m$ matrices, be a function which is homogeneous of degree zero, i.e.,

$$A(x, \lambda p) = A(x,p) \quad \text{for all } (x,p,\lambda) \in \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times (0,\infty)$$

and which satisfies

$$\|A(x,p) - A(y,q)\| \le C_1(|x-y| + |p-q|) \quad \text{for all } x, y \in \overline{\Omega} \text{ and } p, q \in S^{n-1},$$

where $C_1 > 0$ is a constant and $S^{n-1}$ denotes the unit sphere $\{\xi \in \mathbf{R}^n \ : \ |\xi| = 1\}$. It follows that for all $x, y \in \overline{\Omega}$ and $p, q \in \mathbf{R}^n \setminus \{0\}$,

$$\begin{aligned}\|A(x,p) - A(y,q)\| &\le C_1\Big(|x-y| + \Big|\frac{p}{|p|} - \frac{q}{|q|}\Big|\Big) \\ &\le C_1\Big(|x-y| + \frac{|p-q|}{|p| \vee |q|}\Big) \le C_1(|x-y| + 2\rho(p,q)).\end{aligned}$$

Let $b \in C(\overline{\Omega}, \mathbf{R}^n)$ satisfy

$$|b(x) - b(y)| \le C_2|x-y| \quad \text{for all } x, y \in \overline{\Omega}.$$

Furthermore let $c, f \in C(\overline{\Omega}, \mathbf{R})$ be given. Define the function $F \in C(\overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n)$ by

$$F(x,u,p,X) = -\operatorname{tr}[A(x,p)^*A(x,p)X] + \langle b(x), p\rangle + c(x)u + f(x).$$

As is observed in [CIL], if $X, Y \in \mathcal{S}^n$ and $\mu_1, \mu_2 \in [0,\infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$\begin{aligned}-\operatorname{tr}[A(x,p)^*A(x,p)X] - \operatorname{tr}[A(y,q)^*A(y,q)Y] &\le C_3\|A(x,p) - A(y,q)\|^2 \\ &\le 4C_3C_1(|x-y|^2 + \rho(p,q)^2).\end{aligned}$$

It is now easy to see that $F$ satisfies condition (F3). Also, it is immediate to see that condition (F2) is satisfied with $\gamma \le \min_{\overline{\Omega}} c$.

If $A(x,p) = I - |p|^{-2}(p \otimes p)$, $b = 0$, and $c = f = 0$, then it is the case of the mean curvature flow equation and the above conditions on $A$, $b$, $c$, and $f$ are valid.

*Proof of Theorem 2.1.* We may assume by replacing $T > 0$ by a smaller number if necessary that $u$ and $-v$ is bounded above on $[0,T) \times \overline{\Omega}$. For any constant $A \ge \max_{x\in\overline{\Omega}} u(0,x) \vee (-v(0,x))$, if we choose a constant $B > 0$ large enough, then the functions

$$f(t,x) = -A - Bt \quad \text{and} \quad g(t,x) = A + Bt$$

are, respectively, (viscosity) sub- and supersolutions of (1.1)–(1.2). For such functions $f$ and $g$, we set

$$\tilde{u}(t,x) = u(t,x) \vee f(t,x) \qquad \text{and} \qquad \tilde{v}(t,x) = v(t,x) \wedge g(t,x),$$

and observe that $\tilde{u}$ and $\tilde{v}$ are, respectively, sub- and supersolutions of (1.1)–(1.2) and that $\tilde{u}(0,x) \le \tilde{v}(0,x)$ for $x \in \overline{\Omega}$. If we can show that $\tilde{u} \le \tilde{v}$ on $[0,T) \times \overline{\Omega}$ for any such $f$ and $g$, then we see that $u \le v$ on $[0,T) \times \overline{\Omega}$. This observation reduces the proof to the case where $u$ and $v$ are bounded.

