GREEN CURRENTS FOR MODULAR CYCLES IN ARITHMETIC QUOTIENTS OF COMPLEX HYPERBALLS

Algebraic Number Theory and Related Topics

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GREEN CURRENTS FOR MODULAR CYCLES IN ARITHMETIC QUOTIENTS OF COMPLEX HYPERBALLS

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0. INTRODUCTION AND BASIC NOTATIONS

0.1. Introduction. Let $X$ be a complex manifold and $Y$ its analytic subvariety of codimension $r$. The Green current for $Y$ is defined to be a current $G$ of $(r-1,r-1)$-type on $X$ such that $dd^c G + \delta_Y$ is represented by a $C^\infty$-form of $(r,r)$-type on $X$. In the arithmetic intersection theory developed by Gillet and Soulé, the role played by the algebraic cycles in the conventional intersection theory is replaced with the arithmetic cycles. In a heuristic sense, the Green currents are regarded as the 'archimedean' ingredient of such arithmetic cycles ([2]).

Let us consider the case when $X$ is the quotient of a Hermitian symmetric domain $G/K$ by an arithmetic lattice $\Gamma$ in the semisimple Lie group $G$, and $Y$ is a modular cycle stemming from a modular imbedding $H/H \cap K \hookrightarrow G/K$, where $H$ is a reductive subgroup of $G$ such that $H \cap K$ is maximally compact in $H$. Then inspired by the classical works on the resolvent kernel functions of the Laplacian on Riemannian surfaces and also by a series of works of Miatello and Wallach ([5], [6]), T. Oda posed a plan to construct a Green current for $Y$ making use of a 'secondary spherical function' on $H\backslash G$, giving an evidence for divisorial case with some conjectures. Among many possible choices of the Green currents for a modular cycle $Y$, this construction may provide a way to fix a natural one. If $r = 1$, namely $Y$ is a modular divisor, we already obtained a satisfactory result by introducing the secondary spherical functions properly ([7]). Here we focus on the case when $G/K$ is an $n$-dimensional complex hyperball and $H/H \cap K$ is a complex sub-hyperball of codimension $r > 1$, and show that the same method also works well.

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0.2. Notations. The Lie algebra of a Lie group $G$ is denoted by Lie$(G)$. For a complex matrix $X = (x_{ij})_{ij}$, put $X^* = (\overline{x}_{ji})_{ij}$.

1. INVARIANT TENSORS

Let $n$ and $r$ be integers such that $2 \leq r < n/2$.

Let us consider the two involutions $\sigma$ and $\theta$ in the Lie group $G = U(n,1) := \{g \in \operatorname{GL}_{n+1}(\mathbb{C})| g^* \in_{n,1} g = I_{n,1}\}$ defined by $\theta(g) = I_{n,1} g I_{n,1}$ and $\sigma(g) = S^{-1} g S$ respectively. Here $I_{n,1} := \operatorname{diag}(I_n, -1)$ and $S = \operatorname{diag}(I_{n-r}, -I_r, 1)$. Then $K := \{g \in G| \theta(g) = g\} \cong U(n) \times U(1)$ is a maximal compact subgroup in $G$ and $H := \{g \in G| \sigma(g) = g\} \cong U(n-r,1) \times U(r)$ is a symmetric subgroup of $G$ such that $K_H := H \cap K \cong U(n-r) \times U(r) \times U(1)$ is maximally compact in $H$. 

\[\]
The Lie group $G$ acts transitively on the complex hyperball

$$\mathcal{D} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | \sum_{i=1}^{n} |z_i|^2 < 1\}$$

by the fractional linear transformation $g \cdot z = \frac{g_{11}z_1 + g_{12}}{g_{21}z_1 + g_{22}}$, $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in G$, $z \in \mathbb{C}^n$. (Here the matrix $g \in GL_{n+1}(\mathbb{C})$ is partitioned into blocks so that $g_{11}$ is an $n \times n$-matrix and $g_{22}$ is a scalar.) Since $K$ is the stabilizer of the origin $0 \in \mathcal{D}$, we have the identification $G/K \cong \mathcal{D}$ of $G$-manifolds assigning the point $z = g_{0}$ to $g \in G$. Then $H/K \subset G/K$ corresponds to the $H$-orbit of 0 in $\mathcal{D}$, that is $\mathcal{D}^H := \{z \in \mathcal{D} | z_{n-r+1} = \cdots = z_n = 0\}$. In particular the real codimension of $H/K \subset G/K$ is $2r$.

The Lie algebra $\mathfrak{g} := \text{Lie}(G)$ is realized in its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{n+1}(\mathbb{C})$ as an $\mathbb{R}$-subalgebra of all $X \in \mathfrak{g}_{n+1}(\mathbb{C})$ such that $X^* \mathbb{I}_{n+1} + \mathbb{I}_{n+1}X = 0_{n+1}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{t} := \text{Lie}(K)$ in $\mathfrak{g}$ with respect to the $G$-invariant, non-degenerate bi-linear form $\langle X, Y \rangle = 2^{-1} \text{tr}(XY)$ on $\mathfrak{g}$. For $1 \leq i, j \leq n + 1$, let $E_{i,j} := (\delta_{ii} \delta_{ij})_{uu} \in \mathfrak{g}_{n+1}(\mathbb{C})$ be the matrix unit. The operator $J := \text{ad}(\tilde{Z}_0)\mathfrak{p}$ with $\tilde{Z}_0 := \frac{1}{n+1} (\sum_{i=1}^{n} E_{i+i} - nE_{n+1,n+1})$ gives a $K$-invariant complex structure of $\mathfrak{p}$, which induces the $K$-invariant decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_\pm$ the $(\pm \sqrt{-1})$-eigenspace of $J$ in $\mathfrak{p}_{\mathbb{C}}$. Since $\mathfrak{p}$ is identified with the tangent space of $G/K$ at $K$, we can extend $J$ to the $G$-invariant complex structure of $G/K$ making the identification $G/K \cong \mathcal{D}$ bi-holomorphic. Put $X_i := E_{i,n+1} (1 \leq i \leq n-1)$, $X_0 := E_{n,n+1}$. Then $\mathfrak{p}_+ = \sum_{i=0}^{n} \mathbb{C}X_i$, $\mathfrak{p}_- = \sum_{i=0}^{n} \mathbb{C}X_i$ with $X_i = E_{n+1,i}$, $X_0 = E_{n+1,n}$. Let $\{\omega_i\}$ and $\{\tilde{\omega}_i\}$ be the basis of $\mathfrak{p}_+^*$ and $\mathfrak{p}_-^*$ dual to $\{X_i\}$ and $\{\bar{X}_i\}$ respectively.

