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Modular adjacency algebras of the Hamming association schemes

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Abstract

The adjacency algebra of an association scheme is defined over an arbitrary field. This is always semisimple over a field of characteristic 0, but not semisimple over a field of prime characteristic $p$, in general. The structure of the adjacency algebra over a field of prime characteristic was not studied enough before now. Therefore, we considered the structure of the modular adjacency algebra of the Hamming scheme $H(n, q)$, that is one of the most basic and important association schemes.

In this paper, we will decide the structure of the adjacency algebra of $H(n, q)$ over any field for any $n$ and $q$, and describe the algebra as a factor algebra of a polynomial ring.

1 Introduction

In this paper, we consider the modular adjacency algebra of the Hamming association scheme $H(n, q)$. The modular adjacency algebra means an adjacency algebra over a positive characteristic field. For any prime $p$ such that $p 
mid q$, the adjacency algebra of $H(n, q)$ over a field of characteristic $p$ is semisimple (see [2, Theorem 2.3], [1, Theorem 1.1] and [5, Theorem 4.2]). For each prime $p$, the prime
field $\mathbb{F}_p$ of characteristic $p$ is a splitting field for the adjacency algebra of $H(n, p)$ over $\mathbb{F}_p$ (see [4, Theorem 3.4, Corollary 3.5]). For all prime $p$ such that $p \mid q$, $\mathbb{F}_p H(n, p) \cong \mathbb{F}_p H(n, q)$ (see §2.3). Therefore it is enough to decide the structure of $\mathbb{F}_p H(n, p)$ for all prime $p$, for deciding the structure of the modular adjacency algebra of any $H(n, q)$ over any field. It is known that the algebra $\mathbb{F}_p H(n, p)$ is commutative and local, and that any local commutative algebra is isomorphic to a factor algebra of a polynomial ring.

2 Preparation

For the definitions in this section, refer to [2].

2.1 Association schemes

Let $X$ be a finite set with cardinality $n$. We define $R_0 := \{(x, x) \mid x \in X\}$. Let $R_i \subseteq X \times X$ be given. We set $R_i^* := \{(z, y) \mid (y, z) \in R_i\}$. Let $G$ be a partition of $X \times X$ such that $R_0 \in G$ and the empty set $\emptyset \notin G$, and assume that, $R_i^* \in G$ for each $R_i \in G$. Then, the pair $(X, G)$ will be called an association scheme if, for all $R_i, R_j, R_k \in G$, there exists a cardinal number $p_{ijk}$ such that, for all $y, z \in X$

$$(y, z) \in R_k \Rightarrow \#\{x \in X \mid (y, x) \in R_i, (x, z) \in R_j\} = p_{ijk}.$$ 

The elements of $\{p_{ijk}\}$ will be called the intersection numbers of $(X, G)$.

For each $R_i \in G$, we define the $n \times n$ matrix $A_i$ indexed by the elements of $X$,

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$
and this matrix $A_i$ will be called the adjacency matrix of $R_i$.

Let the cardinal number of $G$ be $d + 1$ and let $J$ be the $n \times n$ all 1 matrix. Then, by the definition, it follows that $\sum_{i=0}^{d} A_i = J$. It follows that for all $A_i, A_j$,

$$A_i A_j = \sum_{k=0}^{d} p_{ijk} A_k.$$  

From this fact, we can define an algebra naturally. For the commutative ring $R$ with 1, we put $R(X, G) = \bigoplus_{i=0}^{d} RA_i$ as a matrix ring over $R$, and it will be called the adjacency algebra of $(X, G)$ over $R$.

For all $i, j, k \in \{ 0, 1, \ldots, d \}$, we define the matrix $B_i$ by $(B_i)_{jk} = p_{ijk}$. This matrix $B_i$ will be called the $i$-th intersection matrix. It follows that for all $B_i, B_j, B_i B_j = \sum_{k=0}^{d} p_{ijk} B_k$. Therefore we can define an algebra $RB = \bigoplus_{i=0}^{d} RB_i$ for a commutative ring $R$ with 1, and it will be called the intersection algebra of $(X, G)$ over $R$. Then the mapping from the adjacency algebra to the intersection algebra of $(X, G)$ over $R$, $A_i \mapsto B_i$, is an algebra isomorphism.