Also, the standard technique reduces the proof to the case when $\gamma = 0$ in (F2). Indeed, if $\gamma < 0$, then the functions $\hat{u}(t,x) = e^{\gamma t}u(t,x)$ and $\hat{v}(t,x) = e^{\gamma t}v(t,x)$ are, respectively, sub- and supersolutions of (1.1)–(1.2) with $F(t,x,r,p,X)$ replaced by the function

$$e^{\gamma t}(-\gamma r + F(t,x,e^{-\gamma t}r, e^{-\gamma t}p, e^{-\gamma t}X)).$$

Thus we may assume that $u$ and $v$ are bounded on $[0,T) \times \overline{\Omega}$ and that the function $r \mapsto F(t,x,r,p,X)$ is non-decreasing in $\mathbf{R}$ for each $(t,x,p,X) \in [0,T] \times \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n$.

By virtue of lemma 2.7, there are a function $w \in C^2(\overline{\Omega} \times \overline{\Omega})$ and a positive constant $C$ such that for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

$$\begin{aligned}
&(2.3) \quad && |x-y|^4 \le w(x,y) \le C|x-y|^4, \\
& && |D_x w(x,y)| \vee |D_y w(x,y)| \le C|x-y|^3, \\
&(2.4) \quad && \langle \gamma_i(x), D_x w(x,y) \rangle \ge 0 \quad \text{for all } x \in \partial\Omega, \quad i \in I(x) \\
& && \langle \gamma_i(y), -D_y w(x,y) \rangle \le 0 \quad \text{for all } y \in \partial\Omega, \quad i \in I(y) \\
&(2.5) \quad && |D_x w(x,y) + D_y w(x,y)| \le C|x-y|^4, \\
& && \rho^*(D_x w(x,y), -D_y w(x,y)) \le C|x-y|,
\end{aligned}$$

and for a. e. $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

$$(2.6) \qquad D^2 w(x,y) \le C\left\{|x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right\}.$$

We argue by contradiction. So we suppose that

$$(2.7) \qquad m_0 := \sup\{u(t,x) - v(t,x) \ : \ (t,x) \in [0,T) \times \overline{\Omega}\} > 0.$$

For $\alpha > 0$, $\varepsilon > 0$, $\delta > 0$ we define

$$\begin{aligned}
\Psi(t,x,y) &= \frac{\varepsilon}{T-t} + \alpha w(x,y) + \delta(\varphi(x) + \varphi(y)), \\
\Phi(t,x,y) &= u(t,x) - v(t,y) - \Psi(t,x,y)
\end{aligned}$$

for $(t,x,y) \in [0,T) \times \overline{\Omega} \times \overline{\Omega}$. Here the function $\varphi \in C^2(\overline{\Omega})$ satisfies

$$\varphi > 0 \quad \text{on} \quad \overline{\Omega} \quad \text{and} \quad \langle D\varphi(x), \gamma_i(x) \rangle \ge 1 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad i \in I(x)$$

Actually we can construct the above function $\varphi$. (see [D-I]) From (2.7) we infer that for sufficiently small $\varepsilon > 0$ and $\delta > 0$, the function $\Phi$ attains a maximum greater that $m_0/2$. Fix such $\delta$ and $\varepsilon$, and choose a maximum point $(\hat{t}, \hat{x}, \hat{y})$ of $\Phi$. Note that $\Phi$ and $(\hat{t}, \hat{x}, \hat{y})$ depend on $\alpha$, $\varepsilon$, $\delta$.

It is now well-known (see, e.g., [CIL]) that

$$\lim_{\varepsilon \searrow 0} \lim_{\alpha \to \infty} \lim_{\delta \searrow 0} \Phi(\hat{t}, \hat{x}, \hat{y}) = m_0, \tag{2.8}$$

$$\lim_{\alpha \to \infty} \sup\{\alpha w(\hat{x}, \hat{y}) \ : \ 0 < \delta < 1,\ 0 < \varepsilon < 1\} = 0. \tag{2.9}$$

We will pass to the limit as $\delta \searrow 0$, $\alpha \to \infty$ in this order. Thus, in view of (2.8), we may assume that $\hat{t} > 0$ and that $u(\hat{t}, \hat{x}) > v(\hat{t}, \hat{y})$.

