数理解析研究所講究録 1327 巻 2003 年 10-20

Modular adjacency algebras of the Hamming association schemes

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Abstruct

The adjacency algebra of an association scheme is defined over an arbitrary field. This is always semisimple over a field of characteristic 0, but not semisimple over a field of prime characteristic p, in general. The structure of the adjacency algebra over a field of prime characteristic was not studied enough before now. Therefore, we considered the structure of the modular adjacency algebra of the Hamming scheme $H(n,q)$, that is one of the most basic and important association schemes.

In this paper, we will decide the structure of the adjacency algebra of $H(n,q)$ over any field for any n and q, and describe the algebra as a factor algebra of a polynomial ring.

1 Introduction

In this paper, we consider the modular adjacency algebra of the Hamming association scheme $H(n,q)$. The modular adjacency algebra means an adjacency algebra over a positive characteristic field. For any prime p such that $p\nmid q$, the adjacency algebra of $H(n,q)$ over a field of characteristic p is semisimple (see [2, Theorem 2.3], [1, Theorem 1.1] and [5, Theorem 4.2]). For each prime p, the prime field $\mathbb{F}_p$ of characteristic p is a splitting field for the adjacency algebra of $H(n,p)$ over $\mathbb{F}_p$ (see [4, Theorem 3.4, Corollary 3.5]). For all prime p such that $p \mid q$, $\mathbb{F}_pH(n,p) \cong \mathbb{F}_pH(n,q)$ (see §2.3). Therefore it is enough to decide the structure of $\mathbb{F}_pH(n,p)$ for all prime p, for deciding the structure of the modular adjacency algebra of any $H(n,q)$ over any field. It is known that the algebra $\mathbb{F}_pH(n,p)$ is commutative and local, and that any local commutative algebra is isomorphic to a factor algebra of a polynomial ring.

2 Preparation

For the definitions in this section, refer to [2].

2.1 Association schemes

Let X be a finite set with cardinality n. We define $R_0 := \{ (x,x) \mid x \in X \}$. Let $R_i \subseteq X \times X$ be given. We set $R_i^* := \{ (z,y) \mid (y,z) \in R_i \}$. Let G be a partition of $X \times X$ such that $R_0 \in G$ and the empty set $\emptyset \notin G$, and assume that, $R_i^* \in G$ for each $R_i \in G$. Then, the pair (X,G) will be called an *association scheme* if, for all $R_i, R_j, R_k \in G$, there exists a cardinal number p_{ijk} such that, for all $y,z \in X$

$$(y,z) \in R_k \Rightarrow \sharp\{ x \in X \mid (y,x) \in R_i, (x,z) \in R_j \} = p_{ijk}.$$

The elements of $\{p_{ijk}\}$ will be called the *intersection numbers* of (X,G).

For each $R_i \in G$, we define the $n \times n$ matrix A_i indexed by the elements of X,

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$

and this matrix A_i will be called the *adjacency matrix* of R_i.

Let the cardinal number of G be $d+1$ and let J be the $n \times n$ all 1 matrix. Then, by the definition, it follows that $\sum_{i=0}^{d} A_i = J$. It follows that for all A_i, A_j,

$$A_i A_j = \sum_{k=0}^{d} p_{ijk} A_k.$$

From this fact, we can define an algebra naturally. For the commutative ring R with 1, we put $R(X,G) = \bigoplus_{i=0}^{d} RA_i$ as a matrix ring over R, and it will be called the *adjacency algebra* of (X,G) over R.

For all $i,j,k \in \{ 0,1,\ldots,d \}$, we define the matrix B_i by $(B_i)_{jk} = p_{ijk}$. This matrix B_i will be called the *i-th intersection matrix.* It follows that for all B_i, B_j, $B_iB_j = \sum_{k=0}^{d} p_{ijk} B_k$. Therefore we can define an algebra $RB = \bigoplus_{i=0}^{d} RB_i$ for a commutative ring R with 1, and it will be called the *intersection algebra* of (X,G) over R. Then the mapping from the adjacency algebra to the intersection algebra of (X,G) over R, $A_i \mapsto B_i$, is an algebra isomorphism.

2.2 P-polynomial schemes

A symmetric association scheme is called a *P-polynomial scheme* with respect to the ordering $R_0, R_1, \ldots, R_d$, if there exist some complex coefficient polynomials v_i of degree i $(0 \le i \le d)$ such that $A_i = v_i(A_1)$, where A_i is the adjacency matrix of R_i.

We use the following notation: a tridiagonal matrix

$$B = \begin{pmatrix} a_0 & c_1 & & & 0 \\ b_0 & a_1 & \ddots & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & c_d \\ 0 & & & b_{d-1} & a_d \end{pmatrix}$$

is denoted by

$$\begin{Bmatrix} * & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & * \end{Bmatrix}.$$

Then the following (i) and (ii) are equivalent to each other (see [2, Proposition 1.1]).