### 2.2 $P$-polynomial schemes

A symmetric association scheme is called a $P$-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$, if there exist some complex coefficient polynomials $v_i$ of degree $i$ ($0 \leq i \leq d$) such that $A_i = v_i(A_1)$, where $A_i$ is the adjacency matrix of $R_i$. 
We use the following notation: a tridiagonal matrix

\[
B = \begin{pmatrix}
a_0 & c_1 & 0 \\
b_0 & a_1 & \cdots \\
& b_1 & \cdots & \cdots \\
& & \cdots & \cdots & c_d \\
0 & & & b_{d-1} & a_d
\end{pmatrix}
\]

is denoted by

\[
\{ * \quad c_1 \cdots c_{d-1} c_d \quad a_0 \quad a_1 \cdots a_{d-1} a_d \quad b_0 \quad b_1 \cdots b_{d-1} \quad * \}
\]

Then the following (i) and (ii) are equivalent to each other (see [2, Proposition 1.1]).

(i) \( B_1 \) is a tridiagonal matrix with non-zero off-diagonal entries:

\[
\begin{pmatrix}
\ast & 1 & c_2 \cdots c_{d-1} & c_d \\
0 & a_1 & a_2 \cdots & a_{d-1} & a_d \\
b_0 & b_1 & b_2 \cdots & b_{d-1} & \ast
\end{pmatrix} (b_i \neq 0, c_i \neq 0).
\]

(ii) \((X, \{R_i\}_{0 \leq i \leq d})\) is a P-polynomial scheme with respect to the ordering \( R_0, R_1, \ldots, R_d \), i.e.,

\[ A_i = v_i(A_1) \quad (i = 0, 1, \ldots, d) \]

for some polynomials \( v_i \) of degree \( i \).

### 2.3 Hamming schemes

Let \( \Sigma \) be an alphabet of \( q \) symbols \( \{ 0, 1, \ldots, q - 1 \} \). We define \( \Omega \) to be the set \( \Sigma^n \) of all \( n \)-tuples of elements of \( \Sigma \), and let \( \rho(x, y) \) be the number of coordinate places in which the \( n \)-tuples \( x \) and \( y \)}
differ. Thus $\rho(x, y)$ is the Hamming distance between $x$ and $y$. We set
\[ R_i = \{ (x, y) \in \Omega \times \Omega \mid \rho(x, y) = i \}, \]
and then $(\Omega, \{R_i\}_{0 \leq i \leq n})$ is an association scheme. This will be called the Hamming scheme, and denoted by $H(n, q)$.

We consider the intersection numbers $p_{ijk}^{(n,q)}$ of $H(n, q)$. For the convenience of the argument, we extend the binomial coefficient as follows.
\[
\binom{x}{y} = \begin{cases} 1 & \text{if} \ x = 0, \\ 0 & \text{otherwise}, \end{cases}
\]
and for each integer $x$ and each negative integer $y$,
\[
\binom{x}{y} = 0, \quad \binom{y}{x} = 0.
\]

Then we can obtain that
\[
p_{ijk}^{(n,q)} = \sum_{\beta=0}^{n-k} \binom{k}{k-i+\beta} \binom{i-\beta}{k-j+\beta} \binom{n-k}{\beta} (q-1)^{\beta} (q-2)^{i+j-k-2\beta}.
\]

Therefore if $p|q$ for some prime number $p$, $p_{ijk}^{(n,q)} \equiv p_{ijk}^{(n,p)} \pmod{p}$. Since the intersection numbers are the structure constants of the adjacency algebra, $\mathbb{F}_p H(n, q) \cong \mathbb{F}_p H(n,p)$.

