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GREEN CURRENTS FOR MODULAR CYCLES IN ARITHMETIC QUOTIENTS OF COMPLEX HYPERBALLS

MASAO TSUZUKI

0. Introduction and Basic Notations

0.1. Introduction. Let $X$ be a complex manifold and $Y$ its analytic subvariety of codimension $r$. The Green current for $Y$ is defined to be a current $\mathcal{G}$ of $(r - 1, r - 1)$-type on $X$ such that $dd^c\mathcal{G} + \delta_Y$ is represented by a $C^\infty$-form of $(r, r)$-type on $X$. In the arithmetic intersection theory developed by Gillet and Soulé, the role played by the algebraic cycles in the conventional intersection theory is replaced with the arithmetic cycles. In a heuristic sense, the Green currents is regarded as the 'archimedean' ingredient of such arithmetic cycles ([2]).

Let us consider the case when $X$ is the quotient of a Hermitian symmetric domain $G/K$ by an arithmetic lattice $\Gamma$ in the semisimple Lie group $G$, and $Y$ is a modular cycle stemming from a modular imbedding $H/H \cap K \hookrightarrow G/K$, where $H$ is a reductive subgroup of $G$ such that $H \cap K$ is maximally compact in $H$. Then inspired by the classical works on the resolvent kernel functions of the Laplacian on Riemannian surfaces and also by a series of works of Miatello and Wallach ([5], [6]), T. Oda posed a plan to construct a Green current for $Y$ making use of a 'secondary spherical function' on $H\backslash G$, giving an evidence for divisorial case with some conjectures. Among many possible choices of the Green currents for a modular cycle $Y$, this construction may provide a way to fix a natural one. If $r = 1$, namely $Y$ is a modular divisor, we already obtained a satisfactory result by introducing the secondary spherical functions properly ([7]). Here we focus on the case when $G/K$ is an $n$-dimensional complex hyperball and $H/H \cap K$ is a complex sub-hyperball of codimension $r > 1$, and show that the same method also works well.

Thanks are due to Professor Takayuki Oda for his interest in this work, a constant encouragement and fruitful discussions which always stimulate the author.

0.2. Notations. The Lie algebra of a Lie group $G$ is denoted by Lie($G$). For a complex matrix $X = (x_{ij})_{ij}$, put $X^* := (\bar{x}_{ji})_{ij}$.

1. Invariant Tensors

Let $n$ and $r$ be integers such that $2 \leq r < n/2$.

Let us consider the two involutions $\sigma$ and $\theta$ in the Lie group $G = U(n, 1) := \{ g \in \text{GL}_{n+1}(\mathbb{C}) \mid g^* I_{n,1} g = I_{n,1} \}$ defined by $\theta(g) = I_{n,1} g I_{n,1}$ and $\sigma(g) = S g S$ respectively. Here $I_{n,1} := \text{diag}(I_n, -1)$ and $S = \text{diag}(I_{n-r}, -I_r, 1)$. Then $K := \{ g \in G \mid \theta(g) = g \} \cong U(n) \times U(1)$ is a maximal compact subgroup in $G$ and $H := \{ g \in G \mid \sigma(g) = g \} \cong U(n-r, 1) \times U(r)$ is a symmetric subgroup of $G$ such that $K_H := H \cap K \cong U(n-r) \times U(r) \times U(1)$ is maximally compact in $H$.
The Lie group $G$ acts transitively on the complex hyperball

$$\mathcal{D} = \{z = \sum_{i=1}^{n} z_i | \sum_{i=1}^{n} |z_i|^2 < 1\}$$

by the fractional linear transformation $g.z = \frac{g_{11}z + g_{12}}{g_{21}z + g_{22}}, g = \left[\begin{array}{cc}g_{11} & g_{12} \\ g_{21} & g_{22} \end{array}\right] \in G, \quad z \in \mathbb{C}^n$. (Here the matrix $g \in \text{GL}_{n+1}(\mathbb{C})$ is partitioned into blocks so that $g_{11}$ is an $n \times n$-matrix and $g_{22}$ is a scalar.) Since $K$ is the stabilizer of the origin $0 \in \mathcal{D}$, we have the identification $G/K \cong \mathcal{D}$ of $G$-manifolds assigning the point $z = g.0$ to $g \in G$. Then $H/K_H \subset G/K$ corresponds to the $H$-orbit of $0$ in $\mathcal{D}$, that is $\mathcal{D}^H := \{z \in \mathcal{D} | z_{n-r+1} = \cdots = z_n = 0\}$. In particular the real codimension of $H/K_H$ in $G/K$ is $2r$.

The Lie algebra $\mathfrak{g} := \text{Lie}(G)$ is realized in its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{n+1}(\mathbb{C})$ as an $\mathbb{R}$-subalgebra of all $X \in \mathfrak{g}_{n+1}(\mathbb{C})$ such that $X^* \mathbb{I}_{n+1} + \mathbb{I}_{n+1}X = 0_{n+1}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{t} := \text{Lie}(K)$ in $\mathfrak{g}$ with respect to the $G$-invariant, non-degenerate bi-linear form $\langle X, Y \rangle = 2^{-1}\text{tr}(XY)$ on $\mathfrak{g}$. For $1 \leq i, j \leq n+1$, let $E_{ij} := (\delta_{ui}\delta_{uj})_{uv} \in \mathfrak{g}_{n+1}(\mathbb{C})$ be the matrix unit. The operator $J := \text{ad}(\bar{Z}_0)|_{\mathfrak{p}}$ with $\bar{Z}_0 := \sum_{i=1}^{n} E_{i,1} - n \mathbb{E}_{n+1,n+1}$ gives a $K$-invariant complex structure of $\mathfrak{p}$, which induces the $K$-invariant decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_\pm$ the $(\pm \sqrt{-1})$-eigenspace of $J$ in $\mathfrak{p}_{\mathbb{C}}$. Since $\mathfrak{p}$ is identified with the tangent space of $G/K$ at $K$, we can extend $J$ to the $G$-invariant complex structure of $G/K$ making the identification $G/K \cong \mathcal{D}$ bi-holomorphic. Put $X_i := E_{i,n+1} (1 \leq i \leq n)$, $\mathfrak{X}_0 := E_{n,n+1}$. Then $\mathfrak{p}_+ = \sum_{i=0}^{n} \mathbb{C}X_i$, $\mathfrak{p}_- = \sum_{i=0}^{n} \mathbb{C}\mathfrak{X}_i$ with $\mathfrak{X}_i = E_{n+1,i}$, $\mathfrak{X}_0 = E_{n+1,n}$. Let $\{\omega_i\}$ and $\{\bar{\omega}_i\}$ be the basis of $\mathfrak{p}_+^*$ and $\mathfrak{p}_-^*$ dual to $\{X_i\}$ and $\{\mathfrak{X}_i\}$ respectively.

