A bound for the number of columns $\ell_{(c,a,b)}$ in the intersection array of a distance-regular graph (Algebraic Combinatorics)

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A bound for the number of columns $\ell_{(c,a,b)}$ in the intersection array of a distance-regular graph

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Abstract
In this paper we give a bound for the number $\ell_{(c,a,b)}$ of columns $(c,a,b)^T$ in the intersection array of a distance-regular graph. We also show that this bound is intimately related to the Bannai-Ito Conjecture.

1 Introduction

Suppose that $\Gamma$ is a finite connected graph with vertex set $V\Gamma$. As usual, we define the distance between any two vertices $u$ and $v$ of $\Gamma$ to be the length of any shortest path in $\Gamma$ between $u$ and $v$, and the diameter $d$ of $\Gamma$ to be the largest distance between any pair of vertices in $V\Gamma$. For $u \in V\Gamma$ and $i$ any non-negative integer not exceeding $d$, let $\Gamma_i(u)$ denote the set of vertices in $V\Gamma$ that are at distance $i$ from $u$ and put $\Gamma_{-1}(v) = \Gamma_{d+1}(v) := \emptyset$. The graph $\Gamma$ is called distance-regular if there are integers $b_0, c_i, 0 \leq i \leq d$, so that for any two vertices $u$ and $v$ in $V\Gamma$ at distance $i$, there are precisely $c_i$ neighbors of $v$ in $\Gamma_{i-1}(u)$ and $b_i$ neighbors of $v$ in $\Gamma_{i+1}(u)$. Clearly such a graph is regular with valency $k := b_0$. The numbers $c_i, b_i$, and $a_i$, where

$$a_i := k - b_i - c_i \quad (i = 0, \ldots, d)$$

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is the number of neighbors of \( v \) in \( \Gamma_i(u) \) for \( u, v \in V \Gamma \) at distance \( i \), are called the intersection numbers of \( \Gamma \), and

\[
\begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{d-1} & c_d \\
a_0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\
b_0 & b_1 & b_2 & \cdots & b_{d-1} & b_d
\end{bmatrix}
\]

the intersection array of \( \Gamma \).

Now, suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 2 \), diameter \( d \geq 2 \) and intersection numbers \( c_i, a_i, b_i, 0 \leq i \leq d \). Given integers \( a \geq 0 \) and \( b, c \geq 1 \) with \( a + b + c = k \), we define

\[
\ell_{(c,a,b)} = \ell_{(c,a,b)}(\Gamma) := |\{i \mid 1 \leq i \leq d - 1 \text{ and } (a_i, a_i, b_i) = (c, a, b)\}|
\]

that is, the number of columns \((c,a,b)^T\) in the intersection array of \( \Gamma \), and put

\[
h = h(\Gamma) := \ell_{(1,a_1,b_1)}.
\]

Note that since \( d \geq 2 \) and \( c_1 = 1 \) we have \( h \geq 1 \).

Finding good bounds for \( \ell_{(c,a,b)} \) is a powerful technique for understanding distance-regular graphs. For example, in [1] Bannai and Ito showed that, for a distance-regular graph with valency \( k \geq 3 \), if \( c \) is an integer with \( 0 \leq 2c \leq k \) then \( \ell_{(a,k-2c,c)} \leq 10k2^k \), from which they deduced that there are finitely many distance-regular graphs with valency 3. Also, in [4] Biggs et al. used circuit chasing to considerably improve this bound, which enabled them to classify the distance-regular graphs with valency 3.

In this paper we prove the following theorem.