The Hamming scheme $H(n, q)$ is P-polynomial scheme (see [2]), and
\[
B_1 = \begin{pmatrix}
* & 1 & \cdots & i & \cdots & n \\
0 & q-2 & \cdots & i(q-2) & \cdots & n(q-2) \\
n(q-1) & (n-1)(q-1) & \cdots & (n-i)(q-1) & \cdots & * 
\end{pmatrix}
\]

In this paper, let $p$ be a fixed prime number. Therefore we set $H(n) := H(n, p)$. And we denote the intersection numbers, the ad-
jacency matrices, and the intersection matrices of $H(n)$ respectively by $p_{ijk}^{(n)}, A_{i}^{(n)}, B_{i}^{(n)}$ and so on.

We can consider the elements of $\Sigma^n$ on $H(n)$ as the $p$-adic number of $n$ figures. Therefore we index the adjacency matrices by the ordinary order on the $p$-adic number. Then it follows that

$$A_{i}^{(n+1)} = I \otimes A_{i}^{(n)} + K \otimes A_{i-1}^{(n)}$$

for $\forall i \in \{ 0, 1, \ldots, n + 1 \}$,

where $I$ is the $p \times p$ identity matrix, $K$ is the $p \times p$ matrix such that the diagonal entries are 0 and the others 1, $A_{-1}^{(n)} = A_{n+1}^{(n)} = O$ (the $p^n \times p^n$ zero matrix), and $\otimes$ is the Kronecker product. The Kronecker product $A \otimes B$ of matrices $A$ and $B$ is defined as follows.

Suppose $A = (a_{ij})$. Then $A \otimes B$ is obtained by replacing the entry $a_{ij}$ of $A$ by the matrix $a_{ij}B$, for all $i$ and $j$. The most important property of this product is that, provided the required products exist,

$$(A \otimes B)(X \otimes Y) = AX \otimes BY.$$

3 $H(p^n - 1)$

Since the intersection numbers are the structure constants of the adjacency algebra, if we consider over a field of characteristic $p$, we may consider the intersection numbers in modulo $p$. Since the size of the adjacency matrix of $H(n)$ is $p^n$, the adjacency algebra of $H(n)$ over a field of characteristic $p$ is local and the unique irreducible representation is $A_i \mapsto p_i 1_{n}$ (see [4, Theorem 3.4, Corollary 3.5]). So the prime field $\mathbb{F}_p$ of characteristic $p$ is a splitting field for the adjacency algebra of $H(n)$ over $\mathbb{F}_p$.

In this paper, since we consider the adjacency algebras only over $\mathbb{F}_p$, we set $\mathfrak{A}_n := \mathbb{F}_p H(n)$. 

By the definition,

\[
B_{1}^{(p^r-1)} = \begin{pmatrix}
B_{1}^{(p-1)} & & & \\
& B_{1}^{(p-1)} & & \\
& & \ddots & \\
& & & B_{1}^{(p-1)}
\end{pmatrix},
\]

therefore if we set \( A_{i}^{(p-1)} = v_i(A_{1}^{(p-1)}) \), it follows that for \( 0 \leq \alpha \leq p - 1 \),

\[
A_{pi+\alpha}^{(p^r-1)} = v_\alpha(A_{1}^{(p^r-1)})A_{pi}^{(p^r-1)}.
\]

Then since any \( c_i^{(p-1)} \not\equiv 0 \pmod{p} \), we can define \( v_\alpha \) over \( \mathbb{F}_p \) for \( 0 \leq \alpha \leq p - 1 \). For calculating \( B_{pi+\alpha}^{(p^r-1)} \), we prepare the following theorem and corollary.

**Theorem 1.** (Lucas' theorem [3, Theorem 3.4.1]) Let \( p \) be prime, and let

\[
m = a_0 + a_1p + \cdots + a_kp^k,
\]

\[
n = b_0 + b_1p + \cdots + b_kp^k,
\]

where \( 0 \leq a_i, b_i < p \) for \( i = 0, 1, \ldots, k - 1 \). Then

\[
\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \pmod{p}.
\]

**Corollary 2.** We assume the same condition for theorem 1 and \( 0 \leq \alpha, \beta < p \). Then

\[
\binom{pm + \alpha}{pn + \beta} \equiv \binom{m}{n} \binom{\alpha}{\beta} \pmod{p}.
\]