The exterior algebra $\wedge \mathfrak{p}_{\mathbb{C}}$ is decomposed to the direct sum of subspaces $\wedge^p \mathfrak{p}_{\mathbb{C}} := (\wedge^p \mathfrak{p}_+^*) \wedge (\wedge^p \mathfrak{p}_-^*)$ ($p, q \in \mathbb{N}$). Put

$$\omega := \frac{\sqrt{-1}}{2} \sum_{i=0}^{n-1} \omega_i \wedge \bar{\omega}_i \quad (\in \wedge^{1,1} \mathfrak{p}_{\mathbb{C}} \cap \wedge \mathfrak{p}^*), \quad \text{vol} := \frac{1}{n!} \omega^n \quad (\in \wedge^n \mathfrak{p}_{\mathbb{C}} \cap \wedge \mathfrak{p}^*).$$

The inner product $\langle X, Y \rangle$ on $\mathfrak{p}$ yields the Hermitian inner product $(\cdot | \cdot)$ of $\wedge \mathfrak{p}_{\mathbb{C}}$ in the standard way. Then the Hodge star operator $\ast$ is defined to be the $\mathbb{C}$-linear automorphism of $\wedge \mathfrak{p}_{\mathbb{C}}$ such that $\ast \alpha = \overline{\alpha}$ and such that $(\alpha | \beta) \text{vol} = \alpha \wedge \ast \beta$, $(\forall \alpha, \beta \in \wedge \mathfrak{p}_{\mathbb{C}})$. For $\alpha \in \wedge \mathfrak{p}_{\mathbb{C}}$, let us define the endomorphism $e(\alpha) : \wedge \mathfrak{p}_{\mathbb{C}} \to \wedge \mathfrak{p}_{\mathbb{C}}$ by $e(\alpha)\beta = \alpha \wedge \beta$. As usual, we have the Lefschetz operator $L := e(\omega)$ and its adjoint operator $\Lambda$ acting on the finite dimensional Hilbert space $\wedge \mathfrak{p}_{\mathbb{C}}$ ([8, Chap. V]).

Put $\mathfrak{h} = \text{Lie}(H)$. Then $\theta$ restricts to a Cartan involution of $\mathfrak{h}$ giving the decomposition $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{t}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. The complex structure $J$ of $\mathfrak{p}$ induces that of $\mathfrak{h} \cap \mathfrak{p}$ by restriction giving the decomposition $(\mathfrak{h} \cap \mathfrak{p})_c = (\mathfrak{h} \cap \mathfrak{p})_+ \oplus (\mathfrak{h} \cap \mathfrak{p})_-$ with $(\mathfrak{h} \cap \mathfrak{p})_+ = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{p}_+$, $\mathfrak{h}_{\mathbb{C}} = \sum_{i=1}^{n-r} \mathbb{C} \mathfrak{X}_i$ and $(\mathfrak{h} \cap \mathfrak{p})_- = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{p}_- = \sum_{i=1}^{n-r} \mathbb{C} \mathfrak{X}_i$. We introduce two tensors $\omega_H$ and $\eta$ as

$$\omega_H := \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-r} \omega_i \wedge \bar{\omega}_i, \quad \eta := \frac{\sqrt{-1}}{2} \sum_{j=n-r+1}^{n-1} \omega_i \wedge \bar{\omega}_i = \omega - \omega_H = \frac{\sqrt{-1}}{2} \omega_{0} \wedge \bar{\omega}_{0}.$$
The irreducible decomposition of the $K$-invariant subspaces $\wedge^{p,q} \mathfrak{p}_{\mathbb{C}}^{*}$ is well-known.

**Lemma 1.** Let $p$, $q$ be non-negative integers such that $p + q \leq n$. Put

$$F_{p,q} := \{ \alpha \in \wedge^{p,q} \mathfrak{p}_{\mathbb{C}}^{*} | \wedge(\alpha) = 0 \}.$$

Then $F_{p,q}$ is an irreducible $K$-invariant subspace of $\wedge^{p,q} \mathfrak{p}_{\mathbb{C}}^{*}$.

The $K$-homomorphism $L$ induces a linear injection $\wedge^{p-1,q-1} \mathfrak{p}_{\mathbb{C}}^{*} \rightarrow \wedge^{p,q} \mathfrak{p}_{\mathbb{C}}^{*}$ whose image is the orthogonal complement of $F_{p,q}$ in $\wedge^{p,q} \mathfrak{p}_{\mathbb{C}}^{*}$, i.e.,

$$\wedge^{p,q} \mathfrak{p}_{\mathbb{C}}^{*} = F_{p,q} \bigoplus L(\bigwedge^{p-1,q-1} \mathfrak{p}_{\mathbb{C}}^{*}).$$

The $\mathbb{R}$-subspace $a$ generated by the element $Y_0 := X_0 + \overline{X}_0 \in \mathfrak{p}$ is a maximal abelian subalgebra in $\mathfrak{q} \cap \mathfrak{p}$ with $\mathfrak{q}$ the (-1)-eigenspace of $da$, the differential of $\sigma$. Since $(G, H)$ is an asymmetric pair, by the general theory, the group $G$ is a union of double cosets $Ha_tK (t \geq 0)$ with

$$a_t := \exp(tY_0) = \text{diag} \left( \begin{array}{ll} \text{cosht} & \text{sinht} \\ \text{sinht} & \text{cosht} \end{array} \right), \quad t \in \mathbb{R}.$$

Put $A = \{a_t | t \in \mathbb{R} \}$. Let $M$ be the group of all the elements $k \in H \cap K$ such that $\text{Ad}(k)Y_0 = Y_0$ and put $M = M_0 \cap H$. Then

$$M = \{ \text{diag}(u_1, u_2, u_0, u_0) | u_1 \in U(n-r), u_2 \in U(r-1), u_0 \in U(1) \}.$$

**Proposition 1.** Let $p$ be an integer such that $0 < p < r$. Put

$$v_0^{(p)} = \frac{1}{n-p+1} \sum_{j=0}^{p} c_{p-j}^{(p)} L^{p-j} \left( (n-p-j+1)\eta^j + \frac{\sqrt{-1}}{2} j(r-j) \omega_0 \wedge \overline{\omega}_0 \wedge \eta^{j-1} \right),$$

$$v_1^{(p)} = \frac{-1}{p(n-2p+1)} \sum_{j=0}^{p} c_{p-j}^{(p)} L^{p-j} \left( (p-j)\eta^j + \frac{\sqrt{-1}}{2} j(r-j) \omega_0 \wedge \overline{\omega}_0 \wedge \eta^{j-1} \right)$$

with

$$c_{p-j}^{(p)} = (-1)^j \binom{p}{j} \binom{n-p+1}{j} \binom{r-1}{j}, \quad 0 \leq j \leq p.$$

Then $F_{p,p}^M$ is a two dimensional space generated by $v_0^{(p)}$ and $v_1^{(p)}$.

For convenience, we put $v_0^{(0)} = 1$, $v_1^{(0)} = 0$; these are elements of $F_{0,0} = \mathbb{C}$.

### 2. Secondary spherical functions

Before we state the main theorem of this section, we put a lemma which is important not only here but in the 'global theory' to be developed in §4.

**Lemma 2.** For each integer $p$ with $1 \leq p \leq r$, there exists a unique holomorphic function $s \mapsto v_s^{(p)}$ on the domain $\mathbb{C} - L_p$ with

$$L_p = \{ s \in \sqrt{-1} \mathbb{R} | |\text{Im}(s)| \leq 2\sqrt{(r-p)(n-p-r+2)} \}.$$
which takes a positive real value for $s > 0$ and such that
$$\{\nu_s^{(p)}\}^2 = s^2 + 4(r - p)(n - p - r + 2).$$

We have the functional equation $\nu_{-s}^{(p)} = -\nu_s^{(p)}$, \((s \in \mathbb{C} - L_p)\). If $\text{Re}(s) > 0$, then we have $\text{Re}(\nu_s^{(p)}) > \text{Re}(\nu_s^{(p+1)}) > |\text{Re}(s)|$.