**Theorem 1.1** There exists a function \( k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+ \) so that for all positive integers \( b, c, C \), \( k(b, c, C) \geq \max\{b + c, 3\} \) and for all distance-regular graphs \( \Gamma \) with valency \( k \geq k(b, c, C) \), diameter \( d \geq 2 \) and \( h \geq 2 \),

\[
\ell_{(c,k-b-c,b)} \leq C.
\]

As might be expected from the previously mentioned results for valency 3 distance-regular graphs, this theorem is closely related to the so-called Bannai-Ito Conjecture. Bannai and Ito conjectured that given an integer \( k \geq 3 \) there are finitely many distance-regular graphs with valency \( k \). In a series of papers [1, 2, 3] they showed that their conjecture was true for valency 3 and 4 and also that, for \( k \geq 3 \) an integer, there are finitely many bipartite distance-regular graphs with valency \( k \) [2]. In addition, it was recently shown that the Bannai-Ito Conjecture is true for valencies 5, 6 and 7 [12] and also that there are finitely many triangle-free (i.e., containing no 3-cycles) distance-regular graphs with valency 8, 9 or 10 [13].

Using Theorem 1.1, we now prove that the Bannai-Ito Conjecture is basically equivalent to bounding \( \ell_{(c,k-b-c,b)} \) by a function of \( b \) and \( c \).

**Theorem 1.2** The following statements are equivalent:

1. For each integer \( k \geq 3 \), there are finitely many distance-regular graphs with valency \( k \).
There exists a function \( f : \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+ \) such that for all \( k, b, c \in \mathbb{N}^+ \) and for all distance-regular graphs \( \Gamma \) with valency \( k \geq \max\{b + c, 3\} \), diameter \( d \geq 2 \) and \( h \geq 2 \)

\[ \ell_{(c,k-b-c,b)}(\Gamma) \leq f(b,c). \]

Proof: (1) \( \Rightarrow \) (2): By (1) there is a function \( g : \mathbb{N}^+ \rightarrow \mathbb{N}^+ \) such that, for all distance-regular graphs \( \Gamma \) with valency \( k \geq 3 \), and diameter \( d \geq 2 \),

\[ d \leq g(k). \]

For \( b, c \in \mathbb{N}^+ \) put

\[ f(b,c) := \max\{g(k) | \max\{b + c, 3\} \leq k < k(b,c,1)\}, \]

where \( k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+ \) is a function with the properties given in Theorem 1.1.

Now suppose \( b, c \in \mathbb{N}^+ \) and that \( \Gamma \) is a distance-regular graph with valency \( k \geq \max\{b + c, 3\} \), diameter \( d \geq 2 \), and \( h \geq 2 \). Then

\[ \ell_{(c,k-b-c,b)}(\Gamma) \leq d \leq g(k) \]

and, by Theorem 1.1 applied with \( C = 1 \), if \( k \geq k(b,c,1) \) then

\[ \ell_{(c,k-b-c,b)}(\Gamma) \leq 1. \]

Hence \( \ell_{(c,k-b-c,b)}(\Gamma) \leq f(b,c) \) and so (2) holds.

(2) \( \Rightarrow \) (1): Put

\[ F(k) := \max\{f(b,1) | 1 \leq b \leq k-1\}. \]

Suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 3 \) and diameter \( d \geq 2 \). Note that \( k \geq 1 + b_1 \) since otherwise \( k < b_1 + 1 = k - a_1 \) which is a contradiction. By (2)

\[ h = \ell_{(1,k-b_1-1,b_1)}(\Gamma) \leq F(k) \]

and so, since \( d < \frac{1}{2} k^3 h \) [10, Theorem 1.1],

\[ d < \frac{1}{2} k^3 F(k). \]

It is now straight-forward to check that (1) holds.

In view of results and examples contained in [6] and [8], it is plausible, for a distance-regular graph with \( h = 1 \) and diameter \( d \geq 4 \), that \( c_4 \geq 2 \). If this were indeed the case, then Theorem 1.1 would also hold for \( h = 1 \) and so the condition \( h \geq 2 \) in Theorem 1.2 (2) could be removed. Bearing this in mind, we make the following conjecture.

