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A bound for the number of columns \( \ell_{(c,a,b)} \) in the intersection array of a distance-regular graph (Algebraic Combinatorics)

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A bound for the number of columns $\ell_{(c,a,b)}$ in the intersection array of a distance-regular graph

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Abstract
In this paper we give a bound for the number $\ell_{(c,a,b)}$ of columns $(c,a,b)^T$ in the intersection array of a distance-regular graph. We also show that this bound is intimately related to the Bannai-Ito Conjecture.

1 Introduction

Suppose that $\Gamma$ is a finite connected graph with vertex set $V\Gamma$. As usual, we define the distance between any two vertices $u$ and $v$ of $\Gamma$ to be the length of any shortest path in $\Gamma$ between $u$ and $v$, and the diameter $d$ of $\Gamma$ to be the largest distance between any pair of vertices in $V\Gamma$. For $u \in V\Gamma$ and $i$ any non-negative integer not exceeding $d$, let $\Gamma_i(u)$ denote the set of vertices in $V\Gamma$ that are at distance $i$ from $u$ and put $\Gamma_{-1}(v) = \Gamma_{d+1}(v) := \emptyset$. The graph $\Gamma$ is called distance-regular if there are integers $b_i, c_i$, $0 \leq i \leq d$, so that for any two vertices $u$ and $v$ in $V\Gamma$ at distance $i$, there are precisely $c_i$ neighbors of $v$ in $\Gamma_{i-1}(u)$ and $b_i$ neighbors of $v$ in $\Gamma_{i+1}(u)$. Clearly such a graph is regular with valency $k := b_0$. The numbers $c_i, b_i$, and $a_i$, where

$$a_i := k - b_i - c_i \quad (i = 0, \ldots, d)$$

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is the number of neighbors of $v$ in $\Gamma_i(u)$ for $u, v \in V\Gamma$ at distance $i$, are called the intersection numbers of $\Gamma$, and

$$
\begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{j-1} & c_{j} & \cdots & c_{d-1} & c_{d} \\
a_0 & a_1 & a_2 & \cdots & a_{j-1} & a_{j} & \cdots & a_{d-1} & a_{d} \\
b_0 & b_1 & b_2 & \cdots & b_{j-1} & b_{j} & \cdots & b_{d-1} & b_{d}
\end{bmatrix}
$$

the intersection array of $\Gamma$.

Now, suppose that $\Gamma$ is a distance-regular graph with valency $k \geq 2$, diameter $d \geq 2$ and intersection numbers $c_i, a_i, b_i$, $0 \leq i \leq d$. Given integers $a \geq 0$ and $b, c \geq 1$ with $a + b + c = k$, we define

$$
\ell_{(c,a,b)} = \ell_{(c,a,b)}(\Gamma) := |\{i \mid 1 \leq i \leq d-1 \text{ and } (a_i, a_i, b_i) = (c, a, b)\}|,
$$

that is, the number of columns $(c,a,b)^T$ in the intersection array of $\Gamma$, and put

$$
h = h(\Gamma) := \ell_{(1,a_1,b_1)}.
$$

Note that since $d \geq 2$ and $c_1 = 1$ we have $h \geq 1$.

Finding good bounds for $\ell_{(c,a,b)}$ is a powerful technique for understanding distance-regular graphs. For example, in [1] Bannai and Ito showed that, for a distance-regular graph with valency $k \geq 3$, if $c$ is an integer with $0 \leq 2c \leq k$ then $\ell_{(c,k-2c,c)} \leq 10k2^k$, from which they deduced that there are finitely many distance-regular graphs with valency 3. Also, in [4] Biggs et al. used circuit chasing to considerably improve this bound, which enabled them to classify the distance-regular graphs with valency 3.

In this paper we prove the following theorem.