(i) B_1 is a tridiagonal matrix with non-zero off-diagonal entries:

$$\begin{Bmatrix} * & 1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_{d-1} & * \end{Bmatrix} \ (b_i \neq 0, c_i \neq 0).$$

(ii) $(X, \{R_i\}_{0 \le i \le d})$ is a P-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$, i.e.,

$$A_i = v_i(A_1) \quad (i = 0, 1, \ldots, d)$$

for some polynomials v_i of degree i.

2.3 Hamming schemes

Let Σ be an alphabet of q symbols $\{\ 0, 1, \ldots, q-1\ \}$. We define Ω to be the set Σ^n of all n-tuples of elements of Σ, and let $\rho(x, y)$ be the number of coordinate places in which the n-tuples x and y

differ. Thus $\rho(x,y)$ is the Hamming distance between x and y. we set

$$R_i = \{\ (x,y) \in \Omega \times \Omega \mid \rho(x,y) = i\ \},$$

and then $(\Omega, \{R_i\}_{0\leq i\leq n})$ is an association scheme. This will be called the *Hamming scheme*, and denoted by $H(n,q)$.

We consider the intersection numbers $p_{ijk}^{(n,q)}$ of $H(n,q)$. For the convenience of the argument, we extend the binomial coefficient as follows.

$$\binom{0}{x} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for each integer x and each negative integer y,

$$\binom{x}{y} = 0, \binom{y}{x} = 0.$$

Then we can obtain that

$$p_{ijk}^{(n,q)} = \sum_{\beta=0}^{n-k} \binom{k}{k-i+\beta}\binom{i-\beta}{k-j+\beta}\binom{n-k}{\beta}(q-1)^{\beta}(q-2)^{i+j-k-2\beta}.$$

Therefore if $p|q$ for some prime number p, $p_{ijk}^{(n,q)} \equiv p_{ijk}^{(n,p)} \pmod p$. Since the intersection numbers are the structure constants of the adjacency algebra, $\mathbb{F}_p H(n,q) \cong \mathbb{F}_p H(n,p)$.

The Hamming scheme $H(n,q)$ is P-polynomial scheme (see [2]), and

$$B_1 = \begin{Bmatrix} * & 1 & \cdots & i & \cdots & n \\ 0 & q-2 & \cdots & i(q-2) & \cdots & n(q-2) \\ n(q-1) & (n-1)(q-1) & \cdots & (n-i)(q-1) & \cdots & * \end{Bmatrix}$$

In this paper, let p be a fixed prime number. Therefore we set $H(n) := H(n,p)$. And we denote the intersection numbers, the adjacency matrices, and the intersection matrices of $H(n)$ respectively by $p_{ijk}^{(n)}, A_i^{(n)}$, $B_i^{(n)}$ and so on.

We can consider the elements of Σ^n on $H(n)$ as the p-adic number of n figures. Therefore we index the adjacency matrices by the ordinary order on the p-adic number. Then it follows that

$$A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)} \qquad \text{for } \forall i \in \{ 0, 1, \ldots, n+1 \},$$

where I is the $p \times p$ identity matrix, K is the $p \times p$ matrix such that the diagonal entries are 0 and the others 1, $A_{-1}^{(n)} = A_{n+1}^{(n)} = O$ (the $p^n \times p^n$ zero matrix), and $\otimes$ is the Kronecker product. The Kronecker product $A \otimes B$ of matrices A and B is defined as follows. Suppose $A = (a_{ij})$. Then $A \otimes B$ is obtained by replacing the entry a_{ij} of A by the matrix $a_{ij}B$, for all i and j. The most important property of this product is that, provided the required products exist,

$$(A \otimes B)(X \otimes Y) = AX \otimes BY.$$

3 $H(p^r - 1)$

Since the intersection numbers are the structure constants of the adjacency algebra, if we consider over a field of characteristic p, we may consider the intersection numbers in modulo p. Since the size of the adjacency matrix of $H(n)$ is p^n, the adjacency algebra of $H(n)$ over a field of characteristic p is local and the unique irreducible representation is $A_i \mapsto p_{i\ i^*\ 0}$ (see [4, Theorem 3.4, Corollary 3.5]). So the prime field $\mathbb{F}_p$ of characteristic p is a splitting field for the adjacency algebra of $H(n)$ over $\mathbb{F}_p$.

In this paper, since we consider the adjacency algebras only over $\mathbb{F}_p$, we set $\mathfrak{A}_n := \mathbb{F}_p H(n)$.