Now we want to calculate \( B_{pi+\alpha}^{(p^r-1)} \), that is the coefficients of \( A_{pi+\alpha}^{(p^r-1)}A_{pj+\beta}^{(p^r-1)} \). But it is enough to investigate \( A_{pi}^{(p^r-1)}A_{pj}^{(p^r-1)} \), i.e. \( p_{pi pj k}^{(p^r-1)} \) because we know \( v_\alpha(A_{1}^{(p^r-1)})v_\beta(A_{1}^{(p^r-1)}) \).
Here we set \( k = pk' + k'' \) \((0 \leq k'' \leq p-1)\). Using Lucas' theorem, we can obtain that if \( p \mid k \), \( p_{pi \, pj}^{(p-1)} \equiv p_{ijk}^{(p-1)} \), and if \( p \nmid k \), \( p_{pi \, pj}^{(p-1)} \equiv 0 \).

Thus
\[
A_{pi + \alpha}^{(p^{r-1})} A_{pj + \beta}^{(p^{f-1})} = v_{\alpha}(A^{(p^{r-1})}) v_{\beta}(A^{(p^{r-1})}) A_{pi}^{(p^{r-1})} A_{pj}^{(p^{r-1})}
\equiv \sum_{k=0}^{p-1} \sum_{\gamma=0}^{p^{r-1} - 1} p_{ijk}^{(p^{r-1} - 1)} p_{\alpha\beta\gamma}^{(p-1)} A_{pk + \gamma}^{(p^{r})}.
\]

By the above argument, it follows that
\[
B_{pi + \alpha}^{(p^{r-1})} = B_{i}^{(p^{r-1} - 1)} \otimes B_{\alpha}^{(p-1)}.
\]

Repeating the same argument, we know that for all non-negative integer \( m \) such that \( 0 \leq m \leq p^{r} - 1 \) and \( m = m_{0} p^{0} + m_{1} p^{1} + \cdots + m_{r-1} p^{r-1} \),
\[
B_{m}^{(p^{r-1})} = B_{m_{r-1}}^{(p-1)} \otimes B_{m_{r-2}}^{(p-1)} \otimes \cdots \otimes B_{m_{0}}^{(p-1)}.
\]

From this fact, we obtain that
\[
\mathfrak{A}_{p^{r-1}} \cong \mathfrak{A}_{p-1} \otimes \mathfrak{A}_{p-1} \otimes \cdots \otimes \mathfrak{A}_{p-1}.
\]

**Theorem 3.** \( \mathfrak{A}_{p-1} \cong \mathbb{F}_{p} C_{p} \cong \mathbb{F}_{p}[X]/\langle X^{p} \rangle \)

Therefore the following theorem holds.

**Theorem 4.** *For all positive integer \( r \), \( \mathfrak{A}_{p^{r-1}} \) is isomorphic to the group algebra of the elementary abelian group of order \( p^{r} \) over \( \mathbb{F}_{p} \).*

### 4 The structure of \( \mathfrak{A}_{n} \)

In the previous section, we considered the structure of \( \mathfrak{A}_{p^{r-1}} \). To determine the structure of \( \mathfrak{A}_{n} \), in general, we construct an algebra homomorphism \( \mathfrak{A}_{n+1} \rightarrow \mathfrak{A}_{n} \).
From § 2.3, $A_{i}^{(n+1)} = I \otimes A_{i}^{(n)} + K \otimes A_{i-1}^{(n)}$. This means that $\mathfrak{A}_{n+1}$ is a subalgebra of $\mathfrak{A}_1 \otimes \mathfrak{A}_n$. The unique irreducible representation of $\mathfrak{A}_1$ is $A_{0}^{(1)} \mapsto 1, A_{1}^{(1)} \mapsto -1$.

Therefore we can define naturally the mapping $f_{n+1}$ for each positive integer $n$ by

$$f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$$

$$A_{i}^{(n+1)} = I \otimes A_{i}^{(n)} + K \otimes A_{i-1}^{(n)} \mapsto A_{i}^{(n)} - A_{i-1}^{(n)}.$$ 

**Proposition 5.** For each positive integer $n$, $f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$ above is an algebra epimorphism.