The exterior algebra $\wedge \mathfrak{p}^\mathbb{C}$ is decomposed to the direct sum of subspaces $\wedge^p \mathfrak{p}_{\mathbb{C}} := (\wedge^p \mathfrak{p}_+^*) \wedge (\wedge^q \mathfrak{p}_-^*) (p, q \in \mathbb{N})$. Put

$$\omega := \frac{-1}{2} \sum_{i=0}^{n-1} \omega_i \wedge \bar{\omega}_i \quad (\in \wedge^{1,1} \mathfrak{p}_{\mathbb{C}} \cap \wedge^\mathbb{C} \mathfrak{p}^*), \quad \text{vol} := \frac{1}{n!} \omega^n \quad (\in \wedge^{n,n} \mathfrak{p}_{\mathbb{C}} \cap \wedge^\mathbb{C} \mathfrak{p}^*)$$

The inner product $\langle X, Y \rangle$ on $\mathfrak{p}$ yields the Hermitian inner product $(\cdot | \cdot)$ of $\wedge \mathfrak{p}^\mathbb{C}$ in the standard way. Then the Hodge star operator $\ast$ is defined to be the $\mathbb{C}$-linear automorphism of $\wedge \mathfrak{p}_{\mathbb{C}}$ such that $\ast \alpha = \overline{\alpha}$ and such that $(\alpha | \beta) \text{vol} = \alpha \wedge \ast \beta$, $(\forall \alpha, \beta \in \wedge \mathfrak{p}^\mathbb{C})$. For $\alpha \in \wedge \mathfrak{p}^\mathbb{C}$, let us define the endomorphism $e(\alpha) : \wedge \mathfrak{p}_{\mathbb{C}} \to \wedge \mathfrak{p}_{\mathbb{C}}$ by $e(\alpha) \beta = \alpha \wedge \beta$. As usual, we have the Lefschetz operator $L := e(\omega)$ and its adjoint operator $\Lambda$ acting on the finite dimensional Hilbert space $\wedge \mathfrak{p}^\mathbb{C}$ ([8, Chap. V]).

Put $\mathfrak{h} = \text{Lie}(H)$. Then $\theta$ restricts to a Cartan involution of $\mathfrak{h}$ giving the decomposition $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{t}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. The complex structure $J$ of $\mathfrak{p}$ induces that of $\mathfrak{h} \cap \mathfrak{p}$ by restriction giving the decomposition $(\mathfrak{h} \cap \mathfrak{p})_+ \oplus (\mathfrak{h} \cap \mathfrak{p})_-$ with $(\mathfrak{h} \cap \mathfrak{p})_+ = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{p}_+ = \sum_{i=1}^{n-r} \mathbb{C}X_i$ and $(\mathfrak{h} \cap \mathfrak{p})_- = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{p}_- = \sum_{i=1}^{n-r} \mathbb{C}X_i$. We introduce two tensors $\omega_H$ and $\eta$ as

$$\omega_H := \frac{-1}{2} \sum_{i=1}^{n-r} \omega_i \wedge \bar{\omega}_i, \quad \eta := \frac{-1}{2} \sum_{j=n-r+1}^{n} \omega_i \wedge \bar{\omega}_i = \omega - \omega_H - \frac{-1}{2} \omega_0 \wedge \bar{\omega}_0.$$

The coadjoint representation of $K$ on $\mathfrak{p}^*$ is extended to the unitary representation $\tau : K \to GL(\wedge \mathfrak{p}^\mathbb{C})$ in such a way that $\tau(k)(\alpha \wedge \beta) = \tau(k)\alpha \wedge \tau(k)\beta$ holds for all $\alpha, \beta \in \wedge \mathfrak{p}^\mathbb{C}$ and $k \in K$. The differential of $\tau$ is also denoted by $\tau$. 
The irreducible decomposition of the $K$-invariant subspaces $\wedge^{p,q} \mathfrak{p}_C^*$ is well-known.

**Lemma 1.** Let $p$, $q$ be non-negative integers such that $p + q \leq n$. Put

$$F_{p,q} := \{ \alpha \in \wedge^{p,q} \mathfrak{p}_C^* | \Lambda(\alpha) = 0 \}.$$

Then $F_{p,q}$ is an irreducible $K$-invariant subspace of $\wedge \mathfrak{p}_C^*$. The $K$-homomorphism $L$ induces a linear injection $\wedge^{p-1,q-1} \mathfrak{p}_C^* \rightarrow \wedge^{p,q} \mathfrak{p}_C^*$ whose image is the orthogonal complement of $F_{p,q}$ in $\wedge^{p,q} \mathfrak{p}_C^*$, i.e.,

$$\wedge^{p,q} \mathfrak{p}_C^* = F_{p,q} \bigoplus L(\wedge^{p-1,q-1} \mathfrak{p}_C^*).$$

The $\mathbb{R}$-subspace $a$ of $\mathfrak{g}$ generated by the element $Y_0 := X_0 + \bar{X}_0 \in \mathfrak{p}$ is a maximal abelian subalgebra in $\mathfrak{q} \cap \mathfrak{p}$ with $\mathfrak{q}$ the $(-1)$-eigenspace of $d\sigma$, the differential of $\sigma$. Since $(G, H)$ is a symmetric pair, by the general theory, the group $G$ is a union of double cosets $Ha_tK$ ($t \geq 0$) with

$$a_t := \exp(tY_0) = \text{diag}(1,_{n-1}, \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}), \quad t \in \mathbb{R}.$$ 

Put $A = \{a_t | t \in \mathbb{R}\}$. Let $M$ be the group of all the elements $k \in H \cap K$ such that $\text{Ad}(k)Y_0 = Y_0$ and put $M = M_0 \cap H$. Then

$$M = \{\text{diag}(u_1, u_2, u_0, u_0) | u_1 \in U(n - r), u_2 \in U(r - 1), u_0 \in U(1)\}.$$

**Proposition 1.** Let $p$ be an integer such that $0 < p < r$. Put

$$v_0^{(p)} = \frac{1}{n - p + 1} \sum_{j=0}^{p} c_{p-j}^{(p)} L^{p-j} \left( (n - p - j + 1)\eta^j + \frac{\sqrt{-1}}{2} j(r - j) \omega_0 \Lambda \bar{\omega}_0 \Lambda \eta^{j-1} \right),$$

$$v_1^{(p)} = \frac{-1}{p(n - 2p + 1)} \sum_{j=0}^{p} c_{p-j}^{(p)} L^{p-j} \left( (p - j)\eta^j + \frac{\sqrt{-1}}{2} j(r - j) \omega_0 \Lambda \bar{\omega}_0 \Lambda \eta^{j-1} \right)$$

with

$$c_{p-j}^{(p)} = (-1)^j \binom{p}{j} \binom{n - p + 1}{j} \binom{r - 1}{j}^{-1}, \quad 0 \leq j \leq p.$$ 

Then $F_{p,p}^M$ is a two dimensional space generated by $v_0^{(p)}$ and $v_1^{(p)}$.

For convenience, we put $v_0^{(0)} = 1$, $v_1^{(0)} = 0$; these are elements of $F_{0,0} = \mathbb{C}$.

2. Secondary spherical functions

Before we state the main theorem of this section, we put a lemma which is important not only here but in the 'global theory' to be developed in §4.

**Lemma 2.** For each integer $p$ with $1 \leq p \leq r$, there exists a unique holomorphic function $s \mapsto \nu_s^{(p)}$ on the domain $\mathbb{C} - L_p$ with

$$L_p = \{s \in \sqrt{-1}\mathbb{R} | |\text{Im}(s)| \leq 2\sqrt{(r-p)(n-p-r+2)}\}$$
which takes a positive real value for $s > 0$ and such that
\[ \{\nu_s^{(p)}\}^2 = s^2 + 4(r - p)(n - p - r + 2). \]
We have the functional equation $\nu_{-s}^{(p)} = -\nu_s^{(p)}$, $(s \in \mathbb{C} - L_p)$. If $\Re(s) > 0$, then we have $\Re(\nu_s^{(p)}) > \Re(\nu_s^{(p+1)}) > |\Re(s)|$.

