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<th>Title</th>
<th>Universal bound for isogenies of elliptic curves over number fields (Algebraic Number Theory and Related Topics)</th>
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Kyoto University
Universal bound for isogenies of elliptic curves over number fields

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1 Introduction

Let $E$ and $E'$ be isogenous elliptic curves defined over a number field $k$ of degree $d$. Masser and Wüstholz [6] proved the existence of a constant $c$ depending effectively only on $d$ such that there is an isogeny between $E$ and $E'$ whose degree is at most $c(w(E))^4$, where $w(E) = \max\{1, h(g_2), h(g_3)\}$ when $E$ is identified with its Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$. Here $h$ denotes the absolute logarithmic Weil height. But they did not give an explicit formula of $c$. The purpose of this paper is to express $c$ as an explicit function of $d$ bounded by a polynomial when $E$ has no complex multiplication. The main result is as follows.

**Theorem.** Given a positive integer $d$, there exists a constant $c(d)$ depending only on $d$ with the following property. Let $k$ be a number field of degree at most $d$, and let $E$ be an elliptic curve defined over $k$ without complex multiplication. Suppose $E$ is isogenous to another elliptic curve $E'$ defined over $k$.

(i) Then there is an isogeny between $E$ and $E'$ whose degree is at most $c(d)\{w(E)\}^4$, where

$$c(d) = 6.55 \times 10^{94}\{\max(1.09 \times 10^7d^{1.45}[15.5\max\{\log(88.8d + 2.8), 38.4\} + 342.3]^{1.45}, 1.82 \times 10^{63})\}^{210}(11.4d + 55.3)^{20}.$$  

In particular the function $c(d)$ in $d$ increases as $1.9 \times 10^{1956}d^{325}$ when $d$ goes to infinity.

(ii) $c(1) = 8.2 \times 10^{3415}$ when $d = 1$, i. e., $k = \mathbb{Q}$.

We proceed along the line of [6]. Main devices in calculating $c$ are as follows. First we distinguish five constants which are unified as $c_3$ in [6, Lemma 3.3.] and those in [6, Lemmas 3.4 and 4.4]. Secondly we improve the relative degree of the field generated by the values of Weierstrass $p$-functions and their derivatives over $k$ from 81 to 36.
Pellarin [8] found an upper bound of the form $4.2 \times 10^{61}d^4 \max\{1, \log d\}^2 h(E)^2$, where $h(E) = \max\{1, h(j)\} + \max\{1, h(1, g_2, g_3)\}$ and $j$ is the $j$-invariant of $E$. But his Lemme 3.2 seems to contain some mistakes, because the cardinality of $\mathbb{C}$-linear independent monic monomials $X^\lambda$ on $G$ such that $\lambda \leq \underline{D}$, is $\prod_n (D_n + 1)$ on line 21 of page 219. This lemma is used in the proof of Proposition 3.1, and plays a crucial role in the main estimate. We hope that his proof will be corrected.

2 Preliminary estimates

Let $\Omega$ be a lattice in the complex plane. Let $(\omega_1, \omega_2)$ be a basis of $\Omega$ such that $\tau = \omega_2/\omega_1$ belongs to the standard fundamental region for the modular group. So $|\tau| \geq 1$, $x = \text{Re} \tau$ satisfies $|x| \leq \frac{1}{2}$, and $y = \text{Im} \tau$ satisfies $y \geq \frac{\sqrt{3}}{2}$. Let $\Omega$ be the area of the unit of $\Omega$, which equals $y|\omega_1|^2$. Let $g_2$ and $g_3$ be the invariants of $\Omega$, let $p(z)$ be the corresponding Weierstrass function, and $\gamma = \max\{|\frac{1}{4}g_2|^\frac{1}{2}, |\frac{1}{4}g_3|^\frac{1}{3}\}$.

**Lemma 2.1.** There exists a function $\theta_0(z)$ such that $\theta(z) = \gamma \theta_0(z)$ and $\tilde{\theta}(z) = p(z)\theta_0(z)$ are entire functions, with no common zeros, that satisfy

$$|\log \max\{|\theta(z)|, |\tilde{\theta}(z)|\} - \pi|z|^2/A| < 10.5y$$

for all complex $z$.

**Proof.** This is [4, Lemma 3.1] except for the estimation of the constant on the right-hand side of the inequality, which is 10.5. q. e. d.

**Lemma 2.2.** Let $z$ be a complex number not in $\Omega$, and $||z||$ be the distance from $z$ to the nearest element of $\Omega$. Then

$$|p(z) - p(\omega_2/2)| < 77244||z||^{-2}.$$ 

**Proof.** This is [6, Lemma 3.2] except for the estimation of the constant on the right-hand side of the inequality, which is 77244. q. e. d.

Let $d$ be a positive integer, and $k$ be a number field of degree at most $d$. Moreover, $g_2$ and $g_3$ are assumed to lie in $k$, and $w = \max\{1, h(g_2), h(g_3)\}$.