Note that

$$\langle \gamma_i(x), D_x w(x, y) \rangle \geq \delta \qquad \text{for all } x \in \partial\Omega, \qquad i \in I(x) \tag{2.10}$$

$$\langle \gamma_i(y), -D_y w(x, y) \rangle \leq -\delta \qquad \text{for all } y \in \partial\Omega, \qquad i \in I(x) \tag{2.11}$$

We apply the maximum principle for semi-continuous functions (see [CIL]), to find matrices $X, Y \in \mathcal{S}^n$ such that

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3C\alpha|\hat{x} - \hat{y}|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \left(\alpha|\hat{x} - \hat{y}|^4 + \delta\right) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

where $C$ is the constant from (2.6) and $C_1 = C \vee \sup_{x \in \Omega} \|D^2\psi(x)\|$, and such that

$$\frac{\varepsilon}{(T - \hat{t})^2} + F_*(\hat{t}, \hat{x}, \hat{u}, \hat{p}, X) - F^*(\hat{t}, \hat{y}, \hat{v}, \hat{q}, -Y) \leq 0,$$

where

$$\hat{u} = u(\hat{t}, \hat{x}), \qquad \hat{v} = v(\hat{t}, \hat{y}),$$
$$\hat{p} = \alpha D_x w(\hat{t}, \hat{x}) + \delta D\psi(\hat{x}), \qquad \hat{q} = -\alpha D_y w(\hat{t}, \hat{y}) - \delta D\psi(\hat{y}).$$

Using (2.2) and writing $\omega = \omega_R$, where $R = \sup_{[0,T) \times \overline{\Omega}}(|u| + |v|)$, we get

$$0 \geq \frac{\varepsilon}{(T - \hat{t})^2} + F_*(\hat{t}, \hat{x}, \hat{u}, \hat{p}, X) - F^*(\hat{t}, \hat{y}, \hat{u}, \hat{q}, -Y) \geq \frac{\varepsilon}{T^2} - \omega(r_1 + r_2 + r_3),$$

where

$$r_1 = 3C\alpha|\hat{x} - \hat{y}|^2(|\hat{x} - \hat{y}|^2 + \rho^*(\hat{p}, \hat{q})^2),$$
$$r_2 = C_1(\alpha|\hat{x} - \hat{y}|^4 + \delta),$$
$$r_3 = |\hat{p} - \hat{q}| + |\hat{x} - \hat{y}|(|\hat{p}| \vee |\hat{q}| + 1).$$

Sending $\delta \searrow 0$ along a sequence, we may assume that $\hat{t} \to \bar{t}$, $\hat{x} \to \bar{x}$, $\hat{y} \to \bar{y}$, $\hat{p} \to \bar{p}$, $\hat{q} \to \bar{q}$, and $r_i \to s_i$ for $i = 1, 2, 3$. We then get

$$0 \geq \frac{\varepsilon}{T^2} - \omega(s_1 + s_2 + s_3), \tag{2.12}$$

$$\bar{p} = \alpha D_x w(\bar{x}, \bar{y}), \qquad \bar{q} = -\alpha D_y w(\bar{x}, \bar{y}),$$
$$s_1 \leq 3C\alpha(1 + C)|\bar{x} - \bar{y}|^4 \leq 3C(1 + C)\alpha w(\hat{x}, \hat{y}),$$
$$s_2 \leq C_1 \alpha w(\bar{x}, \bar{y}),$$

$$\begin{aligned} s_3 &\leq |\bar{p} - \bar{q}| + |\bar{x} - \bar{y}|(|\bar{p}| \vee |\bar{q}| + 1) \\ &\leq +C\alpha|\hat{x} - \hat{y}|^4 + |\hat{x} - \hat{y}|(C\alpha|\hat{x} - \hat{y}|^3 + 1) \\ &\leq 2C\alpha w(\bar{x}, \bar{y}) + |\bar{x} - \bar{y}|. \end{aligned}$$

Sending $\alpha \to \infty$ in (2.12), we get a contradiction, which proves that $\sup_{[0,T)\times\overline{\Omega}}(u-v) \leq 0$. □

We next show the existence of a viscosity solution of the initial-boundary value problem

$$u_t + F(t, x, u, Du, D^2u) = 0 \qquad \text{in } Q, \tag{2.13}$$

$$\frac{\partial u}{\partial \gamma} = 0 \qquad \text{in } S = (0, T) \times \partial\Omega, \tag{2.14}$$

$$u(0, x) = g(x) \qquad \text{for } x \in \overline{\Omega}, \tag{2.15}$$

where $g \in C(\overline{\Omega})$ is a given function.

