Canonical subgroups and $p$-adic vanishing cycles on abelian varieties

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This is a report on a joint work with A. Mokrane [1]. Our motivation is to develop a theory of Siegel $p$-adic modular forms (and for other Shimura varieties) on the model of the elliptic theory developed by Dwork [8], Katz [9], Coleman [5, 6], .... The first step, achieved in [1], provides analogues of the compact Atkin operator $U$.

Let $k$ be an algebraically closed field of characteristic $p > 0$, $W = W(k)$ be the ring of Witt vectors with coefficients in $k$ and $\sigma$ be the Frobenius endomorphism of $k$ or $W$. Let $A$ be an ordinary abelian variety over $k$ of dimension $g$ and let $\mathfrak{M}$ be the formal moduli space of deformations of $A$ over artinien $W$-algebras with residue field $k$. By Serre-Tate theorem, there exists a canonical isomorphism of formal $W$-schemes

$\mathfrak{M} \rightarrow \text{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_p\hat{A}(k), \hat{G}_m),$

where $\hat{A}$ is the dual abelian variety of $A$ and $T_p$ is the Tate module. Dwork developed another approach to this structure theorem. He proved that a toric formal Lie group structure on $\mathfrak{M}$ is imposed by a $W$-morphism $\Phi : \mathfrak{M} \rightarrow \mathfrak{M}^{(\sigma)}$ lifting the Frobenius. In particular, the group structure of Serre-Tate is completely determined by the canonical lifting of the Frobenius $\Phi_{\text{can}} : \mathfrak{M} \rightarrow \mathfrak{M}^{(\sigma)}$ defined as follows. Let $A/\mathfrak{M}$ be the universal formal abelian scheme, $pA$ be the kernel of multiplication by $p$ and $pA^0 \subset pA$ be the neutral connected component. Notice that $pA^0$ is the unique closed subgroup scheme of $pA$, finite and flat over $\mathfrak{M}$ of rank $p^g$, that lifts the kernel of the isogeny of Frobenius $A \rightarrow A^{(\sigma)}$. Then the morphism $\Phi_{\text{can}}$ is defined by the isomorphism of formal abelian schemes $\Phi_{\text{can}}^{(\sigma)}(A^{(\sigma)}) \simeq A/pA^0$.

In a global situation, Dwork conjectured that the canonical lifting of the Frobenius is overconvergent. This problem is known as the excellent lifting problem. Deligne, Dwork [7] and Lubin-Tate [9] proved this conjecture for families of elliptic curves. Then Dwork [8] used it to prove that the unit $L$ function of the Legendre family of ordinary elliptic curves has a meromorphic continuation to $\mathbb{C}_p$. In [1], we prove the overconvergence for higher dimensions
under the assumption $p \geq 3$ and we deduce an application to the study of unit $L$ functions attached to Siegel modular varieties.

In this report, we will review only the overconvergence result. We start by reformulating the problem in modular terms. Let $K$ be a complete discrete valuation field of characteristic 0, with perfect residue field $k$ of characteristic $p > 0$, $\mathcal{O}_K$ be its ring of integers and $\nu_p$ be its valuation normalized by $\nu_p(p) = 1$. We put $S = \text{Spec}(\mathcal{O}_K)$ and $S_1 = \text{Spec}(\mathcal{O}_K/p\mathcal{O}_K)$. Let $M$ be a $\varphi$-$\mathcal{O}_{S_1}$-module, i.e. a free $\mathcal{O}_{S_1}$-module of finite type equipped with a semi-linear endomorphism $\varphi : M \rightarrow M$. We define the Hodge height of $M$ as the (truncated) $p$-adic valuation of the determinant of a matrix of $\varphi$. It is a well defined rational number between 0 and 1. Let $A$ be an $S$-abelian scheme of relative dimension $g$, $A_1 = A \times_S S_1$ and $pA$ the kernel of multiplication by $p$. The Frobenius of $A_1$ makes $H^1(A_1, \mathcal{O}_{A_1})$ as a $\varphi$-$\mathcal{O}_{S_1}$-module. The problem is to construct, under the assumption that the Hodge height of $H^1(A_1, \mathcal{O}_{A_1})$ is strictly less than a rational number $b(g) > 0$, a canonical closed subgroup scheme $H_{\text{can}} \subset pA$, finite and flat over $S$ of rank $p^g$. If $A_k$ is ordinary, we require that $H_{\text{can}}$ is the neutral connected component of $pA$. We solve this problem by studying the ramification of finite flat group schemes over $S$ using the ramification theory of Abbes-Saito [2, 3]. Let $G$ be a finite flat $S$-group scheme. We define on $G$ a canonical exhaustive decreasing filtration $(G^a, a \in \mathbb{Q}_{\geq 0})$ by closed subgroup schemes, finite and flat over $S$. For a real number $a \geq 0$, we put $G^{a+} = \cup_{b>a} G^b$ (where $b \in \mathbb{Q}$).