**Theorem 1.1** There exists a function $k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \rightarrow \mathbb{N}^+$ so that for all positive integers $b, c, C$, $k(b, c, C) \geq \max\{b + c, 3\}$ and for all distance-regular graphs $\Gamma$ with valency $k \geq k(b, c, C)$, diameter $d \geq 2$ and $b \geq 2$,

$$
\ell_{(c,k-b-c,b)} \leq C.
$$

As might be expected from the previously mentioned results for valency 3 distance-regular graphs, this theorem is closely related to the so-called Bannai-Ito Conjecture. Bannai and Ito conjectured that given an integer $k \geq 3$ there are finitely many distance-regular graphs with valency $k$. In a series of papers [1, 2, 3] they showed that their conjecture was true for valency 3 and 4 and also that, for $k \geq 3$ an integer, there are finitely many bipartite distance-regular graphs with valency $k$ [2]. In addition, it was recently shown that the Bannai-Ito Conjecture is true for valencies 5, 6 and 7 [12] and also that there are finitely many triangle-free (i.e. containing no 3-cycles) distance-regular graphs with valency 8, 9 or 10 [13].

Using Theorem 1.1, we now prove that the Bannai-Ito Conjecture is basically equivalent to bounding $\ell_{(c,k-b-c,b)}$ by a function of $b$ and $c$.

**Theorem 1.2** The following statements are equivalent:

1. For each integer $k \geq 3$, there are finitely many distance-regular graphs with valency $k$. 


(2) There exists a function $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ such that for all $k, b, c \in \mathbb{N}^+$ and for all distance-regular graphs $\Gamma$ with valency $k \geq \max\{b + c, 3\}$, diameter $d \geq 2$ and $h \geq 2$

$$\ell_{(c, k-b-c, b)} \leq f(b, c).$$

Proof: $(1) \Rightarrow (2)$: By $(1)$ there is a function $g : \mathbb{N}^+ \to \mathbb{N}^+$ such that, for all distance-regular graphs $\Gamma$ with valency $k \geq 3$, and diameter $d \geq 2$,

$$d \leq g(k).$$

For $b, c \in \mathbb{N}^+$ put

$$f(b, c) := \max\{g(k) \mid \max\{b + c, 3\} \leq k < k(b, c, 1)\},$$

where $k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ is a function with the properties given in Theorem 1.1.

Now suppose $b, c \in \mathbb{N}^+$ and that $\Gamma$ is a distance-regular graph with valency $k \geq \max\{b + c, 3\}$, diameter $d \geq 2$, and $h \geq 2$. Then

$$\ell_{(c, k-b-c, b)}(\Gamma) \leq d \leq g(k)$$

and, by Theorem 1.1 applied with $C = 1$, if $k \geq k(b, c, 1)$ then

$$\ell_{(c, k-b-c, b)}(\Gamma) \leq 1.$$

Hence $\ell_{(c, k-b-c, b)}(\Gamma) \leq f(b, c)$ and so $(2)$ holds.

$(2) \Rightarrow (1)$: Put

$$F(k) := \max\{f(b, 1) \mid 1 \leq b \leq k-1\}.$$

Suppose that $\Gamma$ is a distance-regular graph with valency $k \geq 3$ and diameter $d \geq 2$. Note that $k \geq 1 + b_1$ since otherwise $k < b_1 + 1 = k - a_1$ which is a contradiction. By $(2)$

$$h = \ell_{(1,k-b_1-1,b_1)} \leq F(k)$$

and so, since $d < \frac{1}{2}k^3h$ [10, Theorem 1.1],

$$d < \frac{1}{2}k^3F(k).$$

It is now straight-forward to check that $(1)$ holds.

In view of results and examples contained in [6] and [8], it is plausible, for a distance-regular graph with $h = 1$ and diameter $d \geq 4$, that $c_4 \geq 2$. If this were indeed the case, then Theorem 1.1 would also hold for $h = 1$ and so the condition $h \geq 2$ in Theorem 1.2 $(2)$ could be removed. Bearing this in mind, we make the following conjecture.

**Conjecture 1.3** There exists a function $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ such that for all $b, c \in \mathbb{N}^+$ satisfying $b + c \leq k$ and for all distance-regular graph $\Gamma$ with valency $k \geq \max\{b + c, 3\}$

$$\ell_{(c,k-b-c,b)} \leq f(b, c).$$
In [7] Hiraki proved \( \ell_{(1,k-2,1)} \leq 20 \) for every distance-regular graph with valency \( k \geq 3 \), and hence this conjecture is true in case \( b = c = 1 \). Using Theorem 1.1 we now prove a theorem that generalizes Hiraki’s result in case \( h \neq 1 \).