**Theorem 2.6.** *Assume that (F1)–(F3) and (B1)–(B4) hold. Then for each $g \in C(\overline{\Omega})$ there is a (unique) viscosity solution $u \in C([0, T) \times \overline{\Omega})$ of (2.13)–(2.14) satisfying (2.15).*

*Sketch of proof.* We use the Perron method (see [CIL]) to show the existence of a continuous viscosity solution. If we introduce the new unknown $\hat{u}(t, x) = e^{\gamma t}u(t, x)$, where $\gamma \in \mathbf{R}$ is the constant form (F2), then the problem (2.13)–(2.14) is reduced to the case when $\gamma = 0$. Hence, we may assume that $\gamma = 0$. According to lemma 2.7, there is a function $w \in C(\overline{\Omega} \times \overline{\Omega})$ having the following properties:

$$\begin{aligned} &|x - y|^4 \leq w(x, y) \leq C|x - y|^4, \\ &|D_x w(x, y)| \vee |D_y w(x, y)| \leq C|x - y|^3, \end{aligned} \tag{2.16}$$

$$\langle \gamma_i(x), D_x w(x, y) \rangle \leq 0 \qquad \text{for all} \quad x \in \partial\Omega, \qquad i \in I(x) \tag{2.17}$$

$$\langle \gamma_i(y), -D_y w(x,y)\rangle \geq 0 \quad \text{for all} \quad y \in \partial\Omega, \quad i \in I(y)$$

We can construct sub- and supersolutions of (2.13)-(2.14) satisfying (2.15) similarly as [Theorem3.1, I-S].

□

Let each $\Omega_i$ be a bounded domain with $C^3$ boundary $\partial\Omega$ in $R^n$. Then we see there is a positive contant $C_i$ such that

$$\langle \gamma_i(x), x-y\rangle + C_i|x-y|^2 \geq 0 \quad \text{for all} \quad x \in \partial\Omega_i, \quad y \in \overline{\Omega} \tag{2.18}$$

(see [I-L]).

**Lemma 2.7.** *Assume that (B1)–(B4) hold. There are a function $w \in C^2(\overline{\Omega}\times\overline{\Omega})$ and a positive constant $C$ such that for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$,*

$$\begin{aligned} &(2.19) \quad && |x-y|^4 \leq w(x,y) \leq C|x-y|^4, \\ & && |D_x w(x,y)| \vee |D_y w(x,y)| \leq C|x-y|^3, \\ &(2.20) && \langle \gamma_i(x), D_x w(x,y)\rangle \geq 0 \quad \textit{for all } x \in \partial\Omega, \quad i \in I(x) \\ & && \langle \gamma_i(y), -D_y w(x,y)\rangle \leq 0 \quad \textit{for all } y \in \partial\Omega, \quad i \in I(y) \\ &(2.21) && |D_x w(x,y) + D_y w(x,y)| \leq C|x-y|^4, \\ & && \rho^*(D_x w(x,y), -D_y w(x,y)) \leq C|x-y|, \end{aligned}$$

*and for a. e. $(x,y) \in \overline{\Omega}\times\overline{\Omega}$,*

$$D^2 w(x,y) \leq C\left\{|x-y|^2\begin{pmatrix} I & -I \\ -I & I\end{pmatrix} + |x-y|^4\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right\}. \tag{2.22}$$

*Sketch of proof.* We can construct the function $\varphi \in C^2(\overline{\Omega})$ such that

$\varphi > 0$ on $\overline{\Omega}$ and $\langle D\varphi(x), \gamma_i(x)\rangle \geq \max(1, 2C_i \min_{\overline{\Omega}}\varphi)$ for $x \in \partial\Omega$ and $i \in I(x)$

(see [D-S]). We set $w(x,y) = |x-y|^4(\varphi(x)+\varphi(y))$. Then using (2.18) we can check this function $w$ satisfies (2.19)-(2.22) similarly as [G-S].

□

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