Conjecture 1.3 There exists a function \( f : \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+ \) such that for all \( b, c \in \mathbb{N}^+ \) satisfying \( c + b \leq k \) and for all distance-regular graphs \( \Gamma \) with valency \( k \geq \max\{b + c, 3\} \)

\[ \ell_{(c,k-b-c,b)}(\Gamma) \leq f(b,c). \]
In [7] Hiraki proved $\ell_{i,k-2,1} \leq 20$ for every distance-regular graph with valency $k \geq 3$, and hence this conjecture is true in case $b = c = 1$. Using Theorem 1.1 we now prove a theorem that generalizes Hiraki's result in case $h \neq 1$.

**Theorem 1.4** There exists a function $f : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ such that for all $c \in \mathbb{N}^+$ and all distance-regular graphs $\Gamma$ with valency $k \geq \max\{2c, 3\}$, diameter $d \geq 2$ and $h \geq 2$,

$$\ell_{(c,k-2c,c)}(\Gamma) \leq f(c).$$

**Proof:** Suppose that $k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+$ is a function with the properties given in Theorem 1.1. Given $c \in \mathbb{N}^+$, put $k_c := k(c,c,1) - 1$ and define

$$f(c) := 10 k_c 2^{k_c}.$$

Note that if $k \geq \max\{2c, 3\}$, then $k(c,c,1) \geq \max\{2c, 3\}$, and hence $f(c) > 1$.

Now suppose that $\Gamma$ is a distance-regular graph with valency $k \geq \max\{2c, 3\}$ and $h \geq 2$. In view of Bannai and Ito's bound, $\ell_{(c,k-2c,c)} \leq 10 k 2^k$, mentioned above and since $10 k 2^k$ is an increasing function on $[\max\{2c, 3\}, \infty)$, for all $k$ with $\max\{2c, 3\} \leq k \leq k(c,c,1)$,

$$\ell_{(c,k-2c,c)} \leq 10 k 2^k \leq f(c).$$

The theorem now follows since by Theorem 1.1, for $k \geq k(c,c,1)$,

$$\ell_{(c,k-2c,c)} \leq 1 < f(c).$$

This rest of this paper is organized as follows. In Section 2 we present some definitions and results concerning distance-regular graphs. We also present a partial solution to a problem posed on [5, p.189] that is of independent interest and follows from Theorem 1.1. In Section 3 we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph. Finally, in Section 4 we use these bounds to prove Theorem 1.1.

## 2 Distance-Regular Graphs

We begin this section by presenting some basic facts concerning distance-regular graphs (for more details see [5]). Suppose that $\Gamma$ is a distance-regular graph with valency $k \geq 2$, diameter $d \geq 2$ and intersection numbers $c_i, a_i, b_i$, $0 \leq i \leq d$. Clearly, $b_d = c_0 = a_0 = 0$ and $c_1 = 1$. In [5, Section 4.1], it is shown that $\Gamma_i(u)$ contains $k_i$ elements, where

$$k_0 := 1, \quad k_1 := k, \quad k_{i+1} := k_i b_i/c_{i+1}, \quad i = 0, \ldots, d - 1,$$

and in [5, Proposition 4.1.6] that

$$k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-1} > b_d = 0 \quad \text{and} \quad 1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k. \quad (2)$$

Recall that the eigenvalues of $\Gamma$ are the eigenvalues of the adjacency matrix of $\Gamma$. In particular, if $\theta$ is an eigenvalue of $\Gamma$ then $\theta \in [-\kappa, \kappa]$. We now state a result concerning the second largest eigenvalue of a distance regular graph.
Lemma 2.1 [12, Theorem 6.2] Suppose \( b, c \in \mathbb{N}^+ \) and \( k \geq \max\{b + c, 3\} \) is a positive integer. Let \( \Gamma \) be a distance-regular graph with valency \( k \) and put \( \ell := \ell_{(c, k-b-c, b)} \). The second largest eigenvalue \( \theta_1 \) of \( \Gamma \) satisfies

\[
\theta_1 \geq k - b - c + 2\sqrt{bc \cos \left( \frac{2\pi}{\ell + 1} \right)}.
\]