For convenience, we put
$$\mu = r - 1, \quad \lambda = n - 2r + 2.$$

Consider the holomorphic function
$$d(s) := \prod_{p=1}^{r} \Gamma(\nu_s^{(p)})^{-1}\Gamma(2^{-1}(\nu_s^{(p)} - \lambda) + 1)^{-1}, \quad s \in \mathbb{C} - L_1$$
and put
$$D = \{s \in \mathbb{C} - L_1 \mid d(s) \neq 0\}, \quad \tilde{D} = \bigcap_{p=1}^{\mu}\{s \in D \mid \text{Re}(\nu_s^{(p)}) + \text{Re}(\nu_s^{(p+1)}) > 4\}.$$

**Theorem 1.** There exists a unique family of $C^\infty$-functions $\phi_s : G - HK \to \Lambda^{\mu, \mu} \mathfrak{p}_C^*$ \((s \in \tilde{D})\) with the following conditions.

(i) For each $g \in G - HK$, the function $s \mapsto \phi_s(g)$ is holomorphic.
(ii) $\phi_s$ has the $(H, K)$-equivariance
$$\phi_s(hgk) = \tau(k)^{-1}\phi_s(g), \quad h \in H, k \in K, g \in G - HK.$$
(iii) $\phi_s$ satisfies the differential equation
$$\Omega\phi_s(g) = (s^2 - \lambda^2)\phi_s(g), \quad g \in G - HK$$
(iv) We have
$$\lim_{t \to +0} t^{2\mu}\phi_s(a_t) = (\omega - \omega_H)^\mu.$$
(v) If $\text{Re}(s) > n$, then $\phi_s(a_t)$ decays exponentially as $t \to +\infty$.

We call the function $\phi_s$ the secondary spherical function.

2.1. **Construction of $\phi_s$.** We set
$$c(s) := \frac{\Gamma(s + 1)\Gamma(\mu + 2)}{\Gamma((s + n)/2 + 1)\Gamma((s - \lambda)/2 + 1)},$$
and
$$h_s(z) := _2F_1\left(-\frac{s - n}{2} + 1, -\frac{s + \lambda}{2} + 1; \mu + 2; z\right),$$
$$H_s(z) := _2F_1\left(-\frac{s - n}{2}, \frac{s + \lambda}{2}; s + 1; 1 - z\right).$$
Proposition 2. Let \( \{\gamma_p\}_{p=0}^\mu \) be the sequence of real numbers defined by the recurrence relation:

\[
\gamma_\mu = \frac{1}{c_0^{(\mu)}}, \quad \gamma_j c_0^{(j)} = - \sum_{p=j+1}^\mu \gamma_p c_{p-j}^{(p)}, \quad (0 \leq j < \mu).
\]

Then we have

\[
\phi_s(ha_tk) = \mu r \left\{ \sum_{p=1}^\mu \frac{\gamma_p (n-p-r+1)p}{c(\nu_{s}^{(p+1)})c(\nu_{s}^{(p)})} \tau(k)^{-1} \left( \tilde{f}_{01}^{(p)}(s;\tanh^2 t)v_0^{(p)} + \tilde{f}_{11}^{(p)}(s;\tanh^2 t)v_1^{(p)} \right) + \frac{\gamma_0}{c(\nu_{s}^{(1)})}\tilde{f}_{01}^{(0)}(s;\tanh^2 t)v_0^{(0)} \right\} \quad \forall (h, t, k) \in H \times (0, \infty) \times K.
\]

Here the functions \( \tilde{f}_{ij}^{(p)} \) are given as follows.

- For \( p > 0 \),

\[
\tilde{f}_{01}^{(p)}(s;z) = f_{00}^{(p)}(s;z)a_{01}^{(p)}(s;z) + f_{01}^{(p)}(s;z)a_{11}^{(p)}(s;z),
\]

\[
\tilde{f}_{11}^{(p)}(s;z) = f_{10}^{(p)}(s;z)a_{01}^{(p)}(s;z) + f_{11}^{(p)}(s;z)a_{11}^{(p)}(s;z)
\]

with

\[
a_{01}^{(p)}(s;z) = -z^{-\mu}(1-z)^{(\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-1}H_{\nu_{s}^{(p+1)}}(z)H_{\nu_{s}^{(p)}}(z)
\]

\[
+ \int_{1}^{z} w^{-(\mu+1)}(1-w)^{(\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-2}(1+w)H_{\nu_{s}^{(p+1)}}(w)H_{\nu_{s}^{(p)}}(w)dw,
\]

\[
a_{11}^{(p)}(s;z) = z(1-z)^{(-\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-1}h_{\nu_{s}^{(p+1)}}(z)H_{\nu_{s}^{(p)}}(z)
\]

\[
- \int_{0}^{z} (1-w)^{(-\nu_{s}^{(p+1)}+\nu_{s}^{(p)})/2-2}(1+w)h_{\nu_{s}^{(p+1)}}(w)H_{\nu_{s}^{(p)}}(w)dw
\]

and

\[
f_{00}^{(p)}(s;z) = (1-z)^{(-\nu_{s}^{(p+1)}+n)/2+1}h_{\nu_{s}^{(p+1)}}(z),
\]

\[
f_{11}^{(p)}(s;z) = z^{-(\mu+1)}(1-z)^{(\nu_{s}^{(p+1)}+n)/2+1}H_{\nu_{s}^{(p+1)}}(z),
\]

\[
f_{01}^{(p)}(s;z) = - \frac{(1-z)^{(-\nu_{s}^{(p+1)}+n)/2}}{(n-p-r+1)p} \times \left( z(1-z) \frac{d}{dz} + \frac{\nu_{s}^{(p+1)}+n-2p}{2}z + \frac{r-p(n-p+1)}{n-2p+1}(1-z) \right) h_{\nu_{s}^{(p+1)}}(z),
\]

\[
f_{01}^{(p)}(s;z) = - \frac{z^{-(\mu+1)}(1-z)^{(-\nu_{s}^{(p+1)}+n)/2}}{(n-p-r+1)p} \times \left( z(1-z) \frac{d}{dz} + \frac{-\nu_{s}^{(p+1)}+n-2p}{2}z - \frac{p(n-p-r+1)}{n-2p+1}(1-z) \right) H_{\nu_{s}^{(p+1)}}(z).
\]

- For \( p = 0 \),

\[
\tilde{f}_{01}^{(0)}(s;z) = \frac{2}{\nu_{s}^{(1)}+n} \frac{z^{-\mu}(1-z)^{(\nu_{s}^{(1)}+n)/2}}{\nu_{s}^{(1)}+n}\frac{n-r}{2} F_1 \left( \frac{\nu_{s}^{(1)}+n}{2} + 1, \frac{\nu_{s}^{(1)}+n}{2}; \nu_{s}^{(1)}+1; 1-z \right).
\]
2.2. Some properties of the secondary spherical function.

