数理解析研究所講究録 1323 巻 2003 年 183-194

THREE SINGULAR VARIATIONAL PROBLEMS

BY

LAWRENCE C. EVANS*
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720

"Some of the means I use are trivial–and some are quadrivial."

J. Joyce

ABSTRACT. We discuss here a variational viewpoint common to three problems in nonlinear PDE: the construction of optimal Lipschitz extensions, the Monge–Kantorovich problem, and weak KAM theory for Hamiltonian dynamics. We establish also some interesting analytic estimates.

1. Overview.

This expository paper discusses some viewpoints and estimates common to three related singular variational problems, which turn out in asymptotic limits to have quite different interpretations. These are: (I) the construction of optimal Lipschitz extensions of given boundary data, (II) the Monge–Kantorovich optimal mass transfer problem, and (III) a form of weak KAM theory for Hamiltonian dynamics.

The above quotation from Joyce (cited in the book [J]) depends upon the Latin derivation of the word "trivial", from "tri"(= three) and "via"(= road or way). It is interesting that there is a "trivial" (three–way) variational principle behind these apparently rather different problems. Perhaps a fourth application remains to be found, so that our method would then be "quadrivial".

I would like to reemphasize here that this is an expository paper. I have not yet written up fully detailed proofs of some of the assertions in §1.3, which should therefore be regarded as an informal research announcement.

I presented some of these results at an meeting at RIMS in Kyoto, during September, 2003. I thank the organizers, and especially Professor Hitoshi Ishii, for their hospitality.

* Supported in part by NSF Grant DMS-0070480 and by the Miller Institute for Basic Research in Science, UC Berkeley

1.1 Three variational problems.

First of all, fix a parameter $k > 1$, which we will later send to infinity.

Problem I: Optimal Lipschitz Extensions. Assume for our first problem that we are given a bounded, smooth domain $U \subset \mathbb{R}^n$ and a Lipschitz continuous function $g : \mathbb{R}^n \to \mathbb{R}$. Then define

$$I_1[w] := \int_U e^{\frac{k}{2}|Dw|^2}\,dx \tag{1.1}$$

for functions w in the admissible class

$$\mathcal{A}_1 := \{w : \bar{U} \to \mathbb{R} \mid w \text{ is Lipschitz continuous, } w = g \text{ on } \partial U\}. \tag{1.2}$$

We minimize $I_1[\cdot]$ over $\mathcal{A}_1$.

Problem II: Optimal Mass Transfer. Let $U = B(0, R)$ denote the ball in $\mathbb{R}^n$ with center 0 and (large) radius R.

Assume $f : \mathbb{R}^n \to \mathbb{R}$ is summable and has compact support, lying within $B(0, R)$. We write $f = f^+ - f^-$ and suppose the *mass balance* condition that

$$\int_U f^+\,dx = \int_U f^-\,dy = 1.$$

Then define

$$I_2[w] := \int_U e^{\frac{k}{2}(|Dw|^2-1)} - wf\,dx \tag{1.3}$$

for w belonging to

$$\mathcal{A}_2 := \{w : U \to \mathbb{R} \mid w \text{ is Lipschitz continuous}, w = 0 \text{ on } \partial U\}. \tag{1.4}$$

We minimize $I_2[\cdot]$ over $\mathcal{A}_2$.

Problem III: Weak KAM Theory. For our last example, let $\mathbb{T}^n$ denote the flat unit torus in $\mathbb{R}^n$, that is, the unit cube with opposite faces identified. Suppose $P \in \mathbb{R}^n$ is fixed and $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth, $\mathbb{T}^n$-periodic potential.

Define

$$I_3[w] := \int_{\mathbb{T}^n} e^{k(\frac{|P+Dw|^2}{2}+V)}dx \tag{1.5}$$

on the admissible set

$$\mathcal{A}_3 := \{w : \mathbb{R}^n \to \mathbb{R} \mid w \text{ is Lipschitz continuous and } \mathbb{T}^n\text{-periodic}\}. \tag{1.6}$$

Once again, we minimize $I_3[\cdot]$ over $\mathcal{A}_3$.