The standard sequence \( (u_i = u_i(\theta) | 0 \leq i \leq d) \) associated to an eigenvalue \( \theta \) of \( \Gamma \) is defined recursively by the equations

\[
u_0 = 1, \quad \nu_1 = \frac{\theta}{k}, \quad b_{i+1}u_{i+1} - (\theta - a_i)u_i + c_iu_{i-1} = 0 \quad \text{for } i = 1, 2, \ldots, d - 1.
\]

As is well-known, see e.g. [5, Theorem 4.1.4], if \( v := |V\Gamma| \), then the multiplicity \( m(\theta) \) of \( \theta \) is given by

\[
m(\theta) = \frac{v}{M(\theta)},
\]

where

\[
M(\theta) = \sum_{i=0}^{d} k_i u_i(\theta)^2.
\]

Now given a positive integer \( c \), define

\[
\xi_c := \min\{i | 1 \leq i \leq d \text{ and } c_i = c\}, \quad \text{and} \quad \eta_c := |\{i | 1 \leq i \leq d \text{ and } c_i = c\}|.
\]

To prove the next lemma we will use the following relationships between these numbers that were given in [10] (Lemma 2.1 and Proposition 3.2, respectively). If \( c > 1 \) is an integer, then

\[
\eta_c \leq 2\xi_c - 3,
\]

and if \( c \) is a positive integer and \( \eta_c \neq 0 \), then

\[
\xi_c \leq \frac{c^2}{4} \eta_1 + 1.
\]

Put

\[
e := \max\{i | 1 \leq i \leq d - 1 \text{ and } c_i \leq b_i\}.
\]

Lemma 2.2 Suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 3 \) and diameter \( d \geq 2 \), and that \( b, c \) are positive integers with \( k \geq b + c \). If \( \ell_{(c, k-b-c, b)} \geq 1 \), then

\[
d < \begin{cases} 
2(\eta_1 + 1) & \text{if } c_e = 1, \\
\frac{3}{2} \max\{b, c\}^2 \eta_i & \text{if } c_e \geq 2.
\end{cases}
\]

Proof: Since \( c_e+1 > b_{e+1} \), by [5, Proposition 4.1.6 (ii)]

\[
d < 2(e + 1).
\]
Thus, if \( c_{e} = 1 \), then since \( e \leq \eta_{1} \) it follows that \( d \leq 2\eta_{1} + 1 \) holds.

Now suppose \( c_{e} \geq 2 \). Since \( \{i \mid c_{i} = c_{e}\} = \{\xi_{c_{e}}, \xi_{c_{e}} + 1, \ldots, \xi_{c_{e}} + \eta_{c_{e}} - 1\} \),

\[
e \leq \xi_{c_{e}} + \eta_{c_{e}} - 1.
\]

By applying (4) and then (5) to the righthand side of this inequality, we have

\[
e \leq \frac{3}{4}c_{e}^{2}\eta_{1} - 1.
\]

But \( c_{e} \leq \max\{b, c\} \), since \( 1 \leq \ell_{(c,k-b-c,b)} \). Thus, in view of (6) and (7) we have \( d < \frac{3}{4} \max\{b, c\}^{2}\eta_{1} \). This completes the proof. \( \blacksquare \)

## 3 Bounding Terms of the Standard Sequence

In this section we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph that we use in the proof of Theorem 1.1. We begin with some definitions.

Suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 3 \) and diameter \( d \geq 2 \), and that \( \theta \) is an eigenvalue of \( \Gamma \) with \( a_{1} + 2\sqrt{b_{1}} \theta < k \). Let \( 1 \leq p < d \) be the largest integer for which \( c_{p} \leq b_{p} \) and \( a_{p} + 2\sqrt{b_{p}c_{p}} < \theta \) both hold. Define

\[
T := T(\theta) = \{i \mid 0 \leq i \leq p \text{ and } (c_{i}, a_{i}, b_{i}) \neq (c_{i+1}, a_{i+1}, b_{i+1})\}.
\]

Put \( s := |T| - 1 \) and let \( t_{0}, t_{1}, \ldots, t_{s} \) be the ordering of \( T \) with \( 0 = t_{0} < t_{1} < \cdots < t_{s} = p \).

Now, if \( (u_{i} = u_{i}(\theta) \mid 0 \leq i \leq d) \) is the standard sequence associated to \( \theta \) and, for \( 1 \leq i \leq s \), the largest and smallest roots of the equation

\[
b_{t_{i}}u_{t_{i+1}} + (a_{t_{i}} - \theta)u_{t_{i}} + c_{t_{i}}u_{t_{i}-1} = 0
\]

are \( \rho_{i} := \rho_{i}(\theta) \) and \( \sigma_{i} := \sigma_{i}(\theta) \), respectively, then by the theory of three-term recurrences there are numbers \( \gamma_{i} \) and \( \delta_{i} \) with

\[
u_{j} = \gamma_{i}\rho_{i}^{j-t_{i}-1} + \delta_{i}\sigma_{i}^{j-t_{i}-1} \quad (t_{i-1} \leq j \leq t_{i} + 1).
\]

Note that since \( a_{i} + 2\sqrt{b_{i}c_{i}} < \theta < k \), we have \( 0 < \sigma_{i} < \rho_{i} < 1 \), \( 1 \leq i \leq s \).

We now list some inequalities that will be used in the proof of Theorem 1.1.

**Proposition 3.1** Suppose \( 1 \leq i \leq s \) and \( u_{i} \), \( \gamma_{i} \) and \( \rho_{i} \) are as defined just above. Then the following inequalities hold

(i) \( \rho_{i+1} < \rho_{i} \), \( i \neq s \),

(ii) \( u_{t_{i}-1} + 1 > \rho_{t_{i}}u_{t_{i}-1} \),
(iii) $\gamma_i > u_{i-1}$,
(iv) $u_{ti} > \prod_{j=1}^{i} p_j^{t_j - t_{j-1}}$.

Proof: (i): For positive integers $b, c$ satisfying $b + c \leq k$, $c \leq b$ and $k - b - c + 2\sqrt{bc} < \theta$ we define

$$f_{b,c}(x) := bx^2 + (k - b - c - \theta)x + c.$$ 

Let $\rho_{b,c}$ be the largest root of $f_{b,c}(x) = 0$. Since $b \geq c$,

$$\theta > k - b - c + 2\sqrt{bc} > k - (b + 1) - c + 2\sqrt{(b+1)c},$$

and hence both $\rho_{b,c}$ and $\rho_{b+1,c}$ are positive. Moreover, $0 < \rho_{b,c} < 1$ since $k - b - c + 2\sqrt{bc} < \theta < k$.

Hence

$$f_{b+1,c}(\rho_{b,c}) = \rho_{b,c}^2 - \rho_{b,c} = \rho_{b,c}(\rho_{b,c} - 1) < 0$$

and therefore $\rho_{b,c} < \rho_{b+1,c}$. It is straightforward to show in a similar fashion that $\rho_{b,c} < \rho_{b,c-1}$ holds. It now follows in view of (2) that (i) must hold.

