数理解析研究所講究録 1327 巻 2003 年 38-46

Block decomposition of standard modules

信州大学・理学部 花木 章秀 (Akihide Hanaki)

1 Introduction

In this article, we consider the structure of the standard modules of association schemes. Firstly, we consider the relations between representation theory of some algebraic objects. If we consider representation theory of a finite dimensional algebra, we can only use its algebra structure. For a (generalized) table algebra [1] or a group-like algebra [4], we can use its *distinguished basis*. Group-like algebras are defined by Y. Doi as a generalization of adjacency algebras of association schemes from a viewpoint in the theory of bialgebra. For representation theory of the adjacency algebra of an association scheme, we can use *the standard module* (*representation*), which is the main subject in this article. For representation theory of association schemes, we can use the standard module with the distinguished basis. The information of the standard module with the distinguished basis is equivalent to the combinatorial structure, since we can reconstruct the association scheme from it.

If two association schemes have isomorphic adjacency algebras over the complex number field $\mathbb{C}$, then so are the standard modules since they are completely determined by the degrees and the multiplicities of irreducible characters. But this is not true for over a positive characteristic field. We show an example.

Example 1.1. There exist association schemes (X, G) and (X, G') of order 27 and class 2, such that their adjacency algebras are isomorphic over the rational integer ring $\mathbb{Z}$ (so they are isomorphic over an arbitrary commutative ring with 1). Let F be a field of characteristic 3. Then their adjacency algebras are isomorphic to $A = F[x]/(x^3)$, where $F[x]$ is the usual polynomial ring over F. The set of isomorphism classes of indecomposable A-modules is $\{M_1, M_2, M_3\}$, where $\dim_F M_i = i$. The standard modules are

$$FX_{FG} \cong M_3 \oplus 12M_2, \quad FX_{FG'} \cong M_3 \oplus 11M_2 \oplus 2M_1,$$

and they are not isomorphic. We can find similar observations in [2] and [8].

This example shows us that the structure of a standard module plays an important role in representation theory of association schemes. We consider the structure of standard modules, especially their block decompositions.

2 Definitions

We use the notations in the book of Zieschang [9].

Let X be a finite set, and let G be a collection of subsets of $X \times X$. For $g \in G$, we define the *adjacency matrix* σ_g of g as the following. Let σ_g be a matrix over the rational integer ring whose both rows and columns are indexed by X. The (x, y)-entry of σ_g is 1 if $(x, y) \in g$, and 0 otherwise. If $\{\sigma_g \mid g \in G\}$ satisfies the condition (1) - (4), we call (X, G) an *association scheme.*

(1) The matrix $\sum_{g \in G} \sigma_g$ is the all one matrix.

(2) There exists $g \in G$ such that σ_g is the identity matrix (we will denote this g by 1).

(3) For any $g \in G$, there exists $g^* \in G$ such that $\sigma_{g^*} = {}^t\sigma_g$, where ${}^t\sigma_g$ is the transposed matrix of σ_g.

(4) There exist rational integers a_{efg}, such that $\sigma_e \sigma_f = \sum_{g \in G} a_{efg} \sigma_g$.

By the condition (4), we can define a $\mathbb{Z}$-algebra $\bigoplus_{g \in G} \mathbb{Z}\sigma_g$. For an arbitrary commutative ring R with 1, we define

$$RG := \left(\bigoplus_{g \in G} \mathbb{Z}\sigma_g \right) \otimes_{\mathbb{Z}} R,$$

and we call this the *adjacency algebra* of (X, G) over R. Often we consider the adjacency matrix σ_g is a matrix over the coefficient ring R. Note that $\{\sigma_g \mid g \in G\}$ is linearly independent over any commutative ring by the condition (1).

For $g \in G$, we set $n_g := a_{gg^*1}$ and call it the *valency* of g. For a subset S of G, we also denote $n_S := \sum_{g \in S} n_g$. Especially, n_G is equal to the cardinality of X, and we call it the *order* of (X, G). The number $|G| - 1$ is called the *class* of (X, G). Easily, we can check that the map $\sigma_g \mapsto n_g$ is an algebra homomorphism from the adjacency algebra RG to R (R is an arbitrary commutative ring with 1). We call this the *trivial representation* of G over R. Note that, in this article, a representation means a linear representation of an algebra, namely, an algebra homomorphism from an R-algebra to the full matrix ring over R of some degree.