**Lemma 2.3.** There are constants $c_{1,i}$ ($1 \leq i \leq 5$), depending only on $d$, such that

(i) $c_{1,1}^{-w} \leq \gamma < c_{1,1}^w$,

(ii) $y < c_{1,2}w$,
(iii) $A > c_{1,3}^{-w}$,
(iv) $|\omega_{1}| > c_{1,4}^{-w}$,
(v) $A^{-1}|\omega_{2}|^{2} < c_{1,5}w$,

where $c_{1,1} = 2e^{0.5d}$, $c_{1,2} = 3.2d + 1.2$, $c_{1,3} = 16.6e^{3.8d}$, $c_{1,4} = 4.37e^{1.9d}$, and $c_{1,5} = 3.2d + 1.5$.

Proof. This is [6, Lemma 3.3] except for the estimation of the constants $c_{1,i}$ $(1 \leq i \leq 5)$.

Lemma 2.4. There are a constant $c_{2}$ depending only on $d$ and a positive integer $b < 2.22^w$ with the following properties. Suppose $n$ is a positive integer, $\zeta$ is an element of $\Omega/n$ not in $\Omega$, and write $\xi = p(\zeta)$.

Then
(i) $\xi$ is an algebraic number of degree at most $dn^2$ with $h(\xi) < 8.55w$,
(ii) $bn^2 \xi$ is an algebraic integer, and $|\xi| < c_2^w n^2$,

where $c_2 = 2.951 \times 10^6 \exp(3.8d)$.

Proof. When $\frac{1}{4}g_2$ and $\frac{1}{4}g_3$ are algebraic integers, from the proof of [6, Lemma 3.4] $\xi$ has degree at most $dn^2$, and $n^2 \xi$ is an algebraic integer. In the general case we can find a positive integer $b_0 \leq (\sqrt[3]{2}e^{\frac{1}{6}})^w$ such that $\frac{1}{4}b_0^4g_2$ and $\frac{1}{4}b_0^6g_3$ are algebraic integers. These correspond to the lattice $\Omega_0 = \Omega/b_0$ with Weierstrass function $p_0(z) = b_0^2p(b_0z)$. So $\xi_0 = p_0(\zeta/b_0)$ has degree at most $dn^2$, and $n^2 \xi_0$ is an algebraic integer. As $\xi = b_0^{-2}\xi_0$, $n^2 \xi_0 = b_0^2 n^2 \xi$ is an algebraic integer, $b_0^2 n^2 \xi \leq (\sqrt[3]{4}e^{\frac{d}{3}})^w n^2 \xi < 2.22^w n^2 \xi$, and $\xi$ is an algebraic number of degree at most $dn^2$.

The Néron-Tate height $q(P)$ of the point $P$ in $\mathbf{P}^2$ with projective coordinates $(1, p(\zeta), p'(\zeta))$ satisfies $q(P) = 0$. By [3, Lemme 3.4] the Weil height $h(P)$ satisfies $h(P) \leq q(P) + 3w + 8 \log 2 \leq (3 + 8 \log 2)w$. So $h(\xi) \leq h(P) < 8.55w$.

By Lemma 2.2
$$|\xi| < |p(\omega_2/2)| + c_3 \|\zeta\|^{-2},$$

where $c_3 = 77244$. As $p(\omega_2/2)$ is a root of $4x^3 - g_2x - g_3 = 0$, from Cardano's Formula $|p(\omega_2/2)| \leq (|g_3| + \sqrt{|g_3|^2 + |g_2|^3/27})^{\frac{1}{3}} < (1.3e^{\frac{d}{2}})^w$.

By Lemma 2.3(iv) $\|\zeta\|^{-2} \leq n^2 |\omega_1|^{-2} < n^2 c_{1,4}^{2w}$. From (1)
$$|\xi| \leq (1.3e^{\frac{d}{2}})^w + c_3 c_{1,4}^{2w} n^2 < (2.951 \times 10^6 \exp(3.8d))^{w} n^2 = c_2^w n^2.$$
3 The Main Proposition: construction

Let $E$ and $E^*$ be elliptic curves defined over $\mathbb{C}$, and $\Omega$ and $\Omega^*$ be their period lattices respectively. Let $\varphi$ be an isogeny from $E^*$ to $E$. It is said to be normalized if it induces the identity on the tangent spaces. Then $\Omega^* \subset \Omega$, and $[\Omega : \Omega^*]$ is the degree of $\varphi$. It is said to be cyclic if its kernel is a cyclic group.

**Main Proposition.** Given a positive integer $d$, there exists a constant $c_4(d)$ depending only on $d$, with the following property. Let $k$ be a number field of degree at most $d$, and let $E$ and $E^*$ be elliptic curves defined over $k$ without complex multiplication. Suppose there is a normalized cyclic isogeny $\varphi$ from $E^*$ to $E$ of degree $N$. Then there is an isogeny between $E$ and $E^*$ of degree at most $c_4(d)\{w(E) + w(E^*) + \log N\}^4$, where

$$c_4(d) = 1.47 \times 10^{16}\{\max\{(5910d\cdot 15.5 \max\{\log(7.4d + 2.8), 38.4\}, 342.3\})^{1.45}, 1.82 \times 10^{63}\}\}^{42}.$$

Before the proof of Main Proposition we need Lemmas 3.1-3.5. The body of the proof is described in Section 4.