For convenience, we put
\[ \mu = r - 1, \quad \lambda = n - 2r + 2. \]
Consider the holomorphic function
\[ d(s) := \prod_{p=1}^{r} \Gamma(\nu_s^{(p)})^{-1} \Gamma(2^{-1}(\nu_s^{(p)} - \lambda) + 1)^{-1}, \quad s \in \mathbb{C} - L_1 \]
and put
\[ D = \{ s \in \mathbb{C} - L_1 \mid d(s) \neq 0 \}, \quad \tilde{D} = \bigcap_{p=1}^{\mu} \{ s \in D \mid \Re(\nu_s^{(p)}) + \Re(\nu_s^{(p+1)}) > 4 \}. \]

**Theorem 1.** There exists a unique family of $C^\infty$-functions $\phi_s : G - HK \to \wedge^{\mu,\mu} \mathfrak{p}_{\mathbb{C}}^*$ $(s \in \tilde{D})$ with the following conditions.

(i) For each $g \in G - HK$, the function $s \mapsto \phi_s(g)$ is holomorphic.
(ii) $\phi_s$ has the $(H, K)$-equivariance
\[ \phi_s(hgk) = \tau(k)^{-1} \phi_s(g), \quad h \in H, k \in K, g \in G - HK. \]
(iii) $\phi_s$ satisfies the differential equation
\[ \Omega \phi_s(g) = (s^2 - \lambda^2) \phi_s(g), \quad g \in G - HK \]
(iv) We have
\[ \lim_{t \to +0} t^{2\mu} \phi_s(a_t) = (\omega - \omega_H)^\mu. \]
(v) If $\Re(s) > n$, then $\phi_s(a_t)$ decays exponentially as $t \to +\infty$.

We call the function $\phi_s$ the secondary spherical function.

### 2.1. Construction of $\phi_s$.
We set
\[ c(s) := \frac{\Gamma(s + 1) \Gamma(\mu + 2)}{\Gamma((s + n)/2 + 1) \Gamma((s - \lambda)/2 + 1)}, \]
and
\[ h_s(z) := 2F_1 \left( -\frac{s - n}{2} + 1, -\frac{s + \lambda}{2} + 1; \mu + 2; z \right), \]
\[ H_s(z) := 2F_1 \left( \frac{s - n}{2}, \frac{s + \lambda}{2}; s + 1; 1 - z \right). \]
Propostion 2. Let \( \{ \gamma_p \}_{p=0}^{\mu} \) be the sequence of real numbers defined by the recurrence relation:

\[
\gamma_\mu = \frac{1}{c_0^{(\mu)}}, \quad \gamma_j c_0^{(j)} = - \sum_{p=j+1}^{\mu} \gamma_p c_p^{(p)}, \quad (0 \leq j < \mu).
\]

Then we have

\[
\phi_s(ha_tk) = \mu r \left\{ \sum_{p=1}^{\mu} \frac{\gamma_{p}(n-p-r+1)p}{c(\nu_{s}^{(p+1)})c(\nu_{s}^{(p)})} \tau(k)^{-1}(\tilde{f}_{01}^{(p)}(s; \tanh^2 t) v_{0}^{(p)} + \tilde{f}_{11}^{(p)}(s; \tanh^2 t) v_{1}^{(p)}) \right.
\]
\[
\left. + \frac{\gamma_{0}}{c(\nu_{s}^{(1)})} \tilde{f}_{01}^{(0)}(s; \tanh^2 t) v_{0}^{(0)} \right\} \quad \forall (h, t, k) \in H \times (0, \infty) \times K.
\]

Here the functions \( \tilde{f}_{ij}^{(p)} \) are given as follows.

- For \( p > 0 \),
  \[
  \tilde{f}_{01}^{(p)}(s; z) = f_{00}^{(p)}(s; z) a_{01}^{(p)}(s; z) + f_{01}^{(p)}(s; z) a_{11}^{(p)}(s; z),
  \]
  \[
  \tilde{f}_{11}^{(p)}(s; z) = f_{10}^{(p)}(s; z) a_{01}^{(p)}(s; z) + f_{11}^{(p)}(s; z) a_{11}^{(p)}(s; z)
  \]
  with
  \[
  a_{01}^{(p)}(s; z) = -z^{-\mu}(1-z)^{(\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-1} H_{\nu_{s}^{(p+1)}}(z) H_{\nu_{s}^{(p)}}(z)
  \]
  \[
  + \int_{1}^{z} u^{-(\mu+1)}(1-u)^{(\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-2} (1+w) H_{\nu_{s}^{(p+1)}}(w) H_{\nu_{s}^{(p)}}(w) \, dw,
  \]
  \[
  a_{11}^{(p)}(s; z) = z(1-z)^{-(\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-1} h_{\nu_{s}^{(p+1)}}(z) H_{\nu_{s}^{(p)}}(z)
  \]
  \[
  - \int_{0}^{z} (1-u)^{-(\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-2} (1+w) h_{\nu_{s}^{(p+1)}}(w) H_{\nu_{s}^{(p)}}(w) \, dw
  \]
  and
  \[
  f_{10}^{(p)}(s; z) = (1-z)^{-(\nu_{s}^{(p+1)}+n)/2+1} H_{\nu_{s}^{(p+1)}}(z),
  \]
  \[
  f_{11}^{(p)}(s; z) = z^{-(\mu+1)}(1-z)^{(\nu_{s}^{(p+1)}+n)/2+1} H_{\nu_{s}^{(p+1)}}(z),
  \]
  \[
  f_{00}^{(p)}(s; z) = - \frac{(1-z)^{-(\nu_{s}^{(p+1)}+n)/2}}{(n-p-r+1)p}
  \]
  \[
  \times \left( \frac{d}{dz} + \frac{\nu_{s}^{(p+1)}+n-2p}{2} z + \frac{(r-p)(n-p+1)}{n-2p+1} (1-z) \right) H_{\nu_{s}^{(p+1)}}(z),
  \]
  \[
  f_{01}^{(p)}(s; z) = - \frac{z^{-(\mu+1)}(1-z)^{-(\nu_{s}^{(p+1)}+n)/2}}{(n-p-r+1)p}
  \]
  \[
  \times \left( \frac{d}{dz} + \frac{-\nu_{s}^{(p+1)}+n-2p}{2} z - \frac{p(n-p-r+1)}{n-2p+1} (1-z) \right) H_{\nu_{s}^{(p+1)}}(z).
  \]

- For \( p = 0 \),
  \[
  f_{01}^{(0)}(s; z) = \frac{2 z^{-\mu}(1-z)^{(\nu_{s}^{(1)}+n)/2}}{\nu_{s}^{(1)}+n} \, _2F_1\left( \frac{\nu_{s}^{(1)}-n}{2} + 1, \frac{\nu_{s}^{(1)}+\lambda}{2}; \nu_{s}^{(1)}+1; 1-z \right).
  \]
2.2. Some properties of the secondary spherical function.