**Theorem 2.** Let \( \phi_s(s \in \tilde{D}) \) be the secondary spherical function constructed in Theorem 1.

- There exist \( \mu \) polynomial functions \( a_\alpha(s) \) with values in \((\bigwedge^{\mu,\mu} \mathfrak{p}_\mathbb{C}^*)^M\), positive number \( \epsilon \) and \((\bigwedge^{\mu,\mu} \mathfrak{p}_\mathbb{C}^*)^M\)-valued holomorphic functions \( b_i(s, z) \) \((i = 0, 1, 2)\) on \( \{(s, z)|s \in \tilde{D}, |z| < \epsilon\} \) such that
  
  \[ a_0(s) = (\omega - \omega_H)^\mu, \]
  
  \[ a_\alpha(-s) = a_\alpha(s), \quad \deg(a_\alpha(s)) \leq 2\alpha \]

  and such that

  \[ \phi_s(a_t) = \sum_{\alpha=0}^{\mu-1}\frac{a_\alpha(s)}{z^{\mu-\alpha}} + b_0(s; z) + b_1(s; z) \log z + b_2(s; z) z^{\mu+2}(\log z)^2, \]

  \[ s \in \tilde{D}, \ z = \tanh^2 t \in (0, \epsilon). \]

- There exists a positive number \( \epsilon' \), \((\bigwedge^{\mu,\mu} \mathfrak{p}_\mathbb{C}^*)^M\)-valued holomorphic functions \( f^{(p)}(s; y) \) \((0 \leq p \leq \mu)\) on \( \{(s, y)||y| < \epsilon', \Re(s) > n\} \) such that

  \[ \phi_s(a_t) = \sum_{p=0}^{\mu}y^{(\nu_s^{(p)}+n)/2}f^{(p)}(s; y), \quad \Re(s) > n, \ y = \frac{1}{\cosh^2 t} \in (0, \epsilon') \]

2.3. The function \( \psi_s \). For each \( s \in \tilde{D} \), let us define the function \( \psi_s : G -HK \rightarrow \bigwedge^{r,r} \mathfrak{p}_\mathbb{C} \)

by

\[ \psi_s(g) = \sum_{i,j=0}^{n-1}R_{X_i \overline{X}_j} \phi_s(g) \wedge \omega_i \wedge \overline{\omega}_j, \quad g \in G - HK. \]

**Theorem 3.**

- The function \( \psi_s \) is \( C^\infty \) on \( G - HK \) and satisfies

  \[ \psi_s(hgk) = \tau(k)^{-1}\psi_s(g), \quad \forall h \in H, \forall g \in G - HK, \forall k \in K. \]

- There exist \( \mu \) \((\bigwedge^{r,r} \mathfrak{p}_\mathbb{C}^*)^M\)-valued polynomial functions \( \tilde{c}_\alpha(s) \), positive number \( \epsilon \) and \((\bigwedge^{r,r} \mathfrak{p}_\mathbb{C}^*)^M\)-valued holomorphic functions \( d_i(s, z) \) \((i = 0, 1, 2)\) on \( \{(s, z)|s \in \tilde{D}, |z| < \epsilon\} \) such that

  \[ \tilde{c}_0(s) = -\frac{\sqrt{-1}}{2} \frac{(r-1)!}{(n-r)!} (\omega - \omega_H)^r, \]

  \[ \tilde{c}_\alpha(-s) = \tilde{c}_\alpha(s), \quad \deg(\tilde{c}_\alpha(s)) \leq 2\alpha \]

and

\[ \psi_s(a_t) = (s^2 - \lambda^2) \sum_{\alpha=0}^{\mu-1}\frac{\tilde{c}_\alpha(s)}{z^{\mu-\alpha}} + d_0(s; z) + d_1(s; z) \log z + d_2(s; z) z^{\mu}(\log z)^2, \]

\[ s \in \tilde{D}, \ z = \tanh^2 t \in (0, \epsilon). \]
There exists a positive number $\epsilon'$, $(\bigwedge^{r,r} p_{C}')^{M}$-valued holomorphic functions $g^{(p)}(s ; y) (0 \leq p \leq r)$ on $\{(s, y) | \text{Re}(s) > n, |y| < \epsilon'\}$ such that
\[\psi_{s}(a_{t}) = \sum_{p=0}^{r} y^{(u^{(p)}+n)/2} g^{(p)}(s ; y), \quad \text{Re}(s) > n, \quad y = \frac{1}{\cosh^{2}t} \in (0, \epsilon')\]

3. Poicaré series

Let $\Gamma$ be a discrete subgroup of $G$. We assume that $(G, H, \Gamma)$ is arranged as follows. There exists a connected reductive $\mathbb{Q}$-group $G$, a $\mathbb{Q}$-subgroup $H$ of $G$ and an arithmetic subgroup $\Delta$ of $G(\mathbb{Q})$ such that there exists a morphism of Lie groups from $G(\mathbb{R})$ onto $G$ with compact kernel which maps $H(\mathbb{R})$ onto $H$ and $\Delta$ onto $\Gamma$.

3.1. Invariant measures. Let $dk$ and $dk_{0}$ be the Haar measures of compact groups $K$ and $K_{H}$ with total volume 1. Then we can take a unique Haar measure $dg$ (resp. $dh$) of $G$ (resp. $H$) such that the quotient measure $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_{0}}$) corresponds to the measure on the symmetric space $G/K$ (resp. $H/K_{H}$) determined by the invariant volume form $\text{vol}$ (resp. $\text{vol}_{H}$).

Lemma 3. For any measurable functions $f$ on $G$ we have
\[
\int_{G} f(g) \, dg = \int_{H} dh \int_{K} dk \int_{0}^{\infty} f(ha_{t}k) \varrho(t) \, dt \]
with $dt$ the usual Lebesgue measure on $\mathbb{R}$ and
\[\varrho(t) = 2c_{r} (\sinh t)^{2r-1}(\cosh t)^{2n-2r+1}, \quad c_{r} = \frac{\pi^{r}}{\mu!}.\]

3.2. Currents defined by Poicaré series. Let $\mathcal{F}$ denote the set of the families of functions $\{\varphi_{s}\}_{s \in \overline{D}}$ such that $\varphi_{s} = \partial_{s} \phi_{s} (s \in \overline{D})$ or $\varphi_{s} = \partial_{s} \psi_{s} (s \in \overline{D})$ with some differential operator $\partial_{s}$ with holomorphic coefficient on $\overline{D}$.

For $\{\varphi_{s}\} \in \mathcal{F}$, let us introduce the Poincaré seires
\[\tilde{P}(\varphi_{s})(g) = \sum_{\gamma \in \Gamma \backslash H \backslash \Gamma} \varphi_{s}(\gamma g) \quad g \in G,\]
which is the most basic object in our investigation. First of all, we discuss its convergence in a weak sense. Note that $\varphi_{s}$ takes its values in the finite dimensional Hilbert space $\bigwedge^{\bullet} p_{C}$ with the norm $||\alpha|| = (\alpha|\alpha)^{1/2}$.

Theorem 4. The function in $s$ defined by the integral
\[\tilde{P}(||\varphi_{s}||)(g) := \int_{\Gamma \backslash G} \left( \sum_{\gamma \in \Gamma \backslash H \backslash \Gamma} ||\varphi_{s}(\gamma g)|| \right) \, dg\]
is locally bounded on $\text{Re}(s) > n$. For each $s$ with $\text{Re}(s) > n$, the series (3) converges absolutely almost everywhere in $g \in G$ to define an $L^{1}$-function on $\Gamma \backslash G$.