The map $\Gamma_G : RG \to M_{n_G}(R)$ defined by $\Gamma_G(\sigma_g) = \sigma_g$ is also a representation of G. We call this the *standard representation* of G over R. The corresponding right RG-module is called the *(right) standard module*, and we denote it by RX, since we can consider X as an R-basis of it.

It is well known that the adjacency algebra over the complex number field is always semisimple. In this case, all modules are completely reducible and they are determined by their characters. Here the character means the trace function of a representation. We denote the set of all irreducible characters of $\mathbb{C}G$ by $\mathrm{Irr}(G)$. We consider the irreducible decomposition of the standard character γ_G over $\mathbb{C}$:

$$\gamma_g = \sum_{\chi \in \mathrm{Irr}(G)} m_\chi \chi.$$

We call m_χ the *multiplicity* of $\chi \in \mathrm{Irr}(G)$.

Let p be a prime, and let (K, R, F) be a p-modular system. Namely, R is a complete discrete valuation ring with the maximal ideal (π), K is the quotient field of R and its characteristic is 0, and F is the residue field $R/(\pi)$ and its characteristic is p. Details about p-modular systems, see [7]. The simplest example of p-modular systems is $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z})$. Let (X, G) be an association scheme. To simplify our argument, we suppose that the adjacency algebras KG and FG are splitting algebras. In this case, we say (K, R, F) is a splitting p-modular system of G.

Any idempotent in FG is a image of an idempotent of RG by the natural epimorphism from RG to $FG \cong RG/\pi RG$. The primitivity of idempotents is preserved by this correspondence [7, Theorem I.14.2]. Moreover, there exists a natural correspondence between the set of primitive central idempotents of RG and it of FG [3, Proposition 1.12]. Namely, if

$$1 = e_0 + e_1 + \cdots + e_r$$

is the central idempotent decomposition of 1 in RG, then so is

$$\overline{1} = \overline{e_0} + \overline{e_1} + \cdots + \overline{e_r}$$

in FG, where $\overline{e_i}$ is the image of e_i by the natural epimorphism. We call a primitive central idempotent e_i the *block idempotent* of G. In this case,

$$RG = RGe_0 \oplus \cdots \oplus RGe_r$$

is the indecomposable decomposition of RG as two-sided ideals. We call RGe_i the *block* (or *block ideal*) of G. For a right KG- or RG-module M, we say M belongs to a block RGe_i if $Me_i = M$. For a right FG-module M, we say M belongs to a block e_i if $M\overline{e_i} = M$. Any indecomposable module belongs to the unique block. Let M be a right RG-module, and assume $1 = e_0 + e_1 + \cdots + e_r$ is the central idempotent decomposition of 1 in RG. Then we can decompose M :

$$M = Me_0 \oplus \cdots \oplus Me_r.$$

We call this decomposition the *block decomposition* of M. We define block decompositions for KG-modules and FG-modules similarly.

3 Block decompositions

We begin this section with a well known fact in modular representation theory of finite groups. Let F be a field of characteristic $p > 0$, and let G be a finite group of order $p^a m$, where $p \nmid m$. If M is a finitely generated projective right FG-module, then $p^a | \dim_F M$. Especially, $p^a | \dim_F eFG$ for any idempotent e of FG. We want to generalize this fact to adjacency algebras. But easily we can find counter examples.

Example 3.1. Let (X, G) be an association scheme of order p^a, and assume that it is not thin. Take 1 as an idempotent, then $\dim_F FG < p^a$ and $p^a \nmid \dim_F FG$.

Now we consider the standard module. Then we have the following result.

Theorem 3.2. *Let (X, G) be an association scheme of order $p^a m$, where $p \nmid m$. Let F be a field of characteristic p, and let e be an idempotent in FG. Then $p^a \mid \dim_F FXe$. If e is primitive, then $\dim_F FXe$ equals to the multiplicity of the simple FG-module $eFG/J(eFG)$ in FX as an irreducible constituent.*

Proof. The proof is almost the same as [5, Theorem 3.4].