**Theorem 1.4** There exists a function \( f : \mathbb{N}^+ \to \mathbb{N}^+ \) such that for all \( c \in \mathbb{N}^+ \) and all distance-regular graphs \( \Gamma \) with valency \( k \geq \max\{2c, 3\} \), diameter \( d \geq 2 \) and \( h \geq 2 \),

\[
\ell_{(c,k-2c,c)}(\Gamma) \leq f(c).
\]

**Proof:** Suppose that \( k : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+ \) is a function with the properties given in Theorem 1.1. Given \( c \in \mathbb{N}^+ \), put \( k_c := k(c,c,1) - 1 \) and define

\[
f(c) := 10k_c2^{k_c}.
\]

Note that if \( k \geq \max\{2c, 3\} \), then \( k(c,c,1) \geq \max\{2c, 3\} \), and hence \( f(c) > 1 \).

Now suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq \max\{2c, 3\} \) and \( h \geq 2 \). In view of Bannai and Ito’s bound, \( \ell_{(c,k-2c,c)} \leq 10k2^k \), mentioned above and since \( 10k2^k \) is an increasing function on \( [\max\{2c, 3\}, \infty) \), for all \( k \) with \( \max\{2c, 3\} \leq k \leq \max\{2c, 3\} \)

\[
\ell_{(c,k-2c,c)} \leq 10k2^k \leq f(c).
\]

The theorem now follows since by Theorem 1.1, for \( k \geq k(c,c,1) \),

\[
\ell_{(c,k-2c,c)} \leq 1 < f(c).
\]

This rest of this paper is organized as follows. In Section 2 we present some definitions and results concerning distance-regular graphs. We also present a partial solution to a problem posed on [5, p.189] that is of independent interest and follows from Theorem 1.1. In Section 3 we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph. Finally, in Section 4 we use these bounds to prove Theorem 1.1.

## 2 Distance-Regular Graphs

We begin this section by presenting some basic facts concerning distance-regular graphs (for more details see [5]). Suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 2 \), diameter \( d \geq 2 \) and intersection numbers \( c_i, a_i, b_i, 0 \leq i \leq d \). Clearly, \( b_d = c_0 = a_0 = 0 \) and \( c_1 = 1 \). In [5, Section 4.1], it is shown that \( \Gamma_i(u) \) contains \( k_i \) elements, where

\[
k_0 := 1, \ k_1 := k, \ k_{i+1} := k_i b_i/c_{i+1}, \ i = 0, \ldots, d-1,
\]

and in [5, Proposition 4.1.6] that

\[
k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-1} > b_d = 0 \text{ and } 1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k.
\]

Recall that the eigenvalues of \( \Gamma \) are the eigenvalues of the adjacency matrix of \( \Gamma \). In particular, if \( \theta \) is an eigenvalue of \( \Gamma \) then \( \theta \in [-k,k] \). We now state a result concerning the second largest eigenvalue of a distance regular graph.
Lemma 2.1 [12, Theorem 6.2] Suppose $b, c \in \mathbb{N}^+$ and $k \geq \max\{b + c, 3\}$ is a positive integer. Let $\Gamma$ be a distance-regular graph with valency $k$ and put $\ell := \ell_{(c, k-b-c, b)}$. The second largest eigenvalue $\theta_1$ of $\Gamma$ satisfies
\[ \theta_1 \geq k - b - c + 2\sqrt{bc} \cos\left(\frac{2\pi}{\ell + 1}\right). \]

The standard sequence $(u_i = u_i(\theta) \mid 0 \leq i \leq d)$ associated to an eigenvalue $\theta$ of $\Gamma$ is defined recursively by the equations
\[ u_0 = 1, \quad u_1 = \theta/k, \quad b_i u_{i+1} - (\theta - a_i) u_i + c_i u_{i-1} = 0 \quad \text{for} \quad i = 1, 2, \ldots, d-1. \]