If $\Gamma$ is neat, then the quotient space $\Gamma \backslash G/K$ acquires a structure of complex manifold from the one on $G/K \cong \mathbb{D}$. Let $\pi : G/K \rightarrow \Gamma \backslash G/K$ be the natural projection. Let $A(\Gamma \backslash G/K)$ denote the space of $C^{\infty}$-differential forms on $\Gamma \backslash G/K$ and $A_{c}(\Gamma \backslash G/K)$ the
subspace of compactly supported forms. Given \( \alpha \in A(\Gamma \backslash G/K) \), we have a unique \( C^\infty \)-function \( \tilde{\alpha} : G \to \bigwedge p^*_C \) such that \( \tilde{\alpha}(\gamma gk) = \tau(k)^{-1}\tilde{\alpha}(g) \), \((\gamma \in \Gamma, k \in K)\) and such that
\[
\langle (\pi^*\alpha)(gK), (\bigwedge dL_g)(\xi_o) \rangle = \langle \tilde{\alpha}(g), \xi_o \rangle, \quad g \in G, \xi_o \in \bigwedge p = \bigwedge T_o(G/K)
\]
holds. Here \( L_g \) denotes the left translation on \( G/K \) by the element \( g \) and we identify \( p \) with \( T_o(G/K) \), the tangent space of \( G/K \) at \( o = eK \).

For any left \( \Gamma \)-invariant continuous function \( f \) on \( G \), put
\[
J_H(f ; g) = \int_{\Gamma \backslash H} f(hg) \, dh, \quad g \in G.
\]

We already discussed the convergence problem of this integral in [7, 3.2]. For convenience we recall the result. If \( \Gamma \) is co-compact, we take a compact fundamental domain \( \mathcal{S}^1 \) for \( \Gamma \) in \( G \) and \( t_{\mathcal{S}^1} \) the constant function 1. Hence \( G = \Gamma \mathcal{S}^1 \) in this case. If \( \Gamma \) is not co-compact, then one can fix a complete set of representatives \( P^i(1 \leq i \leq h) \) of \( \Delta \)-conjugacy classes of \( \mathbb{Q} \)-parabolic subgroups in \( G \) together with \( \mathbb{Q} \)-split tori \( \Gamma \mathbb{G}_m \cong A^i \) in the radical of \( P^i \) such that an eigencharacter of \( \text{Ad}(t) \) \((t \in \mathbb{G}_m)\) in the Lie algebra of \( P^i \) is one of \( t^j \) \((j = 0, 1, 2)\). For each \( i \), let \( T^i \) and \( N^i \) be the images in \( G \) of \( A^i(\mathbb{R}) \) and the unipotent radical of \( P^i(\mathbb{R}) \) respectively. Then we can choose a Siegel domain \( \mathcal{S}^i \) in \( G \) with respect to the Iwasawa decomposition \( G = N^iT^iK \) for each \( i \) such that \( G \) is a union of \( \Gamma \mathcal{S}^i(1 \leq i \leq h) \). Let \( t_{\mathcal{S}^i} : \mathcal{S}^i \rightarrow (0, \infty) \) be the function \( t_{\mathcal{S}^i}(n_i t_i k) = t \), \((n_i t_i k \in \mathcal{S}^i)\). Here \( t_i \) denote the image of \( t \in \mathbb{G}_m(\mathbb{R}) \cong A^i(\mathbb{R}) \) in \( T_i^i \).

Given \( \delta \in (2rn^{-1}, 1) \), let \( \mathcal{M}_\delta \) be the space of all left \( \Gamma \)-invariant \( C^\infty \)-functions \( f : G \rightarrow \bigwedge p^*_C \) with the \( K \)-equivariance \( f(gk) = \tau(k)^{-1}f(g) \) such that for any \( \epsilon \in (0, \delta) \) and \( D \in U(g_C) \) the estimation
\[
\|R_D \varphi(g)\| \prec t_{\mathcal{S}^i}(g)^{(2-\epsilon)n}, \quad \forall g \in \mathcal{S}^i, \forall i
\]
holds.

**Proposition 3.** Let \( f \in \mathcal{M}_\delta \) with \( \delta \in (2rn^{-1}, 1) \) and \( D \in U(g_C) \).

- **We have**
\[
J_H(\|R_D f\| ; a_t) \prec e^{(2-\epsilon)nt} \quad t \geq 0
\]
for any \( \epsilon \in (2rn^{-1}, \delta) \). The function \( J_H(f ; g) \) is of class \( C^\infty \), belongs to \( C^\infty_{\Gamma} \) and
\[
J_H(R_D f ; g) = R_D J_H(f ; g), \quad g \in G.
\]

- **For any \( \{\varphi_s\} \in \mathcal{F} \), the integral**
\[
\int_{\Gamma \backslash G} |(\tilde{P}(\varphi_s)(g)|R_D f(g))| \, dg
\]
is finite if \( \text{Re}(s) > 3n - 2r \). We have
\[
\int_{\Gamma \backslash G} \langle \tilde{P}(\varphi_s)(g)|R_D f(g) \rangle \, dg = \int_0^\infty e(t) \langle \varphi_s(a_t)|R_D J_H(f ; a_t) \rangle \, dt.
\]
Proposition 4. There exists a unique current $P(\varphi_s)$ on $\Gamma \backslash G/K$ such that

$$\langle \varphi_s, \dot{\tilde{\alpha}} \rangle = \int_{\Gamma \backslash G} (\tilde{P}(\varphi_s)(g)|\dot{\tilde{\alpha}}(g))d\dot{g}$$

$$= \int_0^\infty \rho(t) (\varphi_s(a_t)|\int_H(\tilde{\alpha};a_t))dt, \quad \alpha \in A_c(\Gamma \backslash G/K)$$

Let $\partial_s$ be a holomorphic differential operator on $\tilde{D}$. Then for any $\alpha \in A_c(\Gamma \backslash G/K)$, the function $s \mapsto \langle \varphi_s, \alpha \rangle$ is holomorphic on $\tilde{D}$ and $\partial_s \langle \varphi_s, \alpha \rangle = \langle \partial_s \varphi_s, \alpha \rangle$.

Definition

For $s \in \mathbb{C}$ with $\text{Re}(s) > n$, we put

$$\tilde{G}_s := \tilde{P}(\phi_s), \quad \tilde{\Psi}_s := \tilde{P}(\psi_s),$$

$$G_s := P(\psi_s), \quad \Psi_s := P(\psi_s).$$

The current $G_s$ and $\Psi_s$ on $\Gamma \backslash G/K$ are of type $(r-1, r-1)$ and of type $(r, r)$ respectively.

4. Spectral expansion

In this section we investigate the spectral expansion of the functions $\delta_{j,s} \tilde{G}_s$ with

$$\delta_{j,s} := \frac{1}{j!} \left( -\frac{1}{2s} \frac{d}{ds} \right)^j, \quad j \in \mathbb{N}$$

to obtain a meromorphic continuation of the current-valued function $s \mapsto G_s$, which is already holomorphic on the half plane $\text{Re}(s) > n$.

4.1. Spectral expansion. In order to describe the spectral decomposition of the function $\delta_{\mu,s} G_s$, we need some preparations.