Let e be an idempotent in FG. Then there exists an idempotent f of RG such that $\overline{f} = e$. We have $\dim_F eFG = \operatorname{rank}_R fRG = \operatorname{rank} \Gamma_G(f)$, where Γ_G is the standard representation. Since f is an idempotent, we have $\operatorname{rank} \Gamma_G(f) = \gamma_G(f) = \sum_{\chi\in\mathrm{Irr}(G)} m_\chi \chi(f)$. If $f = \sum_{g\in G} \alpha_g \sigma_g$, then $\gamma_G(f) = \alpha_1 n_G = \alpha_1 p^a m$, so we have $\alpha_1 = \gamma_G(f)/p^a m$. Since $f \in RG$, $\alpha_1 \in R$, so $\gamma_G(f)$ must be divided by p^a. □

Corollary 3.3. *If (X, G) is an association scheme of order $p^a m$, $p \nmid m$, then the number of isomorphism classes of irreducible FG-modules is at most m. Moreover, this bound is best possible.*

Proof. It is enough to show that $FXe \neq 0$ for any primitive idempotent e of FG.

We fix an element x in X. Define a map $\varphi : FG \longrightarrow FX$ by $\varphi(\sigma_g) = x\sigma_g$. Then easily we can verify that φ is an FG-monomorphism. Now $FXe \neq 0$, since $FGe \neq 0$.

The groups algebra of abelian group of order $p^a m$ has m irreducible modules. So this bound is best possible. □

We note that FXe is not an FG-module, in general. But, if e is a central idempotent, then FXe is an FG-module. So we have the following.

Theorem 3.4. *Let (X, G) be an association scheme of order $p^a m$, where $p \nmid m$. For the block decomposition of the standard module*

$$FX = FXe_0 \oplus \cdots \oplus FXe_r,$$

we have $p^a \mid \dim_F FXe_i$ for any i.

For a block B of G, we write the set of irreducible characters belonging to it by $\mathrm{Irr}(B)$.

Corollary 3.5. *If (X, G) is an association scheme of order $p^a m$, $p \nmid m$, then*

$$p^a \,\Big|\, \sum_{\chi\in\mathrm{Irr}(B)} m_\chi \chi(1),$$

for any block B of G.

Proof. Let $B = eRG$. For $\chi \in \mathrm{Irr}(G)$, $\chi(e) = \chi(1)$ if $\chi \in \mathrm{Irr}(B)$, and $\chi(e) = 0$ otherwise. By the proof of Theorem 3.2, we have the result. □

4 Commutative case

If (X, G) is a commutative association scheme, then any block $\overline{e_i}FG$ of FG is a local commutative algebra. So we have the following.

Proposition 4.1. *Let (X, G) be a commutative association scheme. If $\chi, \varphi \in \mathrm{Irr}(G)$, then χ and φ belong to the same block if and only if*

$$\chi(\sigma_g) \equiv \varphi(\sigma_g) \pmod{(\pi)}, \quad \textit{for all } g \in G.$$

The following is a easy consequence of the result in the previous section.

Corollary 4.2. *If (X, G) is a commutative association scheme of order $p^a m$, $p \nmid m$, then*

$$p^a \,\Big|\, \sum_{\chi \in \mathrm{Irr}(B)} m_\chi,$$

for any block B of G.

5 Noncommutative case

For $\chi \in \mathrm{Irr}(G)$, we define $\omega_\chi : Z(KG) \to K$ by $\omega_\chi(z) = \chi(z)/\chi(1)$. Then, if $\chi \neq \varphi$, then $\omega_\chi \neq \omega_\varphi$, and we have

$$\mathrm{Irr}(Z(KG)) = \{\omega_\chi \mid \chi \in \mathrm{Irr}(G)\}.$$

Now we can say a generalization of Proposition 4.1.

Theorem 5.1. *Let (X, G) be a group-like association scheme. If $\chi, \varphi \in \mathrm{Irr}(G)$, then χ and φ belong to the same block if and only if*

$$\omega_\chi(z) \equiv \omega_\varphi(z) \pmod{(\pi)}, \quad \textit{for all } z \in Z(RG).$$

If we want to use this result, then we need a basis of $Z(RG)$. But, in general, we do not know how to calculate a basis of $Z(RG)$. If we assume a property of (X, G), we can decide a good basis of $Z(RG)$. It is stated in the next section.