**Theorem 2.** Let $\phi_s(s \in \mathring{D})$ be the secondary spherical function constructed in Theorem 1.

- There exist $\mu$ polynomial functions $a_\alpha(s)$ with values in $(\wedge^{\mu,\mu} \mathfrak{p}_{\mathbb{C}}^{*})^M$, positive number $\epsilon$ and $(\wedge^{\mu,\mu} \mathfrak{p}_{\mathbb{C}}^{*})^{M}$-valued holomorphic functions $b_i(s, z) (i = 0, 1, 2)$ on $\{(s, z)|s \in \mathring{D}, |z| < \epsilon\}$ such that

$$a_0(s) = (\omega - \omega_H)^\mu,$$

$$a_\alpha(-s) = a_\alpha(s), \quad \text{deg}(a_\alpha(s)) \leq 2\alpha$$

and such that

$$\phi_s(a_i) = \sum_{\alpha=0}^{\mu-1} \frac{a_\alpha(s)}{z^{\mu-\alpha}} + b_0(s, z) + b_1(s, z) \log z + b_2(s, z) z^{\mu+2}(\log z)^2,$$

$$s \in \mathring{D}, \quad z = \tanh^2 t \in (0, \epsilon).$$

- There exists a positive number $\epsilon'$, $(\wedge^{\mu,\mu} \mathfrak{p}_{\mathbb{C}}^{*})^M$-valued holomorphic functions $f^{(p)}(s; y) (0 \leq p \leq \mu)$ on $\{(s, y) | |y| < \epsilon', \Re(s) > n\}$ such that

$$\phi_s(a_i) = \sum_{p=0}^{\mu} y^{(\nu_s^{(p)} + n)/2} f^{(p)}(s; y), \quad \Re(s) > n, \quad y = \frac{1}{\cosh^2 t} \in (0, \epsilon').$$

2.3. The function $\psi_s$. For each $s \in \mathring{D}$, let us define the function $\psi_s : G - HK \rightarrow \wedge^{r, r} \mathfrak{p}_{\mathbb{C}}$ by

$$\psi_s(g) = \sum_{i,j=0}^{n-1} R_{X_i \overline{X}_j} \phi_s(g) \wedge \omega_i \wedge \overline{\omega}_j, \quad g \in G - HK.$$

**Theorem 3.**

- The function $\psi_s$ is $C^\infty$ on $G - HK$ and satisfies

$$\psi_s(hgk) = \tau(k)^{-1} \psi_s(g), \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.$$

- There exist $\mu$ $(\wedge^{r, r} \mathfrak{p}_{\mathbb{C}})^M$-valued polynomial functions $\tilde{c}_\alpha(s)$, positive number $\epsilon$ and $(\wedge^{r, r} \mathfrak{p}_{\mathbb{C}})^M$-valued holomorphic functions $d_i(s, z) (i = 0, 1, 2)$ on $\{(s, z)|s \in \mathring{D}, |z| < \epsilon\}$ such that

$$\tilde{c}_0(s) = -\sqrt{-1} \frac{(r-1)!}{2} \frac{(n-r)!}{(n-r)!} (\omega - \omega_H)^r,$$

$$\tilde{c}_\alpha(-s) = \tilde{c}_\alpha(s), \quad \text{deg}(\tilde{c}_\alpha(s)) \leq 2\alpha$$

and

$$\psi_s(a_t) = (s^2 - \lambda^2) \sum_{\alpha=0}^{\mu-1} \frac{\tilde{c}_\alpha(s)}{z^{\mu-\alpha}} + d_0(s, z) + d_1(s, z) \log z + d_2(s, z) z^\mu(\log z)^2,$$

$$s \in \mathring{D}, \quad z = \tanh^2 t \in (0, \epsilon).$$
3. Poisson series

Let $\Gamma$ be a discrete subgroup of $G$. We assume that $(G, H, \Gamma)$ is arranged as follows. There exists a connected reductive $\mathbb{Q}$-group $G$, a $\mathbb{Q}$-subgroup $H$ of $G$ and an arithmetic subgroup $\Delta$ of $G(\mathbb{Q})$ such that there exists a morphism of Lie groups from $G(\mathbb{R})$ onto $G$ with compact kernel which maps $H(\mathbb{R})$ onto $H$ and $\Delta$ onto $\Gamma$.

3.1. Invariant measures. Let $dk$ and $dk_0$ be the Haar measures of compact groups $K$ and $K_H$ with total volume 1. Then we can take a unique Haar measure $dg$ (resp. $dh$) of $G$ (resp. $H$) such that the quotient measure $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_0}$) corresponds to the measure on the symmetric space $G/K$ (resp. $H/K_H$) determined by the invariant volume form $\text{vol}_H$ (resp. $\text{vol}_H$).

Lemma 3. For any measurable functions $f$ on $G$ we have

$$\int_G f(g) \, dg = \int_H dh \int_K dk \int_0^\infty f(ha_tk) \varphi(t) \, dt$$

with $dt$ the usual Lebesgue measure on $\mathbb{R}$ and

$$\varphi(t) = 2c_r (\sinh t)^{2r-1} (\cosh t)^{2n-2r+1}, \quad c_r = \frac{\pi^r}{\mu!}.$$

3.2. Currents defined by Poisson series. Let $\mathcal{F}$ denote the set of the families of functions $\{\varphi_s\}_{s \in \tilde{D}}$ such that $\varphi_s = \partial_s \phi_s (s \in \tilde{D})$ or $\varphi_s = \partial_s \psi_s (s \in \tilde{D})$ with some differential operator $\partial_s$ with holomorphic coefficient on $\tilde{D}$.

For $\{\varphi_s\} \in \mathcal{F}$, let us introduce the Poisson series

(3) $$\tilde{P}(\varphi_s)(g) = \sum_{\gamma \in \Gamma_H \backslash \Gamma} \varphi_s(\gamma g) \quad g \in G,$$

which is the most basic object in our investigation. First of all, we discuss its convergence in a weak sense. Note that $\varphi_s$ takes its values in the finite dimensional Hilbert space $\wedge^r \mathfrak{p}_*^C$ with the norm $\|\alpha\| = (\alpha|\alpha)^{1/2}$.

Theorem 4. The function in $s$ defined by the integral

$$\tilde{P}(|\varphi_s|)(g) := \int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma_H \backslash \Gamma} \|\varphi_s(\gamma g)\| \right) dg$$

is locally bounded on $\text{Re}(s) > n$. For each $s$ with $\text{Re}(s) > n$, the series (3) converges absolutely almost everywhere in $g \in G$ to define an $L^1$-function on $\Gamma \backslash G$.