Let $(\omega_1, \omega_2)$ and $(\omega_1^*, \omega_2^*)$ be bases of $\Omega$ and $\Omega^*$ respectively such that $\tau = \omega_2/\omega_1$ and $\tau^* = \omega_2^*/\omega_1^*$ lie in the standard fundamental region. Then there are integers $m_{ij}$ ($i, j = 1, 2$) such that

$$\omega_1^* = m_{11}\omega_1 + m_{12}\omega_2, \quad \omega_2^* = m_{21}\omega_1 + m_{22}\omega_2 \quad (2)$$

and $m_{11}m_{22} - m_{12}m_{21} = N$. Write $h = w(E) + w(E^*) \geq 2$.

**Lemma 3.1.** We have $|m_{ij}| < (7.4d + 2.8)N^{\frac{1}{2}}h$ ($i, j = 1, 2$).

**Proof.** This is [6, Lemma 4.1] except for the estimation of the constant on the right-hand side of the inequality, which is $7.4d + 2.8$. q. e. d.

Let $C$ be a sufficiently large constant depending only on $d$, $L = h + \log N$, $D = [C^{20}L^2]$ and $T = [C^{39}L^4]$. Let $p(z)$ and $p^*(z)$ be the Weierstrass functions corresponding to $\Omega$ and $\Omega^*$ respectively. For $t > 0$ and independent variables $z_1$ and $z_2$ let $D_i(t)$ be the set of differential operators of the form

$$\partial = (\partial/\partial z_1)^{t_1}(\partial/\partial z_2)^{t_2} \quad (t_1 \geq 0, \ t_2 \geq 0, \ t_1 + t_2 < t).$$

**Lemma 3.2.** There is a nonzero polynomial $P(X_1, X_2, X_1^*, X_2^*)$ of degree at most $D$ in each variable, whose coefficients are rational integers of absolute values at most $\exp(c_5TL)$, such that the function

$$f(z_1, z_2) = P(p(z_1), p(z_2), p^*(m_{11}z_1 + m_{12}z_2), p^*(m_{21}z_1 + m_{22}z_2))$$
satisfies $\partial f(\omega_1/2, \omega_2/2) = 0$ for all $\partial$ in $D_i(8T)$, where

$$c_5 = 156 \log C + 12 \max\{\log(7.4d + 2.8), 38.4\} + 251.3.$$  

Proof. Let $M$ denote any monomial of degree at most $D$ in each of the four functions appearing in $f$, that is,

$$M = \{p(z_1)\}^{d_1}\{p(z_2)\}^{d_2}\{p^\ast(m_{11}z_1 + m_{12}z_2)\}^{d_3}\{p^\ast(m_{21}z_1 + m_{22}z_2)\}^{d_4}$$

with $0 \leq d_i \leq D$ ($1 \leq i \leq 4$), and let $\partial$ be any operator of $D_i(8T)$. Then $\partial M$ can be written as a polynomial in the four numbers $m_{ij}$ ($i, j = 1, 2$) and the twelve functions obtained from the above four by replacing the Weierstrass functions by their first and second derivatives. From Baker's Lemma [2, Lemma 3]

$$\frac{d^j}{dz^j}\{p(z)\}^k = \sum u(t, t', t^{J}, j, k)\{p(z)\}^t\{p'(z)\}^{t'}\{p'(z)\}^{t'}$$

where the sum is taken over nonnegative integers $t$, $t'$ and $t''$ which satisfy $2t + 3t' + 4t'' = j + 2k$, and $u(t, t', t'', j, k)$ are integers of absolute values at most $j!48^j(7!2^8)^k$. So the total degree of $\partial M$ is at most $3D + 8T - 1 + 0.5 \cdot (8T - 1) + D < 12(D + T)$. And its coefficients are integers of absolute values at most $(8T - 1)!48^{8T-1}(7!2^8)^D < T^{8T}(2^{56} \times 3^8)^{D+T}$.

By Lemma 3.1 we have $\log |m_{ij}| < (\log c_6 + 1)L/2$, where $c_6 = 7.4d + 2.8$. From (2) the twelve functions at $(z_1, z_2) = (\omega_1/2, \omega_2/2)$ take the values

$$p^{(t)}(\omega_j/2), p^{*(t)}(\omega_j^*/2) \ (t = 0, 1, 2; j = 1, 2).$$

By Lemma 2.4 $h(p(\omega_j/2))$ and $h(p^\ast(\omega_j^*/2))$ are at most $8.55L$. Both $p'(\omega_j/2)$ and $p'^\ast(\omega_j^*/2)$ are zero. And

$$h(p''(\omega_j/2)) = h(6p(\omega_j/2)^2 - g_2/2) \\ \leq 2h(p(\omega_j/2)) + h(g_2) + \log 12 + \log 2 < 19.7L.$$  

So does $h(p''(\omega_j^*/2))$. Thus $m_{ij}$ and the values of the twelve functions have heights at most $c_7L$, where

$$c_7 = \max\{0.5 + 0.5\log(7.4d + 2.8), 19.7\}.$$  

As $p(\omega_j/2)$ and $p^\ast(\omega_j^*/2)$ are roots of cubic equations with coefficients in $k$, and $p''(\omega_j/2)$ and $p'^\ast(\omega_j^*/2)$ lie in the field generated by $p(\omega_j/2)$ and $p^\ast(\omega_j^*/2)$ over $k$, these values lie in $k'$ whose degree is at most $36d$.