By Theorem 4, $\mathfrak{A}_{p^r-1}$ is isomorphic to $\mathbb{F}_p(C_p \times C_p \times \cdots \times C_p)$ for all positive integer $r$. Furthermore, there exists the algebra isomorphism $g$ from the quotient ring $\mathfrak{P}_r = \mathbb{F}_p[X_1, X_2, \ldots, X_r]/\langle X_1^p, \cdots, X_r^p \rangle$ of the polynomial ring of $r$ variables over $\mathbb{F}_p$ to $\mathbb{F}_p(C_p \times C_p \times \cdots \times C_p)$ by $g(X_i) = 1 - x_i$. Therefore we can define an algebra isomorphism $s_r : \mathfrak{P}_r \to \mathfrak{A}_{p^r-1}$ by

$$s_r(X_i) = A_{0}^{(p^r-1)} - A_{p^{j-1}}^{(p^r-1)}.$$ 

We define a weight function $wt$ on the set of the monomials of $\mathfrak{P}_r$ by

$$wt(X_i) = p^{i-1}, \quad wt(\prod_j X_j^{k_j}) = \sum_j k_j p^{j-1}.$$ 

**Proposition 6.** For all positive integers $m$ such that $1 \leq m \leq p-1$,

$$(A_{0}^{(p^r-1)} - A_{p^{j}}^{(p^r-1)})^m = m! \sum_{n=0}^{m} \binom{m}{n} (-1)^n A_{np^{j}}^{(p^r-1)}.$$ 

And if $i \neq j, 0 \leq \alpha, \beta \leq p - 1$,

$$A_{\alpha p^{j}}^{(p^r-1)} A_{\beta p^{j}}^{(p^r-1)} = A_{\alpha p^{j} + \beta p^{j}}^{(p^r-1)}.$$
Let $Y_i = X_{i_0}^{k_0} X_{i_1}^{k_1} \cdots X_{i_s}^{k_s}$ be the monomial of $\mathfrak{P}_r$ such that $wt(Y_i) = i$. Then by the above two equations, the following Proposition holds.

**Proposition 7.**

$$s_r(Y_i) = (\prod_{j=0}^{s} k_j!) \sum_{n=0}^{p^r-1} \binom{i}{n} (-1)^n A_n^{(p^r-1)}.$$

Then the following theorem holds that is the main theorem in this paper.

**Theorem 8.** We set $\mathfrak{P} = \mathbb{F}_p[X_1, X_2, \cdots]/\langle X_1^p, X_2^p, \cdots \rangle$, and for all positive integer $n$, we set

$$W_n = \langle x \mid x is the monomial of $\mathfrak{P}$ such that $wt(x) > n \rangle.$$

Then it holds that $\mathfrak{P}/W_n \cong \mathfrak{A}_n$ as algebras.

**Proof.** It is enough that we show that,

$$\mathfrak{P}_r/W_n \cong \mathfrak{A}_n \quad for \ n < p^r.$$

Furthermore it is enough that we show that for each positive integer $n$ such that $n \leq p^r - 1$, $Y_n \in \text{Ker} f_n f_{n+1} \cdots f_{p^r-1}s_r$, but $f_n f_{n+1} \cdots f_{p^r-1}s_r(Y_n) = 0$. \hfill \square

**Remark 1** We set $G_{n,q} = S_q wr S_n, H_{n,q} = S_{q-1} wr S_n$ for positive integers $n, q$. Let $K$ be a field. Then $KH(n, q)$ and the Hecke algebra $\text{End}_{KH(n, q)}(1^{G_{n,q}})$ are isomorphic as algebras (see [2, III.2]). Therefore we also could decide the structure of $\text{End}_{KH(n, q)}(1^{G_{n,q}})$. In particular, Theorem 4 means that for all positive integer $r$, if $n = p^r - 1$, the Hecke algebra $\text{End}_{\mathbb{F}_pG_{n,p}}(1^{G_{n,p}})$ is isomorphic to the group algebra of the elementary abelian group of order $p^r$. 

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