By the definition,

$$B_1^{(p^r-1)} = \begin{pmatrix} B_1^{(p-1)} & & & \\ & B_1^{(p-1)} & & \\ & & \ddots & \\ & & & B_1^{(p-1)} \end{pmatrix},$$

therefore if we set $A_i^{(p-1)} = v_i(A_1^{(p-1)})$, it follows that for $0 \leq \alpha \leq p-1$,

$$A_{pi+\alpha}^{(p^r-1)} = v_\alpha(A_1^{(p^r-1)})A_{pi}^{(p^r-1)}.$$

Then since any $c_i^{(p-1)} \not\equiv 0 \pmod{p}$, we can define v_α over $\mathbb{F}_p$ for $0 \leq \alpha \leq p-1$. For calculating $B_{pi+\alpha}^{(p^r-1)}$, we prepare the following theorem and corollary.

Theorem 1. **(Lucas' theorem** [3, Theorem 3.4.1]) *Let p be prime, and let*

$$m = a_0 + a_1p + \cdots + a_kp^k,$$

$$n = b_0 + b_1p + \cdots + b_kp^k,$$

where $0 \leq a_i, b_i < p$ for $i = 0, 1, \ldots, k-1$. Then

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \quad (\text{mod p}).$$

Corollary 2. *We assume the same condition for theorem 1 and $0 \leq \alpha, \beta < p$. Then*

$$\binom{pm+\alpha}{pn+\beta} \equiv \binom{m}{n}\binom{\alpha}{\beta} \quad (\text{mod p}).$$

Now we want to culculate $B_{pi+\alpha}^{(p^r-1)}$, that is the coefficients of $A_{pi+\alpha}^{(p^r-1)}A_{pj+\beta}^{(p^r-1)}$. But it is enough to investigate $A_{pi}^{(p^r-1)}A_{pj}^{(p^r-1)}$, i.e. $p_{pi\ pj\ k}^{(p^r-1)}$ because we know $v_\alpha(A_1^{(p^r-1)})v_\beta(A_1^{(p^r-1)})$.

Here we set $k = pk' + k''(0 \le k'' \le p-1)$. Using Lucas' theorem, we can obtain that if $p \mid k$, $p_{pi\ pj\ k}^{(p^r-1)} \equiv p_{ijk'}^{(p^{r-1}-1)}$, and if $p \nmid k$, $p_{pi\ pj\ k}^{(p^r-1)} \equiv 0$.

Thus

$$A_{pi+\alpha}^{(p^r-1)} A_{pj+\beta}^{(p^r-1)} = v_\alpha(A_1^{(p^r-1)}) v_\beta(A_1^{(p^r-1)}) A_{pi}^{(p^r-1)} A_{pj}^{(p^r-1)}$$

$$\equiv \sum_{k=0}^{p^{r-1}-1} \sum_{\gamma=0}^{p-1} p_{ijk}^{(p^{r-1}-1)} p_{\alpha\beta\gamma}^{(p-1)} A_{pk+\gamma}^{(p^r-1)}.$$

By the above argument, it follows that

$$B_{pi+\alpha}^{(p^r-1)} = B_i^{(p^{r-1}-1)} \otimes B_\alpha^{(p-1)}.$$

Repeating the same argument, we know that for all non-negative integer m such that $0 \le m \le p^r - 1$ and $m = m_0 p^0 + m_1 p^1 + \cdots + m_{r-1} p^{r-1}$,

$$B_m^{(p^r-1)} = B_{m_{r-1}}^{(p-1)} \otimes B_{m_{r-2}}^{(p-1)} \otimes \cdots \otimes B_{m_0}^{(p-1)}.$$

From this fact, we obtain that

$$\mathfrak{A}_{p^r-1} \cong \overbrace{\mathfrak{A}_{p-1} \otimes \mathfrak{A}_{p-1} \otimes \cdots \otimes \mathfrak{A}_{p-1}}^{r}.$$

Theorem 3. $\mathfrak{A}_{p-1} \cong \mathbb{F}_p C_p \cong \mathbb{F}_p[X]/\langle X^p \rangle$

Therefore the following theorem holds.

Theorem 4. *For all positive integer r, $\mathfrak{A}_{p^r-1}$ is isomorphic to the group algebra of the elementary abelian group of order p^r over $\mathbb{F}_p$.*

4 The structure of $\mathfrak{A}_n$

In the previous section, we considered the structure of $\mathfrak{A}_{p^r-1}$. To determine the structure of $\mathfrak{A}_n$, in general, we construct an algebra homomorphism $\mathfrak{A}_{n+1} \to \mathfrak{A}_n$.