The conditions of Lemma 3.2 amount to $R = 4T(8T + 1)$ homogeneous linear equations in $S = (D + 1)^4$ unknowns with coefficients in $k'$. By
Siegel's Lemma [1, Proposition], if \( S \geq 2 \times 36dR \), these can be solved in rational integers, not all zero, of absolute values at most \( S \exp(c_8) \), where \( c_8 \) is the height of linear equations. To satisfy the condition \( S \geq 72dR \) it suffices that
\[
C^{80}L^8 > 2305dC^{78}L^8, \text{ so } C > 48.1\sqrt{d}. \tag{3}
\]

Next we calculate \( c_8 \). By Lemma 2.4 there is a positive integer \( b \leq 2.22^w \) such that \( 4bp(\omega_j/2) \) is an algebraic integer. Since \( p''(\omega_j/2) = 6p(\omega_j/2)^2 - g_2/2 \), and there is a positive integer \( b_2 \leq e^w \) such that \( b_2g_2 \) is an algebraic integer, \( 16b^2b_2p''(\omega_j/2) \) is an algebraic integer. If we multiply \( \partial M \) at \((z_1, z_2) = (\omega_1/2, \omega_2/2)\) by an integer at most \( (16 \times 2.22^2L^e)^{12(D+T)} \), every term is an algebraic integer. As \( h(\sum_{i=1}^n a_i) \leq \max h(a_i) + \log n \) for algebraic integers \( a_i \),
\[
S \exp(c_8) \leq (D + 1)^4 (16 \times 2.22^2L^e)^{12(D+T)} T^{8T}(2^{56} \times 3^8)^{D+T} \exp\{12c_7(D + T)L\} < \exp(c_9TL).
\]
q. e. d.

Let \( \theta_0(z) \) and \( \theta_0^*(z) \) be the functions in Lemma 2.1 corresponding to \( p(z) \) and \( p^*(z) \) respectively. So the function
\[
\Theta(z_1, z_2) = \{\theta_0(z_1)\theta_0(z_2)\theta_0^*(m_{11}z_1 + m_{12}z_2)\theta_0^*(m_{21}z_1 + m_{22}z_2)\}^D
\]
is entire. Let \( F(z_1, z_2) = \Theta(z_1, z_2)f(z_1, z_2) \).

**Lemma 3.3.** The function \( F(z_1, z_2) \) is entire. Further, for any complex number \( z \) and any operator \( \partial \) in \( D_i(4T + 1) \) we have
\[
|\partial F(\omega_1z, \omega_2z)| < \exp\{c_9L(T + D|z|^2)\},
\]
where
\[
c_9 = 234\log C + 154.8d + 2\log(7.4d + 2.8) + 12\max\{\log(7.4d + 2.8), 38.4\} + 423.5.
\]

**Proof.** Let \( \gamma, \gamma^*, \theta, \theta^*, \tilde{\theta}, \tilde{\theta}^* \) be as in Lemma 2.1 corresponding to \( p, p^* \). Then \( F(z_1, z_2) \) can be expressed as a polynomial in the eight functions
\[
\gamma^{-1}\theta(z_i), \tilde{\theta}(z_i), \gamma^*\theta^*(m_{i1}z_1 + m_{i2}z_2), \tilde{\theta}^*(m_{i1}z_1 + m_{i2}z_2) \quad (i = 1, 2), \tag{4}
\]
so it is entire. It is the quadrihomogenized version of \( P \) in Lemma 3.2.
Let $M_0 = \max |m_{ij}|$, $A_0 = \min (A, A^*)$, and $\delta = M_0^{-1} A_0^{\frac{1}{2}}$, where $A$ and $A^*$ are determinants of $\Omega$ and $\Omega^*$ respectively. For any complex number $z$ let $z_1$ and $z_2$ be complex numbers satisfying

$$|z_i - \omega_i z| = \delta \ (i = 1, 2). \quad (5)$$

We claim that $|F(z_1, z_2)| < \exp\{c_{10} L (T + D|z|^2)\}$, where $c_{10} = 156 \log C + 147.2d + 12 \max\{\log(7.4d + 2.8), 38.4\} + 404.3$. By Lemma 2.1

$$\log \max\{|\theta(z_i)|, |\tilde{\theta}(z_i)|\} < 10.5y + \pi A^{-1}|z_i|^2$$

$$\leq 10.5(y + A^{-1}\delta^2 + A^{-1}|\omega_i|^2|z|^2) \ (i = 1, 2).$$

As $A^{-1}\delta^2 \leq M_0^{-2} \leq 1$, from Lemma 2.3(i)(ii)(v) the first two functions in (4) have absolute values at most

$$c_{1,1} L \exp\{10.5(c_{1,2} L + 1 + c_{1,5} L|z|^2)\} < \exp\{(11.5c_{1,5} + 5.25)L(1 + |z|^2)\},$$

for $c_{1,5} > c_{1,2} > \log c_{1,1}$.