As is well-known, see e.g. [5, Theorem 4.1.4], if $v := |V\Gamma|$, then the multiplicity $m(\theta)$ of $\theta$ is given by
\[ m(\theta) = \frac{v}{M(\theta)}, \]
where
\[ M(\theta) = \sum_{i=0}^{d} k_i u_i(\theta)^2. \]

Now given a positive integer $c$, define
\[ \xi_c := \min\{i \mid 1 \leq i \leq d \text{ and } c_i = c\}, \quad \text{and} \]
\[ \eta_c := |\{i \mid 1 \leq i \leq d \text{ and } c_i = c\}|. \]

To prove the next lemma we will use the following relationships between these numbers that were given in [10] (Lemma 2.1 and Proposition 3.2, respectively). If $c > 1$ is an integer, then
\[ \eta_c \leq 2\xi_c - 3, \quad (4) \]
and if $c$ is a positive integer and $\eta_c \neq 0$, then
\[ \xi_c \leq \frac{c^2}{4} \eta_1 + 1. \quad (5) \]

Put
\[ e := \max\{i \mid 1 \leq i \leq d - 1 \text{ and } c_i \leq b_i\}. \]

Lemma 2.2 Suppose that $\Gamma$ is a distance-regular graph with valency $k \geq 3$ and diameter $d \geq 2$, and that $b, c$ are positive integers with $k \geq b + c$. If $\ell_{(c, k-b-c, b)} \geq 1$, then
\[ d < \begin{cases} 2(\eta_1 + 1) & \text{if } c_e = 1, \\ \frac{3}{2} \max\{b, c\}^2 \eta_1 & \text{if } c_e \geq 2. \end{cases} \]

Proof: Since $c_{e+1} > b_{e+1}$, by [5, Proposition 4.1.6 (ii)]
\[ d < 2(e + 1). \quad (6) \]
Thus, if \( c_e = 1 \), then since \( e \leq \eta_1 \) it follows that \( d \leq 2\eta_1 + 1 \) holds.

Now suppose \( c_e \geq 2 \). Since \( \{i \mid a_i = c_e\} = \{\xi_{c_e}, \xi_{c_e} + 1, \ldots, \xi_{c_e} + \eta_{c_e} - 1\} \),
\[
e \leq \xi_{c_e} + \eta_{c_e} - 1.
\]
By applying (4) and then (5) to the righthand side of this inequality, we have
\[
e \leq \frac{3}{4}c_e^2\eta_1 - 1.
\] (7)
But \( c_e \leq \max\{b, c\} \), since \( 1 \leq \ell_{(c, k-b-c, b)} \).

Thus, in view of (6) and (7) we have \( d < \frac{3}{4} \max\{b, c\}^2\eta_1 \). This completes the proof. \( \blacksquare \)

3 Bounding Terms of the Standard Sequence

In this section we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph that we use in the proof of Theorem 1.1. We begin with some definitions.

Suppose that \( \Gamma \) is a distance-regular graph with valency \( k \geq 3 \) and diameter \( d \geq 2 \), and that \( \theta \) is an eigenvalue of \( \Gamma \) with \( a_1 + 2\sqrt{b_1} < \theta < k \). Let \( 1 \leq p < d \) be the largest integer for which \( c_p \leq b_p \) and \( a_p + 2\sqrt{b_p} \leq \theta \) both hold. Define
\[
T := T(\theta) = \{i \mid 0 \leq i \leq p \text{ and } (c_i, a_i, b_i) \neq (c_{i+1}, a_{i+1}, b_{i+1})\}.
\]
Put \( s := |T| - 1 \) and let \( t_0, t_1, \ldots, t_s \) be the ordering of \( T \) with \( 0 = t_0 < t_1 < \cdots < t_s = p \).