1.2 Euler–Lagrange equations.

For Problems I and II, let u_k denote a minimizer and for Problem III, write $u_k := P \cdot x + v_k$; where v_k is a minimizer. The Euler–Lagrange equations then take these forms:

Problem I:

$$-\operatorname{div}(\sigma_k Du_k) = 0 \quad \text{in } U \tag{1.7}$$

where

$$\sigma_k := e^{k\left(\frac{|Du_k|^2}{2} - \frac{L_k^2}{2}\right)} \tag{1.8}$$

for

$$L_k := \left(\frac{2}{k}\log \int_U e^{\frac{k}{2}|Du_k|^2}dx\right)^{1/2}. \tag{1.9}$$

Problem II:

$$-\operatorname{div}(\sigma_k Du_k) = f \quad \text{in } U \tag{1.10}$$

$$\sigma_k := e^{\frac{k}{2}(|Du_k|^2-1)} \tag{1.11}$$

Problem III:

$$-\operatorname{div}(\sigma_k Du_k) = 0 \quad \text{in } \mathbb{T}^n \tag{1.12}$$

for

$$\sigma_k := e^{k\left(\frac{|Du_k|^2}{2} + V - \bar{H}_k(P)\right)} \tag{1.13}$$

and

$$\bar{H}_k(P) := \frac{1}{k}\log \int_{\mathbb{T}^n} e^{k\left(\frac{|Du_k|^2}{2}+V\right)}dx. \tag{1.14}$$

Remark. We call (1.7), (1.10), (1.12) *continuity* (or *transport*) *equations*. Notice that for Problems I and III we have normalized so that $\sigma_k \geq 0$ satisfies

$$\int_U \sigma_k \, dx = 1, \quad \int_{\mathbb{T}^n} \sigma_k \, dx = 1 \tag{1.15}$$

respectively, but have not done so for Problem II.

Our goal is to understand for each of our problems what happens in the limit as the parameter k goes to infinity.

1.3 Limits as k$\rightarrow\infty$.

We describe next the limiting behaviors of u_k and σ_k:

Problem I: Optimal Lipschitz Extensions.

For our first problem, we assert this asymptotic behavior:
(i) *As $k \to \infty$, $u_k \to u$ uniformly on $\bar{U}$ and u is the unique viscosity solution of*

$$\begin{cases} -u_{x_i}u_{x_j}u_{x_ix_j} = 0 & \text{in } U \\ u = g & \text{in } \partial U. \end{cases} \tag{1.16}$$

(ii) *Furthermore, $L_k \to L$ for*

$$L := \sup\left\{\frac{|g(x)-g(y)|}{|x-y|} \mid x, y \in \partial U,\ x \neq y\right\}.$$

(iii) *Also, $\sigma_k \rightharpoonup \sigma$ weakly as measures, where σ is a probability measure on $\bar{U}$ such that*

$$|Du| = L \quad \sigma\text{-a.e. in } U.$$

(iv) *We have*

$$-\operatorname{div}(\sigma Du) = 0 \quad \text{in } U. \tag{1.17}$$

The idea here is to construct an optimal Lipschitz extension into the domain U of the given boundary values g, following *Aronsson's variational principle* that for each subdomain $V \subseteq U$ we should have

$$||Du||_{L^\infty(V)} \leq ||Dv||_{L^\infty(V)}$$

for each Lipschitz function v satisfying $u \equiv v$ on ∂V. The PDE in (1.16) is in effect the Euler–Lagrange equation for this sup–norm minimization problem. We sometimes write

$$u_{x_i}u_{x_j}u_{x_ix_j} =: \Delta_\infty u,$$

the so–called "infinity Laplacian". See for instance Aronsson [A], Barron [B], Barron–Jensen–Wang [B-J-W] for more detailed explanations.

Problem II: Optimal Mass Transfer.

We examine next Problem II as $k \to \infty$:
(i) *As $k \to \infty$, $u_k \to u$ locally uniformly on U, where*

$$|Du| \leq 1 \ a.e. \tag{1.18}$$

(ii) *Furthermore* $\sigma_k \rightharpoonup \sigma$ *weakly as measures and*

$$-\operatorname{div}(\sigma Du) = f \quad \text{in } U. \tag{1.19}$$

(iii) $\sigma(U)$ *is the Monge–Kantorovich cost of optimally rearranging the probability measure* $d\mu^+ = f^+dx$ *to* $d\mu^- = f^-dy$.