For $q > 0$, let $L^q_{\Gamma}(\tau)$ denote the Banach space of all measurable functions $f : G \to \mathbb{C}$ such that $f(\gamma g k) = \tau(k)^{-1} f(g)$, $(\forall \gamma \in \Gamma, \forall k \in K)$ and $\int_{\Gamma \backslash G} \|f(g)\|^q d\dot{g} < \infty$. For $0 \leq d \leq n$, let $L^q_{\Gamma}(\tau)^{(d)}$ denote the subspace of those functions $f \in L^q_{\Gamma}(\tau)$ with values in $\bigwedge^{d,d} \mathfrak{p}_C^\ast$. The inner product of two functions $f_1$ and $f_2$ in $L^q_{\Gamma}(\tau)^{(d)}$ is given as $\langle f_1|f_2 \rangle = \int_{\Gamma \backslash G} (f_1(g)|f_2(g))d\dot{g}$. Let $\tilde{\Delta}$ be the operator on $L^2_{\Gamma}(\tau)$ whose action on the smooth functions in $L^2_{\Gamma}(\tau)$ is induced by $-R_\Omega$. For each $0 \leq d \leq n$, let $\{ \lambda_n^{(d)} \}_{n \in \mathbb{N}}$ be the increasing sequence of the eigenvalues of the bidegree $(d,d)$-part of $\tilde{\Delta}$ such that each eigenvalue occurs with its multiplicity. Choose an orthonormal system $\{ \tilde{\alpha}_n^{(d)} \}_{n \in \mathbb{N}}$ in $L^2_{\Gamma}(\tau)^{(d)}$ consisting of automorphic forms such that $\tilde{\Delta} \tilde{\alpha}_n^{(d)} = \lambda_n^{(d)} \tilde{\alpha}_n^{(d)}$ for each $n$ and put $L^2_{\Gamma,\tau}(\tau)^{(d)}$ to be the closed span of the functions $\tilde{\alpha}_n^{(d)}$ in $L^2_{\Gamma}(\tau)^{(d)}$. When $\Gamma$ is co-compact we have $L^2_{\Gamma,\tau}(\tau)^{(d)} = L^2_{\Gamma}((\tau)^{(d)})$. Otherwise we need the Eisenstein series to describe the orthogonal complement of $L^2_{\Gamma,\tau}(\tau)^{(d)}$.

Recall the parabolic subgroups $P^i$ used to construct the Siegel domains $\mathfrak{S}^i$ (see 3.2). Let $P^i = M^i_0 T^i N^i$ be its Langlands decomposition with $M^i_0 := Z_K(T^i)$. For each $i$ let $\Gamma_{P^i} = \Gamma \cap P^i$ and $\Gamma_{M^i_0} = M^i_0 \cap (\Gamma P^i N^i)$. Then $\Gamma_{M^i_0}$ is just a finite subgroup of the compact group $M^i_0$. 

For a vector \( \mathbf{u} \in V_{i}^{(d)} := (\wedge^{d,d} \mathfrak{p}^{*}_{\mathbb{C}})_{\Gamma \mathfrak{a}_{i}^{d}} \) and a complex number \( s \), let us define the function \( \varphi_{s}^{i}(\mathbf{u}; g) \) on \( G \) using the Iwasawa decomposition \( G = N^{i}T^{i}K \) by
\[
\varphi_{s}^{i}(\mathbf{u}; n_{i} t_{k}) = t^{s+n} \tau(k)^{-1} \mathbf{u}, \quad n_{i} \in N^{i}, \ t > 0, \ k \in K.
\]
Then the Eisenstein series associated with \( \mathbf{u} \) is defined by the infinite series
\[
E^{i}(s; \mathbf{u}; g) = \sum_{\gamma \in F_{p} \Gamma_{\mathbb{C}}} \varphi_{s}^{i}(\mathbf{u}; \gamma g), \quad g \in G
\]
By the general theory, the series is convergent in \( \text{Re}(s) > n \) normally and the function \( g \mapsto E^{i}(s; \mathbf{u}; g) \) is an automorphic form on \( \Gamma \backslash G \). Moreover there exists a family of linear maps \( E^{i}(s) \) from \( V_{i}^{(d)} \) to the space of automorphic forms on \( \Gamma \backslash G \), which depends meromorphically on \( s \in \mathbb{C} \) and is holomorphic on the imaginary axis, such that \( (E^{i}(s)(\mathbf{u}))(g) = E^{i}(s; \mathbf{u}; g) \) coincides with (4) when \( \text{Re}(s) > n \). For each \( 1 \leq i \leq h \), let \( \Omega_{M_{i}^{0}} \) be the Casimir element of \( M_{i}^{0} \) corresponding to the invariant form \( \langle X, Y \rangle \) on its Lie algebra. Then if \( \mathbf{u} \in V_{i}^{(d)} \) is an eigenvector of \( \tau(\Omega_{M_{i}^{0}}) \) with eigenvalue \( c \in \mathbb{C} \), then \( R_{0} E^{i}(s; \mathbf{u}) = (s^{2} - n^{2} + c) E^{i}(s, \mathbf{u}) \) for any \( s \in \mathbb{C} \) where \( E^{i}(s) \) is regular.

**Lemma 4.** For \( 0 \leq p \leq d \) and \( \epsilon \in \{0, 1\} \), let \( W_{i}^{(d)}(p; \epsilon) \) be the eigenspace of \( \tau(\Omega_{M_{i}^{0}}) \) on \( V_{i}^{(d)} \) corresponding to the eigenvalue \( (2p - \epsilon)(2n - 2p + \epsilon) \). Then we have the orthogonal decomposition
\[
V_{i}^{(d)} = \bigoplus_{p=0}^{\mu} \bigoplus_{\epsilon \in \{0, 1\}} W_{i}^{(d)}(p; \epsilon).
\]
For each index \( (d, i, p, \epsilon) \), fix an orthonormal basis \( B_{i}^{(d)}(p; \epsilon) \) of the space \( W_{i}^{(d)}(p; \epsilon) \).

**4.2. Some properties of Eisenstein period.**

**Proposition 5.**

- For \( 1 \leq i \leq h \) and \( \mathbf{u} \in V_{i}^{(d)} \), there exists a unique \( \wedge^{d,d} \mathfrak{p}^{*}_{\mathbb{C}} \)-valued meromorphic function \( \mathcal{P}_{H}(s; \mathbf{u}) \) on \( \mathbb{C} \) which is regular and has the value given by the absolutely convergent integral \( J_{H}(E^{i}(s; \mathbf{u}); e) \) at any regular point \( s \in \mathbb{C} \) of \( E^{i}(s; \mathbf{u}) \) in \( \{\text{Re}(s)\} < 1 - 2n^{-1} \).
- Let \( 1 \leq i \leq h \) and \( 1 \leq p \leq d \). Then for any \( \mathbf{u} \in W_{i}^{(d)}(p; 1) \), we have \( \mathcal{P}_{H}(s; \mathbf{u}) = 0 \) identically.

**4.3. Meromorphic continuation and functional equations.** Put \( \omega := (\omega - \omega_{H})^{\mu} \).

**Theorem 5.** Let \( \text{Re}(s) > 3n - 2r \). Then there exists \( \epsilon > 0 \) such that the function \( \delta_{\mu,s} \hat{G}_{s}(g) \) belongs to the space \( \mathcal{L}_{2}^{\text{reg}}(\tau)^{(\mu)} \). The spectral expansion of \( \delta_{\mu,s} \hat{G}_{s} \) is given as
\[
\delta_{\mu,s} \hat{G}_{s} = \sum_{m=0}^{\infty} \frac{4(w|J_{H}(\hat{\alpha}_{m}^{(\mu)}); e)}{\mu!(\lambda^{2} - \lambda_{m}^{(\mu)} - s^{2})^{r}} \hat{\alpha}_{m}^{(\mu)}
+ \sum_{p=0}^{\mu} \frac{1}{4\pi \sqrt{-1}} \int_{\gamma_{0}^{-1}} \sum_{i=1}^{h} \sum_{\mathbf{u} \in B_{i}^{(d)}(p; 0)} \frac{4(w|J_{H}(E^{i}(\zeta; \mathbf{u}); e))}{\mu!(\zeta^{2} - (\nu_{2}^{(p+1)})^{2})^{r}} E^{i}(\zeta; \mathbf{u}) \, d\zeta,
\]
where the summations in the right-hand side of this formula are convergent in \( \mathcal{L}_{2}^{\text{reg}}(\tau)^{(\mu)} \).
Let $\mathfrak{X}_\Gamma(\tau)$ be the space of $C^\infty$-functions $\tilde{\beta} : G \to \bigwedge \mathfrak{p}_\mathbb{C}$ with compact support modulo $\Gamma$ such that $\tilde{\beta}(\gamma gk) = \tau(k)^{-1}\tilde{\beta}(g) \quad (\forall \gamma \in \Gamma, \forall k \in K)$.