Now, if \( (u_i = u_i(\theta) \mid 0 \leq i \leq d) \) is the standard sequence associated to \( \theta \) and, for \( 1 \leq i \leq s \), the largest and smallest roots of the equation
\[
b_{t_i}u_{t_i+1} + (a_{t_i} - \theta)u_{t_i} + c_{t_i}u_{t_i-1} = 0
\]
are \( \rho_i := \rho_i(\theta) \) and \( \sigma_i := \sigma_i(\theta) \), respectively, then by the theory of three-term recurrences there are numbers \( \gamma_i \) and \( \delta_i \) with
\[
u_j = \gamma_i\rho_i^{j-t_i-1} + \delta_i\sigma_i^{j-t_i-1} \quad (t_i-1 \leq j \leq t_i + 1).
\] (8)
Note that since \( a_i + 2\sqrt{b_ic_i} < \theta < k \), we have \( 0 < \sigma_i < \rho_i < 1 \), \( 1 \leq i \leq s \).

We now list some inequalities that will be used in the proof of Theorem 1.1.

Proposition 3.1 Suppose \( 1 \leq i \leq s \) and \( u_i, \gamma_i \) and \( \rho_i \) are as defined just above. Then the following inequalities hold
\[
(i) \quad \rho_{i+1} < \rho_i, \quad i \neq s,
(ii) \quad u_{t_i-1} > \rho_iu_{t_i-1}.
\]
(iii) \( \gamma_i > u_{t_{i-1}} \).

(iv) \( u_{t_i} > \prod_{j=1}^{i} \rho_j^{t_j-t_{j-1}} \).

**Proof:** (i): For positive integers \( b, c \) satisfying \( b + c \leq k, c \leq b \) and \( k - b - c + 2\sqrt{bc} < \theta \) we define

\[
f_{b,c}(x) := bx^2 + (k - b - c - \theta)x + c.
\]

Let \( \rho_{b,c} \) be the largest root of \( f_{b,c}(x) = 0 \). Since \( b \geq c \),

\[
\theta > k - b - c + 2\sqrt{bc} > k - (b + 1) - c + 2\sqrt{(b+1)c},
\]

and hence both \( \rho_{b,c} \) and \( \rho_{b+1,c} \) are positive. Moreover, \( 0 < \rho_{b,c} < 1 \) since \( k - b - c + 2\sqrt{bc} < \theta < k \).

Hence

\[
f_{b+1,c}(\rho_{b,c}) = \rho_{b,c}^2 - \rho_{b,c} = \rho_{b,c}(\rho_{b,c} - 1) < 0
\]

and therefore \( \rho_{b,c} < \rho_{b+1,c} \). It is straightforward to show in a similar fashion that \( \rho_{b,c} < \rho_{b,c-1} \) holds. It now follows in view of (2) that (i) must hold.

(ii) and (iii): We will prove that these hold using induction on \( i \). Suppose \( i = 1 \). Then \( u_{t_0} = u_0 = 1 \) and \( u_{t_{0+1}} = u_1 = \frac{\theta}{k} \).

Since \( a_1 + 2\sqrt{b_1} < \theta < k \) and \( \rho_1 \) is the largest root of

\[
b_1 x^2 + (a_1 - \theta)x + 1 = 0,
\]

we have

\[
b_1 \left(\frac{\theta}{k}\right)^2 + (a_1 - \theta)\frac{\theta}{k} + 1 = \left(1 - \frac{\theta}{k}\right) \left(1 + (a_1 + 1)\frac{\theta}{k}\right) > 0.
\]

Hence \( \frac{\theta}{k} > \rho_1 \). Thus \( \gamma_1 > 1 \) since \( \gamma_1 \rho_1 + \delta_1 \sigma_1 = u_1 = \frac{\theta}{k} > \rho_1 \), \( \gamma_1 + \delta_1 = u_0 = 1 \) and \( \rho_1 > \sigma_1 > 0 \). Therefore (ii) and (iii) hold for \( i = 1 \).