(iv) *We also have*

$$-u_{x_i}u_{x_j}u_{x_ix_j} = 0 \quad \text{in } U - \operatorname{spt}(f).$$

The basic Monge–Kantorovich problem asks us to find a mapping **s** to minimize the cost functional

$$C[\mathbf{r}] := \int_{\mathbb{R}^n} |x - \mathbf{r}(x)|\, d\mu^+(x)$$

among one–to–one mappings $\mathbf{r} : \mathbb{R}^n \to \mathbb{R}^n$ that push forward μ^+ into μ^-. As explained in Ambrosio [Am], Caffarelli–Feldman–McCann [C-F-M], [E1], [E-G1], etc., the potential function u can be employed to design an optimal mass allocation plan **s**. The measure σ is called the *transport measure* (or the *transport density*, when it has a density with respect to Lebesgue measure).

Problem III: Weak KAM Theory.

Finally, we address the asymptotic limit of Problem III:

(i) *As* $k \to \infty$, $u_k \to u$ *uniformly on* $\mathbb{T}^n$ *and* u *is a viscosity solution of*

$$-u_{x_i}u_{x_j}u_{x_ix_j} = V_{x_j}u_{x_j} \quad \text{in } \mathbb{T}^n. \tag{1.20}$$

(ii) *Furthermore* $\bar{H}_k(P) \to \bar{H}(P)$, *where* $\bar{H}$ *is the effective Hamiltonian in the sense of Lions–Papanicolaou–Varadhan* [L-P-V].

(iii) *We have* $\sigma_k \rightharpoonup \sigma$ *weakly as measures and*

$$-\operatorname{div}(\sigma Du) = 0 \quad \text{in } \mathbb{T}^n. \tag{1.21}$$

(iv) *In addition,*

$$\frac{|Du|^2}{2} + V = \bar{H}(P) \quad \sigma\text{-a.e.} \tag{1.22}$$

A full proof can be found in [E2]. As explained in Evans–Gomes [E-G2], we can regard (1.22) as the *generalized eikonal equation* and (1.21) as the *continuity equation* corresponding to the dynamics

$$\ddot{\mathbf{x}}(t) = -DV(\mathbf{x}(t)).$$

The support of the measure σ is the projection of the *Mather set* onto $\mathbb{T}^n$. See also Fathi [F], Mather [Mt], Mather–Forni [M-F] for other viewpoints.

2. L^2 and L^p bounds.

An advantage of the common viewpoint set forth above is that we can at least hope to find analytic methods applicable to several of the problems at once.

In this and the next two sections we illustrate some common PDE methods, which apply variously to Problems I–III, for deriving useful estimates.

2.1 An L^2-estimate (Problem III).

Consider from Problem III the Euler–Lagrange equation

$$-\operatorname{div}(\sigma_k Du_k) = 0 \quad \text{in } \mathbb{T}^n \tag{2.1}$$

for

$$\sigma_k = e^{k(\frac{|Du_k|^2}{2}+V-\bar{H})} \tag{2.2}$$

and $u_k = P \cdot x + v_k$, v_k periodic. Write

$$h_k := \frac{|Du_k|^2}{2} + V - \bar{H}. \tag{2.3}$$

Lemma 2.1. *We have the identity*

$$\int_{\mathbb{T}^n} |D^2u_k|^2 + k|Dh_k|^2 d\sigma_k = -\int_{\mathbb{T}^n} \Delta V \, d\sigma_k, \tag{2.4}$$

and consequently

$$\int_{\mathbb{T}^n} |D^2u_k|^2 + k|Dh_k|^2 d\sigma_k \leq C, \tag{2.5}$$

for a constant C independent of k.

Here we write "$d\sigma_k$" for "$\sigma_k dx$".

Proof. To simplify notation, we henceforth drop the subscript k. Owing to (2.1) we have

$$\begin{aligned} 0 = \int_{\mathbb{T}^n} (\sigma u_{x_i})_{x_i} u_{x_j x_j} \, dx &= \int_{\mathbb{T}^n} (\sigma u_{x_i})_{x_j} u_{x_i x_j} \, dx \\ &= \int_{\mathbb{T}^n} \sigma u_{x_i x_j} u_{x_i x_j} + \sigma_{x_j} u_{x_i} u_{x_i x_j} \, dx. \end{aligned}$$

Now $\sigma_{x_j} = k(u_{x_i} u_{x_i x_j} + V_{x_j})\sigma = k h_{x_j} \sigma$. Therefore

$$0 = \int_{\mathbb{T}^n} \sigma |D^2 u|^2 + k\sigma Dh \cdot (Dh - DV) \, dx;$$

and this gives

$$\int_{\mathbb{T}^n} |D^2u|^2 + k|Dh|^2 d\sigma = \int_{\mathbb{T}^n} D\sigma \cdot DV \, dx = -\int_{\mathbb{T}^n} \Delta V \, d\sigma.$$

2.2 An L^p estimate for the transport density (Problem II).

We turn now to Problem II, for which

$$-\operatorname{div}(\sigma_k Du_k) = f \quad \text{in } U \tag{2.6}$$

and

$$\sigma_k = e^{\frac{k}{2}(|Du_k|^2-1)}. \tag{2.7}$$

We assume that u_k has compact support, and $|u_k| \le M$ for some constant M. The following estimate is from the forthcoming paper [D-E-P].