If $\Gamma$ is neat, then the quotient space $\Gamma \backslash G/K$ acquires a structure of complex manifold from the one on $G/K \cong \mathbb{D}$. Let $\pi : G/K \to \Gamma \backslash G/K$ be the natural projection. Let $A(\Gamma \backslash G/K)$ denote the space of $C^\infty$-differential forms on $\Gamma \backslash G/K$ and $A_c(\Gamma \backslash G/K)$ the
subspace of compactly supported forms. Given $\alpha \in A(\Gamma \backslash G / K)$, we have a unique $C^\infty$-function $\tilde{\alpha} : G \to \wedge \mathfrak{p}_C^*$ such that $\tilde{\alpha}(\gamma g k) = \tau(k)^{-1}\tilde{\alpha}(g)$, $(\gamma \in \Gamma, k \in K)$ and such that

$$\langle (\pi^\ast \alpha)(gK), (\wedge dL_g)(\xi_o) \rangle = \langle \tilde{\alpha}(g), \xi_o \rangle, \quad g \in G, \xi_o \in \wedge \mathfrak{p} = \wedge T_o(G/K)$$

holds. Here $L_g$ denotes the left translation on $G/K$ by the element $g$ and we identify $\mathfrak{p}$ with $T_o(G/K)$, the tangent space of $G/K$ at $o = eK$.

For any left $\Gamma$-invariant continuous function $f$ on $G$, put

$$\mathcal{J}_H(f; g) = \int_{\Gamma \backslash H} f(hg) \, dh, \quad g \in G.$$ 

We already discussed the convergence problem of this integral in [7, 3.2]. For convenience we recall the result. If $\Gamma$ is co-compact, we take a compact fundamental domain $\mathcal{S}^1$ for $\Gamma$ in $G$ and $t_{\mathcal{S}^1}$ the constant function 1. Hence $G = \Gamma \mathcal{S}^1$ in this case. If $\Gamma$ is not co-compact, then one can fix a complete set of representatives $P^i (1 \leq i \leq h)$ of $\Delta$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups in $G$ together with $\mathbb{Q}$-split tori $G_m \cong \mathbb{A}^i$ in the radical of $P^i$ such that an eigencharacter of $\text{Ad}(t) (t \in G_m)$ in the Lie algebra of $P^i$ is one of $t^j (j = 0, 1, 2)$. For each $i$, let $T^i$ and $N^i$ be the images in $G$ of $A^i(\mathbb{R})$ and the unipotent radical of $P^i(\mathbb{R})$ respectively. Then we can choose a Siegel domain $\mathcal{S}^i$ in $G$ with respect to the Iwasawa decomposition $G = N^i T^i K$ for each $i$ such that $G$ is a union of $\Gamma \mathcal{S}^i (1 \leq i \leq h)$. Let $t_{\mathcal{S}^i} : \mathcal{S}^i \to (0, \infty)$ be the function $t_{\mathcal{S}^i}(n, \xi_k) = t$, $(n, \xi_k, \in \mathcal{S}^i)$. Here $t_{\mathcal{S}^i}$ denote the image of $t \in G_m(\mathbb{R}) \cong A^i(\mathbb{R})$ in $T^i$.

Given $\delta \in (2rn^{-1}, 1)$, let $\mathcal{M}_\delta$ be the space of all left $\Gamma$-invariant $C^\infty$-functions $f : G \to \wedge \mathfrak{p}_C^*$ with the $K$-equivariance $f(gk) = \tau(k)^{-1}f(g)$ such that for any $\epsilon \in (0, \delta)$ and $D \in U(\mathfrak{g}_C)$ the estimation

$$\|R_D \varphi(g)\| \prec t_{\mathcal{S}^i}(g)^{(2-\epsilon)n}, \quad \forall g \in \mathcal{S}^i, \forall i$$

holds.

**Proposition 3.** Let $f \in \mathcal{M}_\delta$ with $\delta \in (2rn^{-1}, 1)$ and $D \in U(\mathfrak{g}_C)$.

- We have

$$\mathcal{I}_H(\|R_D f\|; a_t) \prec e^{(2-\epsilon)nt} \quad t \geq 0$$

for any $\epsilon \in (2rn^{-1}, \delta)$. The function $\mathcal{I}_H(f; g)$ is of class $C^\infty$, belongs to $C^\ast_r$ and

$$\mathcal{I}_H(R_D f; g) = R_D \mathcal{I}_H(f; g), \quad g \in G.$$

- For any $\{\varphi_s\} \in \mathcal{F}$, the integral

$$\int_{\Gamma \backslash G} |(\tilde{P}(\varphi_s)(g)|R_D f(g))| \, dg$$

is finite if $\text{Re}(s) > 3n - 2r$. We have

$$\int_{\Gamma \backslash G} (\tilde{P}(\varphi_s)(g)|R_D f(g)) \, dg = \int_0^\infty \rho(t) (\varphi_s(a_t)|R_D \mathcal{I}_H(f; a_t)) \, dt.$$
Proposition 4. There exists a unique current $P(\varphi_s)$ on $\Gamma \backslash G/K$ such that

$$\langle P(\varphi_s), \alpha \rangle = \int_{\Gamma \backslash G} (\tilde{P}(\varphi_s)(g)|\tilde{\alpha}(g))d\dot{g} = \int_0^\infty \varrho(t) (\varphi_s(a_t)|J_H(\tilde{\alpha}; a_t))dt, \quad \alpha \in A_c(\Gamma \backslash G/K)$$

Let $\partial_s$ be a holomorphic differential operator on $\tilde{D}$. Then for any $\alpha \in A_c(\Gamma \backslash G/K)$, the function $s \mapsto \langle P(\varphi_s), \alpha \rangle$ is holomorphic on $\tilde{D}$ and $\partial_s \langle P(\varphi_s), \alpha \rangle = \langle P(\partial_s \varphi_s), \alpha \rangle$.

**Definition**

For $s \in \mathbb{C}$ with $\text{Re}(s) > n$, we put

$$\tilde{G}_s := \tilde{P}(\phi_s), \quad \tilde{\Psi}_s := \tilde{P}(\psi_s),$$

$$G_s := P(\phi_s), \quad \Psi_s := P(\psi_s).$$

The current $G_s$ and $\Psi_s$ on $\Gamma \backslash G/K$ are of type $(r - 1, r - 1)$ and of type $(r, r)$ respectively.

**4. Spectral expansion**

In this section we investigate the spectral expansion of the functions $\delta_{j,s} \tilde{G}_s$ with

$$\delta_{j,s} := \frac{1}{j!} \left( - \frac{1}{2s} \frac{d}{ds} \right)^j , \quad j \in \mathbb{N}$$

to obtain a meromorphic continuation of the current-valued function $s \mapsto G_s$, which is already holomorphic on the half plane $\text{Re}(s) > n$.

**4.1. Spectral expansion.** In order to describe the spectral decomposition of the function $\delta_{\mu,s} G_s$, we need some preparations.