(ii) and (iii): We will prove that these hold using induction on $i$. Suppose $i = 1$. Then $u_0 = u_0 = 1$ and $u_{t_{i+1}} = u_1 = \frac{b}{k}$. Since $a_1 + 2\sqrt{b_1} < \theta < k$ and $\rho_1$ is the largest root of

$$b_1 x^2 + (a_1 - \theta)x + 1 = 0,$$

we have

$$b_1 \left(\frac{\theta}{k}\right)^2 + (a_1 - \theta)\frac{\theta}{k} + 1 = \left(1 - \frac{\theta}{k}\right)\left(1 + (a_1 + 1)\frac{\theta}{k}\right) > 0.$$ 

Hence $\frac{\theta}{k} > \rho_1$. Thus $\gamma_1 > 1$ since $\gamma_1 \rho_1 + \delta_1 \sigma_1 = u_1 = \frac{b}{k} > \rho_1$, $\gamma_1 + \delta_1 = u_0 = 1$ and $\rho_1 > \sigma_1 > 0$. Therefore (ii) and (iii) hold for $i = 1$.

Now suppose $2 \leq i < s$ and suppose $u_{t_{i+1}} > \rho_i u_{t_{i-1}}$ and $\gamma_i > u_{t_{i-1}}$ both hold. Then $\delta_i < 0$ since $\gamma_i + \delta_i = u_{t_{i-1}}$. Thus, using equations

$$u_{t_i} = \gamma_i \rho_i^{t_i - t_{i-1}} + \delta_i \sigma_i^{t_i - t_{i-1}}$$

and

$$u_{t_{i+1}} = \gamma_i \rho_i^{t_{i+1} - t_{i-1}} + \delta_i \sigma_i^{t_{i+1} - t_{i-1}} + 1,$$

we obtain

$$\rho_i u_{t_i} < u_{t_{i+1}}.$$ 

(9)

Hence $\rho_{i+1} u_{t_i} < \rho_i u_{t_i} < u_{t_{i+1}}$ by (i) and (9) and so (ii) holds.

Now, in view of

$$u_{t_i} = \gamma_{i+1} + \delta_{i+1}$$

and

$$u_{t_{i+1}} = \gamma_{i+1} \rho_{i+1} + \delta_{i+1} \sigma_{i+1},$$

it follows that

$$\gamma_{i+1} = \frac{u_{t_{i+1}} - \sigma_{i+1} u_{t_i}}{\rho_{i+1} - \sigma_{i+1}}$$

holds, and hence by (i) and (9)

$$\gamma_{i+1} > \frac{\rho_i - \sigma_{i+1}}{\rho_{i+1} - \sigma_{i+1}} u_{t_i} > u_{t_i}.$$
holds. Thus (iii) holds.

(iv) We prove this by using induction on $i$. Suppose $i = 1$. Then by (8), (ii) and (iii) we have

\[ u_{t_1} - \rho_1^{t_1} = (\gamma_1 - 1)\rho_1^{t_1} + \delta_1\sigma_1^{t_1} = (\gamma_1 - 1)\rho_1^{t_1} + \sigma_1^{t_1-1}(u_1 - \gamma_1\rho_1) > \rho_1(\gamma_1 - 1)(\rho_1^{t_1-1} - \sigma_1^{t_1-1}) > 0. \]

Therefore (iv) holds for $i = 1$.

Now, suppose $2 \leq i < s$ and assume

\[ u_{t_i} > \prod_{j=1}^{i} \rho_j^{t_j-t_{j-1}}. \]  \hfill (10)

Then using (iii), $u_{t_i} = \gamma_{i+1} + \delta_{i+1}$ and $u_{t_{i+1}} = \gamma_{i+1}\rho_{i+1}^{t_{i+1}-t_i} + \delta_{i+1}\sigma_{i+1}^{t_{i+1}-t_i}$, we obtain

\[ u_{t_{i+1}} - u_{t_i}\rho_{i+1}^{t_{i+1}-t_i} = \delta_{i+1} \left(\sigma_{i+1}^{t_{i+1}-t_i} - \rho_{i+1}^{t_{i+1}-t_i}\right) > 0. \]