Lemma 2.2. *For each $2 \le p < \infty$, there exists a constant C, depending on p but not k, such that*

$$\int_U \sigma_k^p \, dx \le C\Big(\int_U |f|^p \, dx + 1\Big). \tag{2.8}$$

Proof. 1. We again omit the subscripts k. Let $q = p - 1 \ge 1$. Multiply (2.6) by $\sigma^q u$ to discover

$$\int_U \sigma u_{x_i} (\sigma^q u)_{x_i} \, dx = \int_U f \sigma^q u \, dx.$$

Hence

$$\int_U \sigma^{q+1} |Du|^2 + q\sigma^q Du \cdot D\sigma u \, dx = \int_U f \sigma^q u \, dx.$$

Since $|Du|^2 \ge 1$ if $\sigma \ge 1$, owing to (2.7), we can deduce that

$$\int_U \sigma^{q+1} dx \le C \int_U |f|^{q+1} dx + C \int_U \sigma^q |Du \cdot D\sigma| \, dx + C. \tag{2.9}$$

2. We must control the second term on the right-hand side of (2.9). To do so, we next multiply our PDE (2.6) by $-\operatorname{div}(\sigma^q Du)$ and integrate by parts:

$$\int_U (\sigma u_{x_i})_{x_j} (\sigma^q u_{x_j})_{x_i} \, dx = -\int_U f (\sigma^q u_{x_j})_{x_j} \, dx. \tag{2.10}$$

(We are ignoring here a boundary term, which turns out to have a good sign: see [D-E-P] for details.) The term on the left equals

$$\begin{aligned}
&\int_U (\sigma u_{x_i x_j} + \sigma_{x_j} u_{x_i})(\sigma^q u_{x_i x_j} + q\sigma^{q-1} \sigma_{x_i} u_{x_j}) \, dx \\
&\quad = \int_U \sigma^{q+1} |D^2 u|^2 + q\sigma^{q-1} |Du \cdot D\sigma|^2 + (q+1)\sigma^q \sigma_{x_i} u_{x_j} u_{x_i x_j} \, dx \\
&\quad = \int_U \sigma^{q+1} |D^2 u|^2 + q\sigma^{q-1} |Du \cdot D\sigma|^2 + \frac{(q+1)}{k} \sigma^{q-1} |D\sigma|^2 \, dx,
\end{aligned} \tag{2.11}$$

since (2.7) implies $\sigma_{x_i} = k u_{x_j x_i} u_{x_j} \sigma$. The term on the right-hand side of (2.10) is

$$\begin{aligned}
&-\int_U f(\sigma^q u_{x_j x_j} + q\sigma^{q-1} \sigma_{x_j} u_{x_j}) \, dx \\
&\qquad \le \frac{1}{2} \int_U \sigma^{q+1} |D^2 u|^2 + q\sigma^{q-1} |Du \cdot D\sigma|^2 \, dx + C \int_U f^2 \sigma^{q-1} \, dx.
\end{aligned} \tag{2.12}$$

We combine (2.10)–(2.12) and perform elementary estimates to arrive at (2.8). □

3. Bounds and formulas involving Du.

3.1 Detailed mass balance (Problem II).

Interesting identities sometime result if we multiply the various transport equations by $\Phi(Du)$, where

$$\Phi : \mathbb{R}^n \to \mathbb{R}$$

is an arbitary smooth function. For example, turn again to Problem II:

$$-\operatorname{div}(\sigma_k Du_k) = f \quad \text{in } U, \tag{3.1}$$

where

$$\sigma_k = e^{\frac{k}{2}(|Du_k|^2-1)}. \tag{3.2}$$

Lemma 3.1. *We have the identity*

$$\frac{1}{k}\int_U D\sigma_k \cdot D\Phi(Du_k)\,dx - \int_{\partial U} \sigma_k \frac{\partial u_k}{\partial \nu}\Phi(Du_k)\,d\mathcal{H}^{n-1} = \int_U f\Phi(Du_k)\,dx. \tag{3.3}$$