The last two expressions in (4) are estimated similarly. From (2) and (5) $z_i^* := m_{i1}z_1 + m_{i2}z_2$ satisfy $|z_i^* - \omega_{i^*} z| \leq 2M_0\delta \ (i = 1, 2)$. Thus

$$\log \max\{|\theta^*(z_i^*)|, |\tilde{\theta}^*(z_i^*)|\} < 10.5(y^* + 4M_0^2 A^{-1}\delta^2 + A^{-1}|\omega_{i^*}|^2|z|^2)$$

$$(i = 1, 2).$$

By Lemma 2.3 the last two functions have absolute values at most

$$c_{1,1} L \exp\{10.5(c_{1,2} L + 4 + c_{1,5} L|z|^2)\} < \exp\{(11.5c_{1,5} + 21)L(1 + |z|^2)\}.$$  

By Lemma 3.2

$$|F(z_1, z_2)| < \exp(c_5 TL) \exp\{(46c_{1,5} + 84)DL(1 + |z|^2)\}(D + 1)^4$$

$$< \exp\{c_{10} L(T + D|z|^2)\},$$

which is the claim.

By the Cauchy Integral Formula

$$|\partial F(\omega_1z, \omega_2z)| = \frac{t_1!t_2!}{(2\pi i)^2} \oint_f \oint_f \frac{F(z_1, z_2)}{(z_1 - \omega_1 z)^{t_1+1}(z_2 - \omega_2 z)^{t_2+1}}dz_1dz_2$$

$$< t_1!t_2!\delta^{-(t_1+t_2)}\exp\{c_{10} L(T + D|z|^2)\},$$

where the integrals are around the circles (5). From Lemma 2.3(iii) and Lemma 3.1

$$\delta = M_0^{-1} A_0^{\frac{1}{2}} > (7.4d + 2.8)^{-1} N^{-\frac{1}{2}} h^{-1} c_{1,3}^{-\frac{3}{2}}$$

$$> \{6.72(7.4d + 2.8)^{\frac{1}{2}} \exp(1.9d)\}^{-L} =: c_{11}^{-L}.$$
\[ |\partial F(\omega_1 z, \omega_2 z)| < (4T)!c_{11} 4LT \exp\{c_{10}L(T + D|z|^2)\} \]
\[ < \exp\{c_9L(T + D|z|^2)\}. \]

q. e. d.

Let \( Q \) be the unique integral power of 2 that satisfies
\[ C^{17/8} < Q \leq 2C^{17/8}. \]

**Lemma 3.4.** For any odd integer \( q \) and \( \zeta = q/Q \), we have
\[ |\Theta(\omega_1 \zeta, \omega_2 \zeta)| > \exp(-84DLQ^2). \]

Further, for any \( \partial \) in \( D_i(4T+1) \) such that \( \partial f(\omega_1 \zeta, \omega_2 \zeta) \neq 0 \), we have
\[ |\partial f(\omega_1 \zeta, \omega_2 \zeta)| > \exp(-c_{12}TLQ^8), \]
where \( c_{12} = 16d[290 \log C + 15.5 \max\{\log(7.4d + 2.8), 38.4\} + 342.3]. \)

**Proof.** By Lemma 2.3(i) and Lemma 2.4(i)
\[ \max\{\gamma, |p(\omega_j \zeta)|\} < \exp(8.55dhQ^2) (j = 1, 2). \]

From Lemma 3.1 and Lemma 2.3(ii)
\[ |\theta_0(\omega_j \zeta)| > \exp(-10.5d - 8.55dhQ^2) > \exp\{-10.5d(1 + c_{1,2}/Q^2)hQ^2\}, \]
and the same bound holds for \( |\theta_0^*(\omega_j^* \zeta)| (j = 1, 2). \) Thus
\[ |\Theta(\omega_1 \zeta, \omega_2 \zeta)| > \exp\{-4D \times 10.5d(1 + c_{1,2}/Q^2)hQ^2\} > \exp(-84DLQ^2), \]
for by (3) \( Q^2 > C^{17/4} > 48^4d^2 > 3.2d + 1.2 = c_{1,2}. \)

\( \alpha := \partial f(\omega_1 \zeta, \omega_2 \zeta) \) is estimated as in the proof of Lemma 3.2. \( \alpha \) is a polynomial in the \( m_{ij} \) (\( i, j = 1, 2 \)) and the twelve numbers \( p^{(t)}(\omega_j \zeta), p^{*(t)}(\omega_j^* \zeta) \) (\( j = 1, 2; t = 0, 1, 2 \)). Let \( \partial M \) be as in the proof of Lemma 3.2, and \( \partial \) be any operator of \( D_i(4T+1) \). From Baker's Lemma the total degree of \( \partial M \) is at most \( 6(D + T) \), and the absolute values of its coefficients are at most \( T^{4T}(2^{24} \times 3^4)^{D+T} \).