**Theorem 6.** Let $L_1$ be the interval on the imaginary axis defined by (1). Let $0 \leq j \leq \mu$. Then for each $\tilde{\beta} \in \mathfrak{X}_\Gamma(\tau)$ the holomorphic function $s \mapsto \mathcal{S}_j(s, \tilde{\beta}) := \langle \delta_j s \tilde{G}_s | \tilde{\beta} \rangle$ on $\text{Re}(s) > n$ has a meromorphic continuation to the domain $\mathbb{C} - L_1$. A point $s_0 \in \mathbb{C} - L_1$ with $\text{Re}(s_0) \geq 0$ is a pole of the meromorphic function $\mathcal{S}_j(s, \beta)$ if and only if there exists an $m \in \mathbb{N}$ such that $(w | J_H(\tilde{\alpha}_m^{(\mu)} ; e)) \neq 0$, $\langle \tilde{\alpha}_m^{(\mu)} | \tilde{\beta} \rangle \neq 0$ and $s_0^2 - \lambda^2 = -\lambda_m^{(\mu)}$. In this case, the function

$$\mathcal{S}_j(s, \beta) - \sum_{m \in \mathbb{N}, \lambda_m^{(\mu)} = \lambda^2 - s_0^2} \frac{4(w | J_H(\tilde{\alpha}_m^{(\mu)} ; e)) \langle \tilde{\alpha}_m^{(\mu)} | \tilde{\beta} \rangle}{\mu! (s_0^2 - s^2)^{j+1}}$$

is holomorphic at $s = s_0$. We have the functional equation

$$\mathcal{S}_j(-s, \tilde{\beta}) - \mathcal{S}_j(s, \tilde{\beta}) = (-1)^\mu \delta_j, s \left( \sum_{p=0}^{\mu} \frac{\langle \tilde{\mathcal{E}}_p^{(\mu)}(\nu_s^{(p+1)}) | \tilde{\beta} \rangle}{2 \nu_s^{(p+1)}} \right).$$

5. **GREEN CURRENTS**

We put the Kähler form $\omega$ on $\Gamma \backslash G/K$ such that $\tilde{\omega}(g) = \omega \quad (\forall g \in G)$. The metric on $\Gamma \backslash G/K$ corresponding to $\omega$ defines the Laplacian $\triangle$, the Lefschetz operator and its adjoint $\Lambda$ acting on the space of forms and currents on $\Gamma \backslash G/K$.

5.1. **Currents defined by modular cycles.** Let $D$ be the image of the map $\Gamma_H \backslash H/K_H \to \Gamma \backslash G/K$ induced by the natural holomorphic inclusion $H/K_H \hookrightarrow G/K$. Then $D$, a closed complex analytic subset of $\Gamma \backslash G/K$, defines an $(r, r)$-current $\delta_D$ on $\Gamma \backslash G/K$ by the integration

$$\langle \delta_D, \alpha \rangle = \int_D j^* \alpha, \quad \alpha \in \Lambda_c(\Gamma \backslash G/K).$$

Here $j : D \hookrightarrow \Gamma \backslash G/K$ is the natural inclusion and $D_{\text{ne}}$ is the smooth locus of $D$. Since $\delta_D$ is real and closed, it defines a cycle on $\Gamma \backslash G/K$ of real codimension $2r$ ([4, p.32–33]).

5.2. **Differential equations.**

**Theorem 7.** Let $\text{Re}(s) > n$. Then we have

$$(\triangle + s^2 - \lambda^2)G_s = -4\Lambda \delta_D,$$

$$\triangle \Psi_s = (\lambda^2 - s^2)(\Psi_s - 2\sqrt{-1}\delta_D),$$

$$\partial \bar{\partial} G_s = \Psi_s - 2\sqrt{-1} \delta_D.$$
5.3. Main theorem. Let $A_{(2)}^{p,q}(\Gamma \backslash G/K)$ be the Hilbert space of the measurable $(p, q)$-forms on $\Gamma \backslash G/K$ with the finite $L^2$-norm $||\alpha|| := \int_{\Gamma \backslash G/K} \alpha \Lambda^* \overline{\alpha}$. For each $c \in \mathbb{C}$, let $A_{(2)}^{p,q}(\Gamma \backslash G/K ; c)$ be the $c$-eigenspace of the Laplacian $\Delta$ acting on $A_{(2)}^{p,q}(\Gamma \backslash G/K)$. In particular, $\mathcal{H}_{(2)}^{p,q}(\Gamma \backslash G/K) := A_{(2)}^{p,q}(\Gamma \backslash G/K ; 0)$ is the space of the harmonic $L^2$-forms of $(p, q)$-type. For each $p$, let $\mathcal{E}_{p}^{(\mu)}(\nu)$ be the $C^\infty$-form of $(\mu, \mu)$-type on $\Gamma \backslash G/K$ corresponding to the function $\tilde{\mathcal{E}}_{p}^{(\mu)}(\nu)$ on $G$ defined by (5). Then Theorem 6 immediately gives us the following theorem.

**Theorem 8.** There exists a meromorphic family of $(\mu, \mu)$-currents $G_s (s \in \mathbb{C} - L_1)$ on $\Gamma \backslash G/K$ with the following properties.

- For $s \in \mathbb{C}$ with $\text{Re}(s) > n$, it is given by
  $$\langle G_s, *\bar{\alpha} \rangle = \frac{1}{(r - 1)\pi^r} \int_0^\infty \varrho(t) (\phi_s(a_t)|\mathcal{J}_H(\tilde{\alpha} ; a_t)) \, dt, \quad \alpha \in A_c(\Gamma \backslash G/K).$$

- A point $s_0 \in \mathbb{C} - L_1$ with $\text{Re}(s) \geq 0$ is a pole of $G_s$ if and only if there exists an $L^2$-form $\alpha \in A_{(2)}^{r-1,r-1}(\Gamma \backslash G/K ; (n - 2r + 2)^2 - s_0^2)$ such that
  $$\int_D j^*(\omega \wedge \bar{\alpha}) \neq 0.$$  
  In this case $s_0$ is a simple pole with the residue
  $$\text{Res}_{s=s_0} G_s = \frac{2}{s_0} \sum_m \left( \int_D j^*(\omega \wedge \bar{\alpha}_m) \right) \cdot \alpha_m.$$  
  Here $\{\alpha_m\}$ is an arbitrary orthonormal basis of $A_{(2)}^{r-1,r-1}(\Gamma \backslash G/K ; (n - 2r + 2)^2 - s_0^2)$.

- The functional equation
  $$G_{-s} - G_s = (-1)^{r-1} \sum_{p=0}^{r-1} \frac{\mathcal{E}_{p}^{(r-1)}(\nu_{s}^{(p+1)})}{2\nu_{s}^{(p+1)}},$$  
  $s \in \mathbb{C} - L_1$

holds.