Remark 5.2. If we want do know the block decomposition of $\mathrm{Irr}(G)$, then we can use the following method. Let $\chi \in \mathrm{Irr}(B)$. Consider $\mathcal{S} := \{S \subseteq \mathrm{Irr}(G) \mid \sum_{\varphi \in S} e_\varphi \in RG\}$, where e_φ is the central idempotent in KG corresponding to φ. Then $\bigcap_{S \in \mathcal{S}} S \in \mathcal{S}$ and this is $\mathrm{Irr}(B)$.

6 Group-like case

Let (X, G) be an association scheme. For $g, h \in G$, we define $g \sim h$ if

$$\frac{1}{n_g}\chi(\sigma_g) = \frac{1}{n_h}\chi(\sigma_h), \quad \text{for any } \chi \in \mathrm{Irr}(G).$$

We say (X, G) is *group-like* if the number of $\sim$ equivalence classes is equal to the number of irreducible characters of G (this is different from group-like algebras defined by Y. Doi [4]). For details, see [6]. Suppose that (X, G) is group-like. For $g \in G$, we put $\widetilde{g} = \bigcup_{h\sim g} h$, and $\widetilde{G} = \{\widetilde{g} \mid g \in G\}$. Then $(X, \widetilde{G})$ is an association scheme, and the adjacency algebra $R\widetilde{G}$ is the center of RG.

Theorem 6.1. *Let (X, G) be a group-like association scheme. If $\chi, \varphi \in \mathrm{Irr}(G)$, then χ and φ belong to the same block if and only if*

$$\omega_\chi(\sigma_{\widetilde{g}}) \equiv \omega_\varphi(\sigma_{\widetilde{g}}) \pmod{(\pi)}, \quad \textit{for all } \widetilde{g} \in \widetilde{G}.$$

If (X, G) is thin, namely G is a finite group, then it is group-like and the relation $\sim$ is the conjugacy relation of the group. In this case, our result is well known in representation theory of finite groups.

7 Some examples

In this section, we consider some examples.

Example 7.1. We consider the association schemes defined by permutation groups on the set of prime cardinalities. Let (X, G) be such an association scheme of order p and class d. In this case, d must divide $p-1$. Let F be an algebraically closed field of characteristic p. Then the adjacency algebra FG is isomorphic to $F[x]/(x^d+1)$. The set of isomorphism classes of indecomposable FG-modules is $\{M_i \mid 1 \le i \le d+1\}$, where $\dim_F M_i = i$. Now the standard module is

$$FX_{FG} \cong M_{d+1} \oplus \left(\frac{p-1}{d} - 1\right) M_d.$$

In this case, $M_{d+1} \cong FG$ as FG-modules.

For many examples, the standard module FX_{FG} contains the regular module FG_{FG} as a direct summand. But this is not true, in general.

Example 7.2. Let $H(2,2)$ be the Hamming scheme, and let F be a field of characteristic 2. Then the standard module of $H(2,2)$ over F is indecomposable. Especially, it does not contain the regular module as a direct summand.

We consider a general situation. There exists an FG-module monomorphism from FG to FX as we see in the proof of Corollary 3.3. This does not split, in general. If FG is self-injective (equivalently a quasi-Frobenius algebra), then this monomorphism splits.

Proposition 7.3. *If FG is self-injective, then FG_{FG} is a direct summand of FX_{FG}.*

References

[1] Z. Arad, E. Fisman, M. Muzychuk, Generalized table algebras, Israel J. Math. **114**, 29–60, 1999.

[2] A. E. Brouwer, C. A. van Eyl, On the p-rank of the adjacency matrices of strongly regular graphs, J. Alg. Comb. **1**, 329–346, 1992.

[3] E. C. Dade, Block extensions, Illinois J. Math. **17**, 198–272, 1973.

[4] Y. Doi, Group-like algebras, The 35 th Symposium on Ring Theory and Representation Theory, Okayama, 2002.

[5] A. Hanaki, Locality of a modular adjacency algebra of an association scheme of prime power order, Arch. Math. 79, 167–170, 2002.

[6] A. Hanaki, Characters of association schemes and normal closed subsets, to appear in Graphs Comb.

[7] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, New York, 1989.

[8] R. Peeters, On the p-rank of the adjacency matrices of distance-regular graphs, J. Alg. Comb. **15**, 127–149, 2002.

[9] P.-H. Zieschang, An Algebraic Approach to Association Schemes, Springer-Verlag, Berlin/New York, 1996.