Proof. We calculate

$$\int_U \sigma u_{x_i}\Phi(Du)_{x_i}\,dx - \int_{\partial U} \sigma \frac{\partial u}{\partial \nu}\Phi(Du)\,d\mathcal{H}^{n-1} = \int_U f\Phi(Du)\,dx,$$

and the first term on the left is

$$\int_{\mathbb{R}^n} \sigma u_{x_i}\Phi_{p_j}(Du)u_{x_j x_i}\,dx = \int_{\mathbb{R}^n} \frac{\sigma_{x_j}}{k}\Phi_{p_j}(Du)\,dx.$$

□

Remark. This formula suggests that in the limit $k \to \infty$, we should have

$$\int_{\mathbb{R}^n} \Phi(Du)f\;dx = 0,$$

if σ_k goes to zero on ∂U. Consequently,

$$\int_{\mathbb{R}^n} \Phi(Du)f^+dx = \int_{\mathbb{R}^n} \Phi(Du)f^-dy \tag{3.4}$$

for all smooth $\Phi : \mathbb{R}^n \to \mathbb{R}$.

This is a form of *detailed mass balance* for the Monge–Kantorovich problem: see [E-G1]. The basic insight is that the mass of the measure $d\mu^+ = f^+dx$ is optimally rearranged into $d\mu^- = f^-dy$ by "moving each mass point in the direction $-Du$". If we formally take $\Phi = \chi_B$, where B is some set of directions in the unit sphere, the identity (3.4) reads

$$\int_A f^+dx = \int_A f^-dy$$

for $A := \{x \in \mathbb{R}^n \mid Du(x) \in B\}$, and this is consistent with the foregoing interpretation.

3.2 Gradient bounds (Problem III).

For Problem III we can as in [E2] bound the term in the exponential:

Lemma 3.2. *We have the estimate*

$$\frac{|Du_k|^2}{2} + V \leq \bar{H}_k(P) + \frac{C \log k}{k}. \tag{3.5}$$

Proof. Somewhat as in our proof of Lemma 2.2, we multiply the PDE by $\operatorname{div}(\sigma^q Du)$ and integrate by parts:

$$\int_{\mathbb{T}^n} (\sigma u_{x_i})_{x_j} (\sigma^q u_{x_j})_{x_i} \, dx = 0.$$

As before the term on the left is

$$\int_{\mathbb{T}^n} \sigma^{q+1} |D^2 u|^2 + q\sigma^q |Du \cdot D\sigma| + (q+1)\sigma^q \sigma_{x_i} u_{x_j} u_{x_i x_j} \, dx.$$

Now $\sigma = e^{k(\frac{|Du|^2}{2} + V - \bar{H})}$ and so $\sigma_{x_i} = k(u_{x_i} u_{x_i x_j} + V_{x_i})\sigma$. Hence

$$\frac{1}{k} \int_{\mathbb{T}^n} \sigma^{q-1} |D\sigma|^2 dx \leq \int_{\mathbb{T}^n} \sigma^q |D\sigma \cdot DV| \, dx$$

and therefore

$$\int_{\mathbb{T}^n} \sigma^{q-1} |D\sigma|^2 dx \leq Ck^2 \int_{\mathbb{T}^n} \sigma^{q+1} dx. \tag{3.6}$$

Using Sobolev's inequality, we deduce

$$\left(\int_{\mathbb{T}^n} \sigma^{(q+1)(1+\theta)} dx\right)^{\frac{1}{1+\theta}} \leq C \int_{\mathbb{T}^n} \left|D\sigma^{\frac{q+1}{2}}\right|^2 + \sigma^{q+1} \, dx \leq C(q+1)^2 k^2 \int_{\mathbb{T}^n} \sigma^{q+1} dx.$$

A standard Moser iteration implies that

$$\|\sigma\|_{L^\infty} \leq Ck^\alpha$$

for some power $\alpha > 0$. But then

$$k\left(\frac{|Du|^2}{2} + V - \bar{H}\right) = \log \sigma \leq C + \alpha \log k$$

and estimate (3.5) follows. □

This estimate, combined with a minimax formula explained in [E2], implies

$$\bar{H}_k(P) \leq \bar{H}(P) \leq \bar{H}_k(P) + \frac{C \log k}{k}.$$

Therefore the normalization factor (1.14) provides an approximation to the effective Hamil-