**Theorem 1** Assume that $p \geq 3$ and let $e$ be the absolute ramification index of $K$ and $j = e/(p-1)$. Let $A$ be an $S$-abelian scheme of relative dimension $g$ such that the Hodge height of $H^1(A_1, \mathcal{O}_{A_1})$ is strictly less than

$$\inf \left( \frac{1}{p(p-1)}, \frac{p-2}{(p-1)(2g(p-1)-p)} \right).$$

Then the level $pA^{j+}$ of the canonical filtration of $pA$ is finite and flat over $S$ of rank $p^g$. Moreover, if $A_k$ is ordinary, then $pA^{j+}$ is the neutral connected component of $pA$.

Let $K$ be an algebraic closure of $K$, $\mathcal{O}_K$ the integral closure of $\mathcal{O}_K$ in $K$, $\overline{S} = \text{Spec}(\mathcal{O}_K)$ and $\overline{s}$ and $\overline{\eta}$ be its closed and generic points. In order to prove Theorem 1, we give a description of the canonical filtration of $pA$ using differential forms. We proceed in two steps. First, we describe the dual filtration on $H^1(A_{\overline{s}}, \mathbb{Z}/p\mathbb{Z})$ via the spectral sequence of $p$-adic vanishing cycles, in terms of filtration by symbols ([4] Section 1). Then by a syntomic calculus, we deduce a description of the level $pA^{j+}(\overline{K})$. In particular, we prove that $pA^{j+}(\overline{K}) = \ker(\theta(-1))$, where

$$\theta : H^1(A_{\overline{K}}, \mathbb{Z}/p\mathbb{Z}(1)) \rightarrow H^0(A, \Omega^1_{A/S}) \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K$$
is a classical homomorphism in Kummer theory. Notice that this simple description is not enough to compute the rank of $pA^{1+}$.

Finally we review the result on $p$-adic vanishing cycles. Let $\overline{A} = A \times_S \overline{S}$. Consider the cartesian diagram

$$
\begin{array}{ccc}
A_\overline{s} & \rightarrow & A_\overline{t} \\
\downarrow & & \downarrow \\
\overline{s} & \rightarrow & \overline{t}
\end{array}
$$

and the étale sheaves on $A_\overline{s}$

$$\Psi^q = i^* R^q j_* (\mathbb{Z}/p\mathbb{Z}(q)).$$

The Kummer exact sequence $0 \rightarrow \mu_p \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$ on $A_{\overline{t}}$ induce a symbol map

$$h_{\overline{A}} : i^* j_* \mathcal{O}^\times_{A_{\overline{t}}} \rightarrow \Psi^1.$$ 

We put $U^0 \Psi^1 = \Psi^1$ and $U^a \Psi^1 = h_{\overline{A}}(1 + m_a i^* (\mathcal{O}_{\overline{A}}))$ for a rational number $a > 0$, where $m_a = \{ x \in \mathcal{O}_{\overline{K}}; v(x) \geq a \}$ and the valuation $v$ is normalized by $v(K) = \mathbb{Z}$.

There is a spectral sequence

$$E_2^{t,t} = H^t(A_\overline{s}, \Psi^t)(-t) \Rightarrow H^{t+t}(A_{\overline{t}}, \mathbb{Z}/p\mathbb{Z})$$

that induces the exact sequence

$$0 \rightarrow H^1(A_\overline{s}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1(A_{\overline{t}}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{u} H^0(A_{\overline{t}}, \Psi^1)(-1)$$

**Theorem 2** Let $e' = ep/(p-1)$. Under the canonical pairing

$$pA(K) \times H^1(A_{\overline{t}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow \mathbb{Z}/p\mathbb{Z},$$

we have, for any rational number $a > 0$,

$$pA^{a+}(K) = \begin{cases} 
\left( u^{-1}(H^0(A_\overline{s}, U^{e'-a} \Psi^1)(-1)) \right) & \text{if } 0 \leq a < e', \\
H^1(A_{\overline{t}}, \mathbb{Z}/p\mathbb{Z}) & \text{if } a \geq e'.
\end{cases}$$

**References**


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