Now suppose \( 2 \leq i < s \) and suppose \( u_{t_{i-1}+1} > \rho_i u_{t_{i-1}} \) and \( \gamma_i > u_{t_{i-1}} \) both hold. Then \( \delta_i < 0 \) since \( \gamma_i + \delta_i = u_{t_{i-1}} \). Thus, using equations

\[
u_{t_i} = \gamma_i \rho_i^{t_i-t_{i-1}} + \delta_i \sigma_i^{t_i-t_{i-1}} \quad \text{and} \quad u_{t_{i+1}} = \gamma_i \rho_i^{t_{i+1}-t_{i-1}+1} + \delta_i \sigma_i^{t_{i+1}-t_{i-1}+1},
\]

we obtain

\[
\rho_i u_{t_i} < u_{t_{i+1}}.
\]

Hence \( \rho_{i+1} u_{t_i} < \rho_i u_{t_i} < u_{t_{i+1}} \) by (i) and (9) and so (ii) holds.

Now, in view of

\[
u_{t_i} = \gamma_{i+1} + \delta_{i+1} \quad \text{and} \quad u_{t_{i+1}} = \gamma_{i+1} \rho_{i+1} + \delta_{i+1} \sigma_{i+1},
\]

it follows that

\[
\gamma_{i+1} = \frac{u_{t_{i+1}} - \sigma_{i+1} u_{t_i}}{\rho_{i+1} - \sigma_{i+1}}
\]

holds, and hence by (i) and (9)

\[
\gamma_{i+1} > \frac{\rho_i - \sigma_{i+1}}{\rho_{i+1} - \sigma_{i+1}} u_{t_i} > u_{t_i}.
\]
holds. Thus (iii) holds.

(iv) We prove this by using induction on $i$. Suppose $i = 1$. Then by (8), (ii) and (iii) we have

$$u_{t_{1}} - \rho_{1}^{i_{1}} = (\gamma_{1} - 1)\rho_{1}^{i_{1}} + \delta_{1}\sigma_{1}^{i_{1}} = (\gamma_{1} - 1)\rho_{1}^{i_{1}} + \gamma_{1}(u_{1} - \gamma_{1}\rho_{1}) > \rho_{1}(\gamma_{1} - 1)(\rho_{1}^{i_{1} - 1} - \sigma_{1}^{i_{1} - 1}) > 0.$$ 

Therefore (iv) holds for $i = 1$.

Now, suppose $2 \leq i < s$ and assume

$$u_{t_{i}} > \prod_{j=1}^{i} \rho_{j}^{j_{j} - j_{j-1}}. \tag{10}$$

Then using (iii), $u_{t_{i}} = \gamma_{i+1} + \delta_{i+1}$ and $u_{t_{i+1}} = \gamma_{i+1}\rho_{i+1}^{i_{i+1} - i_{i}} + \delta_{i+1}\sigma_{i+1}^{i_{i+1} - i_{i}}$, we obtain

$$u_{t_{i+1}} - u_{t_{i}}\rho_{i+1}^{i_{i+1} - i_{i}} = \delta_{i+1}(\sigma_{i+1}^{i_{i+1} - i_{i}} - \rho_{i+1}^{i_{i+1} - i_{i}}) > 0.$$ 

But by (10) it then follows that

$$u_{t_{i+1}} > u_{t_{i}}\rho_{i+1}^{i_{i+1} - i_{i}} > \prod_{j=1}^{i} \rho_{j}^{j_{j} - j_{j-1}} \rho_{i+1}^{i_{i+1} - i_{i}} = \prod_{j=1}^{i+1} \rho_{j}^{j_{j} - j_{j-1}}$$

holds. This completes the proof of (iv)

\[\Box\]

4 Proof of Theorem 1.1

Before proving the theorem, we first present some definitions. Suppose that $b$, $c$ and $C$ are arbitrary positive integers. Put

$$\phi = \phi_{b,c} := -b - c - 2\sqrt{bc} \quad \text{and} \quad \phi' = \phi'_{b,c,C} := -b - c + 2\sqrt{bc} \cos\left(\frac{2\pi}{C+2}\right).$$

Note

$$\phi < -b - c - \sqrt{bc} \leq \phi'.$$

For each $c'$ with $1 \leq c' \leq c$, let $\beta_{c'}$ be the smallest positive integer satisfying both $\beta_{c'} \geq c'$ and $\phi \geq -\beta_{c'} - c' + 2\sqrt{\beta_{c'}c'}$.