By Lemma 2.4 there is a positive integer \( b < 2.22^w \) such that \( bQ^2p(\omega_j \zeta) \) is an algebraic integer. Since \( p'(\omega_j \zeta)^2 = 4p(\omega_j \zeta)^2 - g_2p(\omega_j \zeta) - g_3 \), and there is a positive integer \( b_3 \leq e^w \) such that \( b_3g_3 \) is an algebraic integer, \( (b^3b_2b_3)^{1/2}Q^3p'(\omega_j \zeta) \) is an algebraic integer. And \( 2b^2b_2Q^4p''(\omega_j \zeta) \) is an algebraic integer. If we multiply \( \partial M \) at \( (z_1, z_2) = (\omega_1 \zeta, \omega_2 \zeta) \) by
a positive integer at most $(2 \times 2.22^{2L} e^{1.5L} Q^4)^6(D+T)$, every term is an algebraic integer. By Lemma 2.4 $h(p(\omega_j \zeta))$ and $h(p^*(\omega_j^* \zeta))$ are at most $8.55L$,

$$h(p'(\omega_j \zeta)) \leq \frac{1}{2} \{3h(p(\omega_j \zeta)) + \log 4 + h(g_2) + h(p(\omega_j \zeta)) + h(g_3) + \log 3\} < 2 \times 8.55L + L + \log 3 < 19.7L,$$

and $h(p''(\omega_j^* \zeta))$, $h(p'''(\omega_j \zeta))$ and $h(p''''(\omega_j^* \zeta))$ are at most $19.7L$. Thus if $(z_1, z_2) = (\omega_1 \zeta, \omega_2 \zeta)$,

$$\exp(h(\partial M)) \leq (2 \times 2.22^{2L} e^{1.5L} Q^4)^{12(D+T)} 17H_{6(D+T)} T^{4T}(2^{24} \times 3^4)^{D+T} \exp\{6c_7(D+T)L\}.$$  

$\alpha$ is a linear combination of $\partial M$ with rational integer coefficients whose absolute values are at most $\exp(c_5TL)$. So

$$h(\alpha) \leq \log(D + 1)^4 + c_5 TL + h(\partial M) \leq [290 \log C + 15.5 \max\{\log(7.4d + 2.8), 38.4\}] + 342.3TL.$$

Next we estimate the degree of $\alpha$, $\deg \alpha$. Since

$$Q(\alpha) = Q(p(t)(\omega_j \zeta), p^*(t)(\omega_j^* \zeta)) \ (j = 1, 2; \ t = 0, 1, 2) \subset k(p(\omega_j \zeta), p^*(\omega_j^*), p'(\omega_j \zeta), p''(\omega_j^* \zeta)),$$

the degrees of $p(\omega_j \zeta)$ and $p^*(\omega_j^* \zeta)$ are at most $dQ^2$ by Lemma 2.4(i), and $[k(p(\omega_j \zeta), p'(\omega_j \zeta)) : k(p(\omega_j \zeta))] \leq 2$,

$$\deg \alpha = [Q(\alpha) : Q] \leq d(Q^2)^42^4 = 16dQ^8.$$  

Hence $|\alpha| \geq \exp\{-(\deg \alpha)h(\alpha)\} > \exp(-c_{12}TLQ^8)$. q. e. d.

Lemma 3.5. If $C$ satisfies $C > (256/\log 2)c_{12}$ with the constant $c_{12}$ in Lemma 3.4, then for any odd integer $q$ and any $\partial$ in $D_i(4T + 1)$ we have $\partial f(q\omega_1/Q, q\omega_2/Q) = 0$.

Proof. Assume that there exist an odd integer $q$ and an operator $\partial$ in $D_i(4T + 1)$ such that $\alpha = \partial f(\omega_1 \zeta, \omega_2 \zeta) \neq 0$ for $\zeta = q/Q$. We can suppose that $0 < \zeta < 1$, and that

$$\Theta(\omega_1 \zeta, \omega_2 \zeta) = G(\zeta),$$

where $G(z) = \partial F(\omega_1 z, \omega_2 z)$ and $\partial$ is of minimal order.

$G(t)(z)$ is a linear combination of the $\partial f(\omega_1 z, \omega_2 z)$ for $\partial$ in $D_i(t + 1 + 4T)$, so by Lemma 3.2 and periodicity

$$G(t)(s + 1/2) = 0.$$
for any integer $t$ with $0 \leq t < 4T$ and any integer $s$. We apply the Schwarz Lemma to (7) for $0 \leq s < S$, where $S = [C^{18}L]$. Then $|G(\zeta)| \leq 2^{-4TS}M_1$, where $M_1$ is the supremum of $|G(z)|$ for $|z| \leq 5S$. By Lemma 3.3 $M_1 < \exp\{25c_9L(T + DS^2)\} < \exp(50c_9LDS^2)$. If $C > (25/\log 2)c_9$, then $\exp(50c_9LDS^2) < 2^{2TS}$, so $|G(\zeta)| < 2^{-2TS}$. By (6) and Lemma 3.4

$$|\alpha| < 2^{-2TS}\exp(84DLQ^2) < 2^{-TS},$$  \hspace{1cm} (8)

where the second inequality follows, because $C > (84/\log 2)^{4/131}$. But also from Lemma 3.4 we have the lower bound

$$|\alpha| > \exp(-c_{12}TLQ^8).$$  \hspace{1cm} (9)

If

$$C > (256/\log 2)c_{12}$$

$$= 5909d[290 \log C + 15.5 \max\{\log(7.4d + 2.8), 38.4\} + 342.3]$$  \hspace{1cm} (10)

then $2^{TS} > \exp(c_{12}TLQ^8)$, which contradicts (8) and (9). As $256c_{12} > 25c_9$, (10) implies that $C > (25/\log 2)c_9$. q. e. d.