But by (10) it then follows that

\[ u_{t_{i+1}} > u_{t_i}\rho_{i+1}^{t_{i+1}-t_i} > \prod_{j=1}^{i} \rho_j^{t_j-t_{j-1}} \rho_{i+1}^{t_{i+1}-t_i} = \prod_{j=1}^{i+1} \rho_j^{t_j-t_{j-1}} \]

holds. This completes the proof of (iv)

\[ \square \]

4 Proof of Theorem 1.1

Before proving the theorem, we first present some definitions. Suppose that $b, c$ and $C$ are arbitrary positive integers. Put

\[ \phi = \phi_{b,c} \quad := \quad -b - c - 2\sqrt{bc} \quad \text{and} \quad \phi' = \phi'_{b,c,C} \quad := \quad -b - c + 2\sqrt{bc} \cos\left(\frac{2\pi}{C+2}\right). \]

Note

\[ \phi < -b - c - \sqrt{bc} \leq \phi'. \]

For each $c'$ with $1 \leq c' \leq c$, let $\beta_{c'}$ be the smallest positive integer satisfying both $\beta_{c'} \geq c'$ and $\phi \geq -\beta_{c'} - c' + 2\sqrt{\beta_{c'}c'}$.

Now, for $l, m$ any positive integers and for any real number $\lambda \geq -l - m - 2\sqrt{lm}$, let $\eta_m(\lambda)$ denote the largest root of the equation

\[ lx^2 - (l + m + \lambda)x + m = 0. \]
Note that since $2\sqrt{\beta_{c'}c'} \leq \phi + \beta_{c'} + c' < \phi' + \beta_{c'} + c'$, it follows that

$$0 < \frac{c'}{\beta_{c'}} < \tau_{\beta_{c'},c'}(\phi') < 1.$$  \hfill (11)

Define

$$\rho = \rho_{b,c,C} := \min\{\tau_{\beta_{c'},c'}(\phi') \mid 1 \leq c' \leq c\} \text{ and }$$

$$\alpha := \max\{\frac{\beta_{c'}}{c'} \mid 1 \leq c' \leq c\}.$$  

By (11) and $\beta_{1} \geq 9$, we have \(\rho < 1\) and $9 \leq \alpha$. \hfill (12)

**Proof of Theorem 1.1:** We define a function $k$ and prove that it has the required properties. For $b$, $c$ and $C$ arbitrary positive integers, put

$$k(b,c,C) := \max\left\{\frac{\alpha^{20}}{\rho^{12}}, \frac{2}{\rho^{3}}, \frac{\alpha^{2\max\{b,c\}^{2}}}{\rho} \right\}, b + c, 3\}.$$  

Now suppose that $\Gamma$ is a distance-regular graph with $h(\Gamma) \geq 2$, valency $k \geq \max\{b + c, 3\}$, diameter $d \geq 2$ and

$$\ell_{(c,k-b-c,b)} > C.$$  

We prove

$$k < \begin{cases} \frac{\alpha^{20}}{\rho^{12}} & \text{if } c = 1, \\ \frac{2}{\rho^{3}} \left(\frac{\alpha^{2\max\{b,c\}^{2}}}{\rho}\right)^{9} & \text{if } c \geq 2, \end{cases}$$

from which the theorem immediately follows.

Let $w$ be the largest non-negative integer so that $t := t_{w}$ is the largest element of $T(\theta_{1})$ with

$$k - b_{t} - c_{t} + 2\sqrt{b_{t}c_{t}} < k - b - c + 2\sqrt{bc}.$$  \hfill (13)

Note that this last equation implies $c_{t} \leq c$.