**Theorem 9.** There exists a meromorphic family of $(r, r)$-currents $\Psi_s (s \in \mathbb{C} - L_1)$ on $\Gamma \backslash G/K$ with the following properties.

- For $s \in \mathbb{C}$ with $\text{Re}(s) > n$, it is given by
  $$\langle \Psi_s, *\bar{\alpha} \rangle = \frac{1}{(r - 1)\pi^r} \int_0^\infty \varrho(t) (\psi_s(a_t)|\mathcal{J}_H(\tilde{\alpha} ; a_t)) \, dt, \quad \alpha \in A_c(\Gamma \backslash G/K).$$

- $\Psi_s$ is holomorphic at $s = n - 2r + 2$.

**Definition**

We define the $(r - 1, r - 1)$-current $G$ on $\Gamma \backslash G/K$ to be the quarter of the constant term of the Laurent expansion of $G_s$ at $s = \lambda$. Namely, if $\{\alpha_m\}$ is any orthonormal basis of $\mathcal{H}_{(2)}^{r-1,r-1}(\Gamma \backslash G/K)$, then we put

$$G(x) = \frac{1}{4} \lim_{s \rightarrow \lambda} \left( G_s(x) - \frac{2}{n - 2r + 2} \sum_m \int_D j^*(\omega \wedge \bar{\alpha}_m) \cdot \frac{\alpha_m(x)}{s - (n - 2r + 2)} \right).$$
Theorem 10. We have the equation
\[ dd_{c} \mathcal{G} = \frac{\sqrt{-1}}{2} \Psi_{n-2r+2} + \delta_{D}, \quad \Delta \Psi_{n-2r+2} = 0 \]
The current \( \Psi_{n-2r+2} \) is represented by an element of \( A^{\tau} \Gamma \backslash G / K \).

6. THE CURRENT \( \Psi_{s} \)

We remark that \( *\text{vol}_{H} = \frac{1}{r!} (\omega - \omega_{H})^{r} \) with \( \text{vol}_{H} = \frac{1}{(n-r)!} \omega_{H}^{n-r} \) the 'volume form' of \( H / K_{H} \).

Theorem 11. Let \( \text{Re}(s) > 3n - 2r \). Then there exists \( \epsilon > 0 \) such that the function \( \delta_{\mu,s}(s^{2} - \lambda^{2})^{-1} \tilde{\Psi}_{s} \) belongs to the space \( L^{2+\epsilon}_{\Gamma} (\tau)^{(r)} \). The spectral expansion of \( \delta_{\mu,s}(s^{2} - \lambda^{2})^{-1} \tilde{\Psi}_{s} \) is given as
\[
\delta_{\mu,s}(\frac{\tilde{\Psi}_{s}}{s^{2}-\lambda^{2}}) = \sum_{m=0}^{\infty} \frac{-2\sqrt{-1}(*\text{vol}_{H}[\mathcal{H}(\tilde{\alpha}_{m}^{(r)} ; e))}{(\lambda^{2} - \lambda_{m}^{(r)} - s^{2})^{r}}, \tilde{\alpha}_{m}^{(r)} + \sum_{p=0}^{r} \frac{1}{4\pi\sqrt{-1}} \int_{\sqrt{-1}R} \sum_{i=1}^{h} \sum_{u \in B^{(r)}(pj0)} (*\text{vol}_{H}[\mathcal{O}^{i}(-\overline{\nu};u);e)) \mathcal{O}^{i}(\nu;u;g) d\zeta,
\]
where the summations in the right-hand side of this formula are convergent in \( L^{2}_{\Gamma} (\tau)^{(r)} \).

Theorem 12. Let \( L_{1} \) be the interval on the imaginary axis defined by (1). Let \( 0 \leq j \leq \mu \). Then for each \( \tilde{\gamma} \in \mathcal{K}_{\Gamma}(\tau) \) the holomorphic function \( s \mapsto \mathcal{F}_{j}(s, \tilde{\beta}) := (\delta_{j,s}(s^{2} - \lambda^{2})^{-1} \tilde{\Psi}_{s}) \tilde{\beta} \) on \( \text{Re}(s) > n \) has a meromorphic continuation to the domain \( \mathbb{C} - L_{1} \). A point \( s_{0} \in \mathbb{C} - L_{1} \) with \( \text{Re}(s_{0}) \geq 0 \) is a pole of the meromorphic function \( \mathcal{F}_{j}(s, \tilde{\beta}) \) if and only if there exists an \( m \in \mathbb{N} \) such that \( (*\text{vol}_{H}[\mathcal{H}(\tilde{\alpha}_{m}^{(r)} ; e)) \neq 0 \), \( \langle \tilde{\alpha}_{m}^{(r)} | \tilde{\beta} \rangle \neq 0 \) and \( s_{0}^{2} - \lambda^{2} = -\lambda_{m}^{(r)} \). In this case, the function
\[
\mathcal{F}_{j}(-s, \tilde{\beta}) - \mathcal{F}_{j}(s, \tilde{\beta}) = (-1)^{\mu} \delta_{j,s} \left( \sum_{p=0}^{r} \frac{\langle \tilde{\alpha}_{p}^{(r)}(-\nu;u) | \tilde{\beta} \rangle}{2\nu_{s}^{(p+1)}} \right).
\]
is holomorphic at \( s = s_{0} \). We have the functional equation
\[
\mathcal{F}_{j}(-s, \tilde{\beta}) - \mathcal{F}_{j}(s, \tilde{\beta}) = (-1)^{\mu} \delta_{j,s} \left( \sum_{p=0}^{r} \frac{\langle \tilde{\alpha}_{p}^{(r)}(-\nu;u) | \tilde{\beta} \rangle}{2\nu_{s}^{(p+1)}} \right).
\]

Theorem 13. A point \( s_{0} \in \mathbb{C} - L_{1} \) with \( \text{Re}(s) \geq 0 \), \( s_{0} \neq n - 2r + 2 \) is a pole of \( \Psi_{s} \) if and only if there exists an \( L^{2} \)-form \( \alpha \in A_{(2)}^{\tau} (\Gamma \backslash G / K ; (n-2r+2)^{2} - s_{0}^{2}) \) such that
\[
\int_{D} j^{*} \alpha \neq 0.
\]
In this case $s_0$ is a simple pole with the residue
\[
\text{Res}_{s=s_0} \Psi_s = \frac{\sqrt{-1}(s_0^2 - (n - 2r + 2)^2)}{s_0} \sum_m \left( \int_D j^* \alpha_m \right) \cdot \alpha_m.
\]
Here $\{\alpha_j\}$ is an arbitrary orthonormal basis of $A_{(2)}^{r,r}(\Gamma\backslash G/K; (n-2r+2)^2-s_0^2)$.

- We have
\[
\Psi_{n-2r+2} = 2\sqrt{-1} \sum_m \left( \int_D j^* \beta_m \right) \cdot \beta_m
\]
with $\{\beta_m\}$ an arbitrary orthonormal basis of $\mathcal{H}_{(2)}^{r,r}(\Gamma\backslash G/K)$. In particular $\Psi_{n-2r+2} \in \mathcal{H}_{(2)}^{r,r}(\Gamma\backslash G/K)$.

The equations in Theorem 10 means the fundamental class $[\delta_D] \in \mathcal{H}^{r,r}(\Gamma\backslash G/K; \mathbb{C})$ of $D$ has the harmonic $L^2$-representative $\Psi_{n-2r+2}$.

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