For $q > 0$, let $L^q_{\Gamma}(\tau)$ denote the Banach space of all measurable functions $f : G \to \bigwedge^d \mathfrak{p}_c^*$ such that $f(\gamma g k) = \tau(k)^{-1} f(g)$, $(\forall \gamma \in \Gamma, \forall k \in K)$ and $\int_{\Gamma \backslash G} \|f(g)\|^q d\dot{g} < \infty$. For $0 \leq d \leq n$, let $L^q_{\Gamma}(\tau)^{(d)}$ denote the subspace of those functions $f \in L^q_{\Gamma}(\tau)$ with values in $\bigwedge^{d,d} \mathfrak{p}_c^*$. The inner product of two functions $f_1$ and $f_2$ in $L^q_{\Gamma}(\tau)^{(d)}$ is given as

$$\langle f_1|f_2 \rangle = \int_{\Gamma \backslash G} (f_1(g)|f_2(g))d\dot{g}.$$  

Let $\tilde{\Delta}$ be the operator on $L^2_{\Gamma}(\tau)$ whose action on the smooth functions in $L^2_{\Gamma}(\tau)$ is induced by $-R_\Omega$. For each $0 \leq d \leq n$, let $\{\lambda_n^{(d)}\}_{n \in \mathbb{N}}$ be the increasing sequence of the eigenvalues of the bidegree $(d,d)$-part of $\tilde{\Delta}$ such that each eigenvalue occurs with its multiplicity. Choose an orthonormal system $\{\tilde{\alpha}_n^{(d)}\}_{n \in \mathbb{N}}$ in $L^2_{\Gamma}(\tau)^{(d)}$ consisting of automorphic forms such that $\tilde{\Delta} \tilde{\alpha}_n^{(d)} = \lambda_n^{(d)} \tilde{\alpha}_n^{(d)}$ for each $n$ and put $L^2_{\Gamma,\text{dis}}(\tau)^{(d)}$ to be the closed span of the functions $\tilde{\alpha}_n^{(d)}$ in $L^2_{\Gamma}(\tau)^{(d)}$. When $\Gamma$ is co-compact we have $L^2_{\Gamma,\text{dis}}(\tau)^{(d)} = L^2_{\Gamma}(\tau)^{(d)}$. Otherwise we need the Eisenstein series to describe the orthogonal complement of $L^2_{\Gamma,\text{dis}}(\tau)^{(d)}$.

Recall the parabolic subgroups $P^i$ used to construct the Siegel domains $\mathfrak{S}^i$ (see 3.2).

Let $P^i = M_0^i T^i N^i$ be its Langlands decomposition with $M_0^i := Z_K(T^i)$. For each $i$ let $\Gamma_{P^i} = \Gamma \cap P^i$ and $\Gamma_{M_0^i} = M_0^i \cap (\Gamma_{P^i} N^i)$. Then $\Gamma_{M_0^i}$ is just a finite subgroup of the compact group $M_0^i$. 
For a vector $u \in V_i^{(d)} := \left( \bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}}^* \right)^{\Gamma_0}$ and a complex number $s$, let us define the function $\varphi_s^i(u; g)$ on $G$ using the Iwasawa decomposition $G = N^i T^i K$ by

$$
\varphi_s^i(u; n_t \tilde{k}) = t^{s+n} \tau(k)^{-1} u, \quad n_t \in N^i, \; t > 0, \; k \in K.
$$

Then the Eisenstein series associated with $u$ is defined by the infinite series

$$(4) \quad \mathcal{E}^i(s; u; g) = \sum_{\gamma \in \Gamma \backslash \Gamma} \varphi_s^i(u; \gamma g), \quad g \in G$$

By the general theory, the series is convergent in $\text{Re}(s) > n$ normally and the function $g \mapsto \mathcal{E}^i(s; u; g)$ is an automorphic form on $\Gamma \backslash G$. Moreover there exists a family of linear maps $\mathcal{E}^i(s)$ from $V_i^{(d)}$ to the space of automorphic forms on $\Gamma \backslash G$, which depends meromorphically on $s \in \mathbb{C}$ and is holomorphic on the imaginary axis, such that $(\mathcal{E}^i(s)(u))(g) = \mathcal{E}^i(s; u; g)$ coincides with (4) when $\text{Re}(s) > n$. For each $1 \leq i \leq h$, let $\Omega_{M_0}$ be the Casimir element of $M_0$ corresponding to the invariant form $(X, Y)$ on its Lie algebra. Then if $u \in V_i^{(d)}$ is an eigenvector of $\tau(\Omega_{M_0})$ with eigenvalue $c \in \mathbb{C}$, then $R_0 \mathcal{E}^i(s; u) = (s^2 - n^2 + c) \mathcal{E}^i(s, u)$ for any $s \in \mathbb{C}$ where $\mathcal{E}^i(s)$ is regular.

**Lemma 4.** For $0 \leq p \leq d$ and $\epsilon \in \{0, 1\}$, let $W_i^{(d)}(p; \epsilon)$ be the eigenspace of $\tau(\Omega_{M_0})$ on $V_i^{(d)}$ corresponding to the eigenvalue $(2p - \epsilon)(2n - 2p + \epsilon)$. Then we have the orthogonal decomposition

$$
V_i^{(d)} = \bigoplus_{p=0}^{\mu} \bigoplus_{\epsilon \in \{0, 1\}} W_i^{(d)}(p; \epsilon).
$$

For each index $(d, i, p, \epsilon)$, fix an orthonormal basis $B_i^{(d)}(p; \epsilon)$ of the space $W_i^{(d)}(p; \epsilon)$.

### 4.2. Some properties of Eisenstein period.

**Proposition 5.** 
- For $1 \leq i \leq h$ and $u \in V_i^{(d)}$, there exists a unique $\bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}}^*$-valued meromorphic function $P_H^i(s; u)$ on $\mathbb{C}$ which is regular and has the value given by the absolutely convergent integral

$$
\mathcal{I}_H^i(E^i(s; u); e) \quad \text{at any regular point } s \in \mathbb{C} \text{ of } E^i(s; u)
$$

in $|\text{Re}(s)| < 1 - 2rn^{-1}$.

- Let $1 \leq i \leq h$ and $1 \leq p \leq d$. Then for any $u \in W_i^{(d)}(p; 1)$, we have $P_H^i(s; u) = 0$ identically.

### 4.3. Meromorphic continuation and functional equations.

Put $w := (\omega - \omega_H)^\mu$.

**Theorem 5.** Let $\text{Re}(s) > 3n - 2r$. Then there exists $\epsilon > 0$ such that the function $\delta_{\mu,s} \tilde{G}_s(g)$ belongs to the space $L_{-}^{2+\epsilon}(\tau)^{(\mu)}$. The spectral expansion of $\delta_{\mu,s} \tilde{G}_s$ is given as

$$
\delta_{\mu,s} \tilde{G}_s = \sum_{m=0}^{\infty} \frac{4(|J_H(\tilde{\alpha}_m^{(\mu)}); e|)}{\mu! (\lambda^2 - \lambda_m^{(\mu)} - s^2)^r} \tilde{\alpha}_m^{(\mu)}
$$

$$
+ \sum_{p=0}^{\mu} \frac{1}{4\pi \sqrt{-1}} \int_{\sqrt{-1} \mathbb{R}} \sum_{\zeta \in \mathbb{C}} \frac{4(|J_H(E^i(\zeta; u); e|)}{\mu! (\lambda^2 - (\nu^{(p+1)}_D)^2)^r} E^i(\zeta; u) d\zeta,
$$

where the summations in the right-hand side of this formula are convergent in $L_{-}^{2}(\tau)^{(\mu)}$. 

Let $\mathcal{X}_{\Gamma}(\tau)$ be the space of $C^\infty$-functions $\tilde{\beta} : G \to \Lambda \mathfrak{p}^*_\mathbb{C}$ with compact support modulo $\Gamma$ such that $\tilde{\beta}(\gamma g k) = \tau(k)^{-1}\tilde{\beta}(g) \quad (\forall \gamma \in \Gamma, \forall k \in K)$.