From § 2.3, $A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)}$. This means that $\mathfrak{A}_{n+1}$ is a subalgebra of $\mathfrak{A}_1 \otimes \mathfrak{A}_n$. The unique irreducible representation of $\mathfrak{A}_1$ is $A_0^{(1)} \mapsto 1, A_1^{(1)} \mapsto -1$.

Therefore we can define naturally the mapping f_{n+1} for each positive integer n by

$$f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$$
$$A_i^{(n+1)} = I \otimes A_i^{(n)} + K \otimes A_{i-1}^{(n)} \mapsto A_i^{(n)} - A_{i-1}^{(n)}.$$

Proposition 5. *For each positive integer n, $f_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$ above is an algebra epimorphism.*

By Theorem 4, $\mathfrak{A}_{p^r-1}$ is isomorphic to $\mathbb{F}_p(\underbrace{C_p \times C_p \times \cdots \times C_p}_{r})$ for all positive integer r. Furthermore, there exists the algebra isomorphism g from the quotient ring $\mathfrak{P}_r = F_p[X_1, X_2, \ldots, X_r]/\langle X_1^p, \cdots, X_r^p\rangle$ of the polynomial ring of r variables over $\mathbb{F}_p$ to $\mathbb{F}_p(\underbrace{C_p \times C_p \times \cdots \times C_p}_{r})$ by $g(X_i) = 1 - x_i$. Therefore we can define an algebra isomorphism $s_r : \mathfrak{P}_r \to \mathfrak{A}_{p^r-1}$ by

$$s_r(X_i) = A_0^{(p^r-1)} - A_{p^{i-1}}^{(p^r-1)}.$$

We define a weight function wt on the set of the monomials of $\mathfrak{P}_r$ by

$$wt(X_i) = p^{i-1}, \ wt(\prod_j X_j^{k_j}) = \sum_j k_j p^{j-1}.$$

Proposition 6. *For all positive integers m such that $1 \le m \le p-1$,*

$$(A_0^{(p^r-1)} - A_{p^i}^{(p^r-1)})^m = m! \sum_{n=0}^{m} \binom{m}{n} (-1)^n A_{np^i}^{(p^r-1)}.$$

And if $i \neq j, 0 \le \alpha, \beta \le p-1$,

$$A_{\alpha p^i}^{(p^r-1)} A_{\beta p^j}^{(p^r-1)} = A_{\alpha p^i + \beta p^j}^{(p^r-1)}.$$

Let $Y_i = X_{i_0}^{k_0} X_{i_1}^{k_1} \cdots X_{i_s}^{k_s}$ be the monomial of $\mathfrak{P}_r$ such that $wt(Y_i) = i$. Then by the above two equations, the following Proposition holds.

Proposition 7.

$$s_r(Y_i) = (\prod_{j=0}^{s} k_j!) \sum_{n=0}^{p^r-1} \binom{i}{n} (-1)^n A_n^{(p^r-1)}.$$

Then the following theorem holds that is the main theorem in this paper.

Theorem 8. *We set* $\mathfrak{P} = \mathbb{F}_p[X_1, X_2, \cdots]/\langle X_1^p, X_2^p \cdots \rangle$, *and for all positive integer* n, *we set*

$$W_n = \langle\ x \mid x \text{ is the monomial of } \mathfrak{P} \text{ such that } wt(x) > n \rangle.$$

Then it holds that $\mathfrak{P}/W_n \cong \mathfrak{A}_n$ *as algebras.*

Proof. It is enough that we show that,

$$\mathfrak{P}_r/W_n \cong \mathfrak{A}_n \quad \text{for } n < p^r.$$

Furthermore it is enough that we show that for each positive integer n such that $n \leq p^r - 1$, $Y_n \in \mathrm{Ker} f_n f_{n+1} \cdots f_{p^r-1} s_r$, but $f_n f_{n+1} \cdots f_{p^r-1} s_r(Y_n) = 0$. □

Remark 1 We set $G_{n,q} = S_q\ wr\ S_n$, $H_{n,q} = S_{q-1}\ wr\ S_n$ for positive integers n, q. Let K be a field. Then $KH(n,q)$ and the Hecke algebra $\mathrm{End}_{KG_{n,q}}(1_{H_{n,q}}^{G_{n,q}})$ are isomorphic as algebras (see [2, III.2]). Therefore we also could decide the structure of $\mathrm{End}_{KG_{n,q}}(1_{H_{n,q}}^{G_{n,q}})$. In particular, Theorem 4 means that for all positive integer r, if $n = p^r - 1$, the Hecke algebra $\mathrm{End}_{\mathbb{F}_p G_{n,p}}(1_{H_{n,p}}^{G_{n,p}})$ is isomorphic to the group algebra of the elementary abelian group of order p^r.

Acknowledgement

The author thanks to A. Hanaki for valuable suggestions and comments.

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