4. Monotonicity formulas.

4.1 Monotonicity (Problem I).

We write Problem I in the form

$$-\operatorname{div}(\sigma_k Du_k) = 0 \quad \text{in } U, \tag{4.1}$$

for

$$\sigma_k = e^{\frac{k}{2}|Du_k|^2}. \tag{4.2}$$

Lemma 4.1. *For each ball $B(y,r) \subset U$ we have the identity*

$$\int_{B(y,r)} (|Du_k|^2 - \frac{n}{k})\sigma_k \, dx = r \int_{\partial B(y,r)} ((\frac{\partial u_k}{\partial \nu})^2 - \frac{1}{k})\sigma_k \, d\mathcal{H}^{n-1}. \tag{4.3}$$

Proof. The Euler-Lagrange equation (1.7) says $-(e^{\frac{k}{2}|Du|^2} u_{x_i})_{x_i} = 0$, and therefore

$$(e^{\frac{k}{2}|Du|^2}(\delta_{ij} - k u_{x_i} u_{x_j}))_{x_i} = 0 \tag{4.4}$$

for $j = 1, \ldots, n$. Assume $y = 0$ and the ball $B(0,r)$ lies within U. Multiply (4.4) by $x_j \phi(|x|)$ where $\phi \equiv 1$ on $[0, r-\varepsilon]$, $\phi \equiv 0$ on $[r, \infty)$ and ϕ is linear on $[r-\varepsilon, r]$. We find

$$0 = \int_{B(0,r)} e^{\frac{k}{2}|Du|^2}(\delta_{ij} - k u_{x_i} u_{x_j})(\delta_{ij}\phi + \frac{x_j x_i}{|x|}\phi')dx.$$

Hence

$$\int_{B(0,r)} \sigma(n - k|Du|^2)\phi \, dx = \frac{1}{\varepsilon} \int_{B(0,r)\setminus B(0,r-\varepsilon)} \sigma(|x| - k\frac{|Du \cdot x|^2}{|x|})dx. \tag{4.5}$$

Let $\varepsilon \to 0$ to derive (4.3). □

We can formally interpret this by first renormalizing so that $\sigma_k(B(y,r)) \equiv 1$ and letting $\sigma_k \rightharpoonup \sigma$. Then (4.3) should imply

$$\int_{B(y,r)} |Du|^2 \, d\sigma = r \int_{\partial B(y,r)} (\frac{\partial u}{\partial \nu})^2 \, d\tau,$$

where τ denotes the restriction of σ to the sphere $\partial B(y,r)$. If for instance $\sigma(B^0(y,r)) = 0$, then our passing to limits in (4.3) as $k \to \infty$, before sending $\epsilon \to 0$, allows us to guess that $\frac{\partial u}{\partial \nu} = 0$ almost everywhere on $\partial B(y,r)$ with respect to the measure τ.

4.2 Monotonicity (Problem II).

The Euler–Lagrange PDE for Problem II reads

$$(4.6) \qquad -\operatorname{div}(\sigma_k Du_k) = f \quad \text{in } U,$$

for

$$(4.7) \qquad \sigma_k = e^{\frac{k}{2}(|Du_k|^2-1)}.$$

Lemma 4.2. *For each ball $B(y,r) \subset U$,*

$$(4.8) \quad \int_{B(y,r)} (|Du_k|^2 - \frac{n}{k})\sigma_k\,dx = r\int_{\partial B(y,r)} ((\frac{\partial u_k}{\partial \nu})^2 - \frac{1}{k})\sigma_k\,d\mathcal{H}^{n-1} + \int_{B(x,r)} f\,x\cdot Du_k\,dx.$$

Proof. We have $-(e^{\frac{k}{2}(|Du|^2-1)}u_{x_i})_{x_i} = f$ and so

$$(e^{\frac{k}{2}(|Du|^2-1)}(\delta_{ij} - ku_{x_i}u_{x_j}))_{x_i} = kfu_{x_j}$$

for $j = 1,\dots,n$. Again suppose $y = 0$ and take ϕ as above. Then

$$k\int_{B(0,r)} f\,x\cdot Du\phi\,dx = \int_{B(0,r)} \sigma(k|Du|^2-n)\phi\,dx + \frac{1}{\varepsilon}\int_{B(0,r)-B(0,r-\varepsilon)} \sigma(k\frac{|Du\cdot x|^2}{|x|} - |x|)\,dx.$$

Let $\varepsilon \to 0$ and divide by k. □

At least formally, this identity in the limit $k \to \infty$ provides some analytic control over the transport density, although we do not here attempt to provide any details.

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