Now, since $\ell_{(c,k-b-c,b)} \geq C + 1 \geq 2$, by Lemma 2.1 the second largest eigenvalue $\theta_{1}$ of $\Gamma$ satisfies

$$\theta_{1} \geq k + \phi'.$$

Hence, in view of the definitions of $\rho_{i}$ and $\rho$,

$$\rho_{w}(\theta_{1}) \geq \rho_{w}(k + \phi') = \tau_{w_{c}}(\phi') \geq \rho.$$  

Therefore, since $\rho_{i}(\theta_{1}) \geq \rho$ for $1 \leq i \leq w$, it follows by Proposition 3.1 (i) and (iv) that

$$w_{t} > \rho^{t}.$$  \hfill (14)
Thus, by (3) and (14) we have

\[ m(\theta_1) < \frac{v}{k_1u_t^2} < \frac{v}{k_t \rho^{2t}}. \]  

(15)

Moreover, since \( b_1 \geq \frac{1}{2}k \) and \( h \geq 2 \), the Terwilliger Tree bound [11, Proposition 3.3] implies

\[ 2\left(\frac{k}{2}\right)^{\frac{3}{2}h} \leq 2(b_1)^{\frac{1}{2}h} \leq m(\theta_1). \]  

(16)

In addition, by (1) and (2) we have

\[
\begin{align*}
  k_t &\leq k_t \leq \alpha^i k_t & & 0 \leq i \leq t-1, \\
  k_{t+i} &\leq \alpha^{i} k_t \leq \alpha^{i+t} k_t & & 0 \leq i \leq d-t,
\end{align*}
\]

and so, as \( d \geq 2 \) and \( \alpha \geq 2 \),

\[ v \leq k_t \sum_{j=0}^{d} \alpha^j = k_t \left[ \frac{\alpha^{d+1} - 1}{\alpha - 1} \right] < k_t \alpha^{\frac{3}{2}d}. \]  

(17)

Thus, by (12), (15), (16), (17) and \( h \geq 2 \),

\[ k < 2 \left( \frac{\alpha^{\frac{3}{2}d}}{2 \rho^{2t}} \right)^{\frac{h}{2}}. \]  

(18)

Now, suppose \( c = 1 \). Since \( c_t \leq c = 1 \) we have \( t \leq \eta_1 \). Hiraki [9, Theorem 2] has shown that if \( h = h(\Gamma) \geq 2 \), then

\[ \eta_1 \leq 2(h + 1). \]  

(19)

Thus Lemma 2.2 implies \( d \leq 2\eta_1 + 1 \leq 4h + 5 \) and so

\[ \frac{\alpha^{\frac{3}{2}d}}{2 \rho^{2t}} < \frac{\alpha^{6h+8}}{2 \rho^{4h+4}}. \]

So, by (18) and \( h \geq 2 \), we obtain

\[ k < \frac{2\alpha^{12}}{\rho^8} \left( \frac{\alpha^{16}}{4 \rho^8} \right)^{\frac{h}{2}} \leq \frac{\alpha^{20}}{\rho^{12}}. \]

Now, to complete the proof, suppose \( c \geq 2 \). Since \( c_t \leq c \), by (4), (5) and (19), we have

\[ t < \xi_c + \eta_c \leq \frac{3}{2} c^2 (h + 1). \]

Also, by Lemma 2.2 and (19),

\[ d < \frac{3}{2} \max\{b, c\}^2 \eta_1 \leq 3 \max\{b, c\}^2 (h + 1). \]

Thus by (18), \( h \geq 2 \) and the last two bounds on \( t \) and \( d \),

\[ k < 2 \left( \frac{\alpha^{\frac{3}{2} \max\{b, c\}^2 (h+1)}}{2 \rho^{2c^2 (h+1)}} \right)^{\frac{3}{2}} h = 2^{1 - \frac{3}{2} h} \left( \frac{\alpha^{\frac{3}{2} \max\{b, c\}^2}}{\rho^{c^2}} \right)^{\frac{6h+11}{h}} \leq 2 \left( \frac{\alpha^{2 \max\{b, c\}^2}}{\rho^{c^2}} \right)^{9}. \]

This completes the proof. \( \blacksquare \)
References


[13] J. H. Koolen and V. Moulton, There are finitely many triangle-free distance-regular graphs with degree 8, 9 or 10, *submitted*