**Theorem 6.** Let $L_1$ be the interval on the imaginary axis defined by (1). Let $0 \leq j \leq \mu$. Then for each $\tilde{\beta} \in \mathcal{X}_{\Gamma}(\tau)$ the holomorphic function $s \mapsto \mathcal{S}_j(s, \tilde{\beta}) := \langle \delta_{j,s} \tilde{G}_s | \tilde{\beta} \rangle$ on $\text{Re}(s) > n$ has a meromorphic continuation to the domain $\mathbb{C} - L_1$. A point $s_0 \in \mathbb{C} - L_1$ with $\text{Re}(s_0) \geq 0$ is a pole of the meromorphic function $\mathcal{S}_j(s, \tilde{\beta})$ if and only if there exists an $m \in \mathbb{N}$ such that $(\mathcal{W} | \mathcal{J}_H(\tilde{\alpha}_m^{(\mu)}; e)) \neq 0$, $\langle \tilde{\alpha}_m^{(\mu)} | \tilde{\beta} \rangle \neq 0$ and $s_0^2 - \lambda^2 = -\lambda_m^{(\mu)}$. In this case, the function

$$\mathcal{S}_j(s, \tilde{\beta}) - \sum_{m \in \mathbb{N}; \lambda_m^{(\mu)} = \lambda^2 - s_0^2} \frac{4(\mathcal{W} | \mathcal{J}_H(\tilde{\alpha}_m^{(\mu)}; e)) \langle \tilde{\alpha}_m^{(\mu)} | \tilde{\beta} \rangle}{\mu! (s_0^2 - s^2)^{j+1}}$$

is holomorphic at $s = s_0$. We have the functional equation

$$\mathcal{S}_j(-s, \tilde{\beta}) + \mathcal{S}_j(s, \tilde{\beta}) = (-1)^\mu \delta_{j,s} \left( \sum_{p=0}^{\mu} \frac{\langle \tilde{\mathcal{E}}_p^{(\mu)}(\nu^{(p+1)}); e \rangle \langle \tilde{\alpha}_m^{(\mu)} | \tilde{\beta} \rangle}{2 \nu^{(p+1)}} \right).$$

with

$$(5) \quad \tilde{\mathcal{E}}_p^{(\mu)}(\nu; g) := \frac{4}{\mu!} \sum_{i=1}^{h} \sum_{\nu \in \mathbb{C}} \langle \mathcal{W} | \mathcal{J}_H^{i}(\nu^{(-1)}; u) \rangle \mathcal{E}^{i}(\nu; u; g), \quad g \in G, \nu \in \mathbb{C}.$$

### 5. Green Currents

We put the Kähler form $\omega$ on $\Gamma \backslash G/K$ such that $\tilde{\omega}(g) = \omega(\forall g \in G)$. The metric on $\Gamma \backslash G/K$ corresponding to $\omega$ defines the Laplacian $\triangle$, the Lefschetz operator and its adjoint $\Lambda$ acting on the space of forms and currents on $\Gamma \backslash G/K$.

#### 5.1. Currents defined by modular cycles.

Let $D$ be the image of the map $\Gamma_H \backslash H/K_H \to \Gamma \backslash G/K$ induced by the natural holomorphic inclusion $H/K_H \hookrightarrow G/K$. Then $D$, a closed complex analytic subset of $\Gamma \backslash G/K$, defines an $(r, r)$-current $\delta_D$ on $\Gamma \backslash G/K$ by the integration

$$(6) \quad \langle \delta_D, \alpha \rangle = \int_{D_{\text{ne}}} j^* \alpha, \quad \alpha \in A_c(\Gamma \backslash G/K).$$

Here $j : D \hookrightarrow \Gamma \backslash G/K$ is the natural inclusion and $D_{\text{ne}}$ is the smooth locus of $D$. Since $\delta_D$ is real and closed, it defines a cycle on $\Gamma \backslash G/K$ of real codimension $2r$ ([4, p.32–33]).

#### 5.2. Differential equations.

**Theorem 7.** Let $\text{Re}(s) > n$. Then we have

$$(\triangle + s^2 - \lambda^2)G_s = -4\Lambda \delta_D,$$

$$\triangle \Psi_s = (\lambda^2 - s^2)(\Psi_s - 2\sqrt{-1} \delta_D),$$

$$\partial\bar{\partial}G_s = \Psi_s - 2\sqrt{-1} \delta_D.$$
5.3. Main theorem. Let $A_{(2)}^{p,q}(\Gamma\backslash G/K)$ be the Hilbert space of the measurable $(p, q)$-forms on $\Gamma\backslash G/K$ with the finite $L^2$-norm $\|\alpha\| := \int_{\Gamma\backslash G/K} \alpha \wedge \overline{\alpha}$. For each $c \in \mathbb{C}$, let $A_{(2)}^{p,q}(\Gamma\backslash G/K; c)$ be the $c$-eigenspace of the Laplacian $\Delta$ acting on $A_{(2)}^{p,q}(\Gamma\backslash G/K)$. In particular, $\mathcal{H}_{(2)}^{p,q}(\Gamma\backslash G/K) := A_{(2)}^{p,q}(\Gamma\backslash G/K; 0)$ is the space of the harmonic $L^2$-forms of $(p, q)$-type. For each $p$, let $\mathcal{E}_{p}^{(\mu)}(\nu)$ be the $C^\infty$-form of $(\mu, \mu)$-type on $\Gamma\backslash G/K$ corresponding to the function $\tilde{\mathcal{E}}_{p}^{(\mu)}(\nu)$ on $G$ defined by (5). Then Theorem 6 immediately gives us the following theorem.

Theorem 8. There exists a meromorphic family of $(\mu, \mu)$-currents $G_{s}(s \in \mathbb{C} - L_{1})$ on $\Gamma\backslash G/K$ with the following properties.

- For $s \in \mathbb{C}$ with Re$(s) > n$, it is given by
  \[ \langle G_{s}, *\overline{\alpha}\rangle = \frac{1}{(r-1)\pi^{r}} \int_{0}^{\infty} \rho(t) (\phi_{s}(a_{t})|J_{H}(\tilde{\alpha} ; a_{t})) \, dt, \quad \alpha \in A_{c}(\Gamma\backslash G/K). \]

- A point $s_{0} \in \mathbb{C} - L_{1}$ with Re$(s) \geq 0$ is a pole of $G_{s}$ if and only if there exists an $L^2$-form $\alpha \in A_{(2)}^{r-1,r-1}(\Gamma\backslash G/K; (n-2r+2)^2 - s_{0}^2)$ such that
  \[ \int_{D} j^{*}(\omega \wedge \tilde{\alpha}) \neq 0. \]
  In this case $s_{0}$ is a simple pole with the residue
  \[ \text{Res}_{s=s_{0}} G_{s} = \frac{2}{s_{0}} \sum_{m} \left( \int_{D} j^{*}(\omega \wedge \tilde{\alpha}_{m}) \right) \cdot \alpha_{m}. \]
  Here $\{\alpha_{m}\}$ is an arbitrary orthonormal basis of $A_{(2)}^{r-1,r-1}(\Gamma\backslash G/K; (n-2r+2)^2 - s_{0}^2).