4 Proof of Main Proposition: deconstruction

Let $G = E^2 \times E^*^2$ embedded in $\mathbb{P}^{81}$ by Segre embedding. Let $\epsilon$ be the exponential map from $\mathbb{C}^4$ to $G$ obtained from the functions $p(z_1), p(z_2), p^*(z_1^*), p^*(z_2^*)$ and their derivatives for independent complex variables $z_1, z_2, z_1^*, z_2^*$. Define a subspace $Z$ of $\mathbb{C}^4$ by the equations

$$z_1^* = m_{11}z_1 + m_{12}z_2, \quad z_2^* = m_{21}z_1 + m_{22}z_2.$$  

Write $O_G$ for the zero of $G$, and let $\Sigma$ and $\Sigma_0$ be the sets of even and odd multiples of the point $\sigma = \epsilon(\omega_1/Q, \omega_2/Q, \omega_1^*/Q, \omega_2^*/Q)$ in $G$ respectively. We use Philippon's zero estimate.

**Lemma 4.** There is a connected algebraic subgroup $H = \epsilon(W) \neq G$ of $G$ such that

$$T^\rho R\Delta < c_{13}D^r,$$  \hspace{1cm} (11)

where $W$ is a subspace of $\mathbb{C}^4$, $\rho$ is the codimension of $Z \cap W$ in $Z$, $R$ is the number of points in $\Sigma$ distinct modulo $H$, $\Delta$ is the degree of $H$, $r$ is the codimension of $H$ in $G$, and $c_{13} = 4.032 \times 10^7$. 
Proof. By Lemma 3.5 there is a polynomial, homogeneous of degree $D$, that vanishes to order at least $4T + 1$ along $\epsilon(Z)$ at all points of $\Sigma_0$, but does not vanish identically on $G$. Let $\Sigma(4) = \{\sum_{i=1}^{4} \sigma_i \mid \sigma_i \in \Sigma\}$, so $\Sigma_0 = \sigma + \Sigma(4)$. From [5, Lemma 1] translations on an elliptic curve are described by homogeneous polynomials of degree 2. According to Philippon’s zero estimate [9, Théorème 1], there exists a connected algebraic subgroup $H = \epsilon(W) \neq G$ of $G$ such that

$$T^p R \Delta \leq \deg G \times 2^{\dim G} (2D)^r.$$  

As $\deg G = 3^{2\dim G} \times 4! = 2^3 \times 3^9$ and $r \leq 4$, $T^p R \Delta < c_{13} D^r$. q. e. d.

Now we can give the proof of Main Proposition. We want to find a nontrivial graph subgroup of an isogeny $E \to E^*$ of small degree. We consider the three cases $\rho = 2$, 1, 0 in (11).

When $\rho = 2$, $T^2 R \Delta < c_{13} D^r$. So

$$R < c_{13} D^r T^{-2} < 4.04 \times 10^7 C^2 D^{r-4} =: c_{14} C^2 D^{r-4}.$$  

Thus $r = 4$, $H = O_G$, and $R = Q/2$. If

$$C > 2^8 c_{14} 8 \approx 1.817 \times 10^{63},$$  

then $Q/2 > C^{17/8}/2 > c_{14} C^2$ contradicting (12). Hence the case $\rho = 2$ is ruled out under (13).

Next when $\rho = 1$, $Z \cap W$ has dimension 1, so $r \leq 3$. If $H$ is nonsplit, then by [8, Lemma 2.2] there is an isogeny of degree at most $9\Delta^2$ between $E$ and $E^*$. From (11) $\Delta < c_{13} D^2 T^{-1} < 4.04 \times 10^7 C^{21} L^2$. Thus we get an isogeny of degree at most

$$9 \times (4.04 \times 10^7)^2 C^{42} L^4 \approx 1.469 \times 10^{16} C^{42} L^4.$$  

If $H$ is split, we can not have $r = 3$ by the proof of [6, Proposition]. If $r \leq 2$, then $R = Q/2$ by [6, Lemma 5.2], and $R < c_{13} D^2 T^{-1} < c_{14} C$. The assumption of no complex multiplication is used to prove [6, Lemma 5.2] in applying Kolchin’s Theorem. Since $C > (2c_{14})^{8/9}$ from (13), $Q/2 > C^{17/8}/2 > c_{14} C$. Hence a contradiction.

Lastly when $\rho = 0$, then $Z \subset W$ and $r \leq 2$. If $r = 2$, then from the proof of [6, Proposition] $N \leq 9\Delta < 9c_{13} D^2 \leq 9c_{13} C^{40} L^4$, so the original isogeny $\varphi$ satisfies the required estimate.

If $r = 1$, then by the proof of [6, Proposition] $H$ is nonsplit, and there is an isogeny of degree at most $9\Delta^2$ between $E$ and $E^*$. As by (11)
\[ \Delta < c_{13} D \leq c_{13} C^{20} L^2, \] we get an isogeny of degree at most \( 9 \times (4.04 \times 10^7)^2 C^{40} L^4 = 1.469 \times 10^{16} C^{40} L^4. \)

Next we estimate \( C \), the conditions for which are (10) and (13), for (10) implies (3). Let \( C_0 \) be the solution of the equation

\[
C_0 = 5910 d [290 \log C_0 + 15.5 \max \{ \log(7.4d + 2.8), 38.4 \} + 342.3].
\]

Let \( x_0 = \log C_0 \), \( A_1 = 5910 \times 290d \), \( A_2 = 5910 d \{ 15.5 \max \{ \log(7.4d + 2.8), 38.4 \} + 342.3 \} \), and \( f(x) = e^x - A_1 x - A_2 \), so \( f(x_0) = 0 \). If \( x_1 = \{ A_2/(A_2 - A_1) \} \log A_2 \), then \( f(x_1) > 0 \). As \( f(x) \) increases monotonously, \( x_0 < x_1 \), that is, \( C_0 < \exp x_1 < A_2^{1.45} \).

Thus \( C = \max \{ A_2^{1.45}, 1.82 \times 10^{63} \} \) satisfies both (10) and (13). From (14) we have proved Main Proposition with \( c_4(d) = 1.47 \times 10^{16} C^{42} \).

## 5 Proof of Theorem

We normalize the isogeny by Lemma 5 to apply Main Proposition.

**Lemma 5.** Given a positive integer \( d \), there exists a constant \( c_{15} \) with the following property. Let \( k \) be a number field of degree at most \( d \), let \( E \) and \( E_1^* \) be elliptic curves defined over \( k \), and let \( \varphi \) be an isogeny from \( E \) to \( E_1^* \) of degree \( N \). Suppose \( k' \) is the smallest extension field of \( k \) over which \( \varphi \) is defined. Then \( [k' : k] \leq 12 \), and there is an elliptic curve \( E^* \), defined over \( k' \) and isomorphic over \( k' \) to \( E_1^* \), such that the induced isogeny from \( E \) to \( E^* \) is normalized. Further we have

\[
w(E^*) < (11.4d + 54.3)w(E) + 13 \log N =: c_{15} w(E) + 13 \log N.
\]

**Proof.** This is [6, Lemma 3.2] except for the estimation of the constant on the right-hand side of the inequality, which is \( 11.4d + 54.3 \). q. e. d.

Now we give the proof of Theorem. Let \( N \) be the smallest degree of any isogeny between \( E \) and \( E' \). By [6, Lemma 6.2] there is a cyclic isogeny from \( E \) to \( E' \) of degree \( N \). According to Lemma 5 there are an extension \( k' \) of \( k \) with \( [k' : k] \leq 12 \) and an elliptic curve \( E^* \) defined over \( k' \) and isomorphic to \( E' \) such that the induced isogeny \( \varphi \) from \( E \) to \( E^* \) is normalized and \( w(E^*) < c_{15} \{ w(E) + \log N \} \).

As \( \varphi \) is cyclic, by Main Proposition there is an isogeny between \( E \) and \( E^* \) whose degree \( N_1 \) satisfies

\[
N_1 \leq c_4(12d) \{ w(E) + w(E^*) + \log N \}^4 < c_4(12d)(c_{15} + 1)^4 \{ w(E) + \log N \}^4.
\]
So there is an isogeny of degree \( N_1 \) between \( E \) and \( E' \), and
\[
N \leq N_1 < c_4(12d)(c_{15} + 1)^4 \{w(E) + \log N\}^4.
\]
Thus \( N < c_{16} \{w(E)\}^4 \) for a constant \( c_{16} \) depending only on \( d \).

Lastly we estimate \( c_{16} \). Let \( c_{17} = c_4(12d)(c_{15} + 1)^4, w = w(E), N_0 \) satisfy \( N_0 = c_{17}(w + \log N_0)^4 \), and \( c_{18} = N_0/w^4 \). Then \( N < N_0 \), and \( c_{18}w^4 = c_{17}(w + 4\log w + \log c_{18})^4 \). Therefore
\[
c_{18} = c_{17}(1 + 4\log w/w + \log c_{18}/w)^4 < c_{17}(5 + \log c_{18})^4.
\]
Let \( c_{19} \) satisfy \( c_{19} = c_{17}(5 + \log c_{19})^4 \). Then \( c_{18} < c_{19} \), and \( c_{19} \) is estimated similarly as \( C_0 \) in the proof of Main Proposition. So \( c_{19} < 5^{20} c_{17}^5 \), and
\[
N < N_0 = c_{18}w^4 < c_{19}w^4 < 5^{20} c_{17}^5 w^4 = 5^{20} \{c_4(12d)\}^5(c_{15} + 1)^{20}w^4.
\]
Hence \( c_{16} = 5^{20} \{c_4(12d)\}^5(c_{15} + 1)^{20} < c(d) \).

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