- The functional equation
  \[ G_{-s} - G_{s} = (-1)^{r-1} \sum_{p=0}^{r-1} \frac{\mathcal{E}_{p}^{(r-1)}(\nu_{s}^{(p+1)})}{2\nu_{s}^{(p+1)}}, \quad s \in \mathbb{C} - L_{1} \]
  holds.

Theorem 9. There exists a meromorphic family of $(r, r)$-currents $\Psi_{s}(s \in \mathbb{C} - L_{1})$ on $\Gamma\backslash G/K$ with the following properties.

- For $s \in \mathbb{C}$ with Re$(s) > n$, it is given by
  \[ \langle \Psi_{s}, *\overline{\alpha}\rangle = \frac{1}{(r-1)\pi^{r}} \int_{0}^{\infty} \rho(t) (\psi_{s}(a_{t})|J_{H}(\tilde{\alpha} ; a_{t})) \, dt, \quad \alpha \in A_{c}(\Gamma\backslash G/K). \]

- $\Psi_{s}$ is holomorphic at $s = n - 2r + 2$.

Definition

We define the $(r-1, r-1)$-current $\mathcal{G}$ on $\Gamma\backslash G/K$ to be the quarter of the constant term of the Laurent expansion of $G_{s}$ at $s = \lambda$. Namely, if $\{\alpha_{m}\}$ is any orthonormal basis of $\mathcal{H}_{(2)}^{r-1,r-1}(\Gamma\backslash G/K)$, then we put
\[
\mathcal{G}(x) = \frac{1}{4} \lim_{s \to \lambda} \left( G_{s}(x) - \frac{2}{n-2r+2} \sum_{m} \int_{D} j^{*}(\omega \wedge \tilde{\alpha}_{m}) \cdot \frac{\alpha_{m}(x)}{s - (n-2r+2)} \right).
\]
Theorem 10. We have the equation

$$dd_c \mathcal{G} = \frac{-1}{2} \Psi_{n-2r+2} + \delta_D, \quad \Delta \Psi_{n-2r+2} = 0$$

The current $\Psi_{n-2r+2}$ is represented by an element of $A^{r,r}(\Gamma \backslash G/K)$.

6. THE CURRENT $\Psi_s$

We remark that $*\text{vol}_H = \frac{1}{r!}(\omega - \omega_H)^r$ with $\text{vol}_H = \frac{1}{(n-r)!}\omega_H^{n-r}$ the 'volume form' of $H/K_H$.

Theorem 11. Let $\text{Re}(s) > 3n - 2r$. Then there exists $\epsilon > 0$ such that the function $\delta_{\mu,s}((s^2 - \lambda^2)^{-1}\tilde{\Psi}_s)$ belongs to the space $L^2+\epsilon_{\Gamma}^{(r)}(\tau)^{(r)}$. The spectral expansion of $\delta_{\mu,s}((s^2 - \lambda^2)^{-1}\tilde{\Psi}_s)$ is given as

$$\delta_{\mu,s}\left(\frac{\tilde{\Psi}_s}{s^2 - \lambda^2}\right) = \sum_{m=0}^{\infty} \frac{-2\sqrt{-1}(\text{vol}_H|_H(\tilde{\alpha}_m^{(r)}; e))}{(\lambda - \lambda_m^{(r)} - s^2)^r}$$

$$\sum_{p=0}^{r} \frac{1}{4\pi\sqrt{-1} \int_{-1}^{1} \sum_{i=1}^{H} \sum_{u \in B^{(r)}(p;0)} (*\text{vo}1_{H}|_H(\tilde{\alpha}_m^{(r)}; e)) E^{i}(\nu; u; g)}{(\zeta^2 - (\nu_s^{(p+1)})^2)^r},$$

where the summations in the right-hand side of this formula are convergent in $L^2_{\Gamma}(\tau)^{(r)}$.

Theorem 12. Let $L_1$ be the interval on the imaginary axis defined by (1). Let $0 \leq j \leq \mu$. Then for each $\beta \in \mathcal{K}_\Gamma(\tau)$ the holomorphic function $s \mapsto \mathcal{F}_j(s, \beta) := \langle \delta_{j,s}(s^2 - \lambda^2)^{-1}\tilde{\Psi}_s|\tilde{\beta}\rangle$ on $\text{Re}(s) > n$ has a meromorphic continuation to the domain $\mathbb{C} - L_1$. A point $s_0 \in \mathbb{C} - L_1$ with $\text{Re}(s_0) \geq 0$ is a pole of the meromorphic function $\mathcal{F}_j(s, \beta)$ if and only if there exists an $m \in \mathbb{N}$ such that $(*\text{vol}_H|_H(\tilde{\alpha}_m^{(r)}; e)) \neq 0$, $\langle \tilde{\alpha}_m^{(r)}|\tilde{\beta}\rangle \neq 0$ and $s_0^2 - \lambda^2 = -\lambda_m^{(r)}$. In this case, the function

$$\mathcal{F}_j(s, \beta) - \sum_{m \in \mathbb{N}, \lambda_m^{(r)} = \lambda^2 - s_0^2} \frac{2\sqrt{-1}(\text{vol}_H|_H(\tilde{\alpha}_m^{(r)}; e))\langle \tilde{\alpha}_m^{(r)}|\tilde{\beta}\rangle}{(s_0^2 - s^2)^{j+1}}$$

is holomorphic at $s = s_0$. We have the functional equation

$$\mathcal{F}_j(-s, \tilde{\beta}) - \mathcal{F}_j(s, \tilde{\beta}) = (-1)^\mu \delta_{j,s} \left( \sum_{p=0}^{r} \frac{\tilde{\mathcal{E}}_p^{(r)}(\nu_s^{(p+1)}; g)}{2 \nu_s^{(p+1)}} \right).$$

with

$$\tilde{\mathcal{E}}_p^{(r)}(\nu; g) := -2\sqrt{-1} \sum_{i=1}^{H} \sum_{u \in B^{(r)}(p;0)} (*\text{vol}_H|_H(E^{i}(-\nu; u); e)) E^{i}(\nu; u; g), \quad g \in G.$$
In this case $s_0$ is a simple pole with the residue
\[
\text{Res}_{s=s_0} \Psi_s = \frac{\sqrt{-1}(s_0^2 - (n - 2r + 2)^2)}{s_0} \sum_m \left( \int_D j^* \overline{\alpha}_m \right) \cdot \alpha_m.
\]
Here $\{\alpha_j\}$ is an arbitrary orthonormal basis of $A_{(2)}^{r,r}(\Gamma \backslash G/K ; (n-2r+2)^2 - s_0^2)$.

• We have
\[
\Psi_{n-2r+2} = 2\sqrt{-1} \sum_m \left( \int_D j^* \overline{\beta}_m \right) \cdot \beta_m
\]
with $\{\beta_m\}$ an arbitrary orthonormal basis of $\mathcal{H}_{(2)}^{r,r}(\Gamma \backslash G/K)$. In particular $\Psi_{n-2r+2} \in \mathcal{H}_{(2)}^{r,r}(\Gamma \backslash G/K)$.

The equations in Theorem 10 means the fundamental class $[\delta_D] \in H^{r,r}(\Gamma \backslash G/K ; \mathbb{C})$ of $D$ has the harmonic $L^2$-representative $\Psi_{n-2r+2}$.

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