Now, for $l$, $m$ any positive integers and for any real number $\lambda \geq -l - m - 2\sqrt{lm}$, let $\eta_{m}(\lambda)$ denote the largest root of the equation

$$lx^{2} - (l + m + \lambda)x + m = 0.$$
Note that since \(2\sqrt{\beta_c'd} \leq \phi + \beta_d' + d' < \phi' + \beta_d' + d'\), it follows that
\[
0 < \sqrt{\frac{c'}{\beta_c'}} < \tau_{\beta_d', c'}(\phi') < 1. 
\] (11)

Define
\[
\rho = \rho_{b,c,C} := \min\{\tau_{\beta_{d'}, c'}(\phi') \mid 1 \leq d' \leq c\} \text{ and } \alpha := \max\left\{\frac{\beta_{d'}}{d'} \mid 1 \leq d' \leq c\right\}.
\]

By (11) and \(\beta_1 \geq 9\), we have \(\rho < 1\) and \(9 \leq \alpha\). (12)

**Proof of Theorem 1.1:** We define a function \(k\) and prove that it has the required properties. For \(b, c\) and \(C\) arbitrary positive integers, put
\[
k(b, c, C) := \max\left\{\frac{\alpha^{20}}{\rho^{12}}, 2\left(\frac{\alpha^{2\max\{b,c\}^2}}{\rho^3}\right)^9, b + c, 3\right\}.
\]

Now suppose that \(\Gamma\) is a distance-regular graph with \(\h(\Gamma) \geq 2\), valency \(k \geq \max\{b + c, 3\}\), diameter \(d \geq 2\) and
\[
\ell_{(c,k-b-c,b)} > C.
\]

We prove
\[
k < \left\{\begin{array}{ll}
\frac{\alpha^{20}}{\rho^{12}} & \text{if } c = 1, \\
2\left(\frac{\alpha^{2\max\{b,c\}^2}}{\rho^3}\right)^9 & \text{if } c \geq 2,
\end{array}\right.
\]
from which the theorem immediately follows.

Let \(w\) be the largest non-negative integer so that \(t := t_w\) is the largest element of \(T(\theta_1)\) with
\[
k - b_1 - c_t + 2\sqrt{b_k c_t} < k - b - c + 2\sqrt{bc}.
\] (13)

Note that this last equation implies \(c_t \leq c\).

Now, since \(\ell_{(c,k-b-c,b)} \geq C + 1 \geq 2\), by Lemma 2.1 the second largest eigenvalue \(\theta_1\) of \(\Gamma\) satisfies
\[
\theta_1 \geq k + \phi'.
\]

Hence, in view of the definitions of \(\rho_i\) and \(\rho\),
\[
\rho_w(\theta_1) \geq \rho_w(k + \phi') = \tau_{\beta_{c_t}}(\phi') \geq \rho.
\]

Therefore, since \(\rho_i(\theta_1) \geq \rho\) for \(1 \leq i \leq w\), it follows by Proposition 3.1 (i) and (iv) that
\[
u_t > \rho^t. 
\] (14)
Thus, by (3) and (14) we have
\[ m(\theta_1) < \frac{v}{k_t u_t^2} < \frac{v}{k_t \rho^{2t}}. \]  
(15)

Moreover, since \( b_1 \geq \frac{1}{2} k \) and \( h \geq 2 \), the Terwilliger Tree bound [11, Proposition 3.3] implies
\[ 2 \left( \frac{k}{2} \right)^{\frac{3}{2} h} \leq 2 (b_1)^{\frac{1}{2} h} \leq m(\theta_1). \]  
(16)

In addition, by (1) and (2) we have
\[
\begin{align*}
  k_t &\leq k \leq c i \kappa_t, & 0 &\leq i \leq t - 1, \\
  k_{t+i} &\leq c i + 1 k_t & 0 &\leq i \leq d - t,
\end{align*}
\]
and so, as \( d \geq 2 \) and \( \alpha \geq 2 \),
\[
\sum_{j=0}^{d} \alpha^j = k_t \left[ \frac{\alpha^{d+1} - 1}{\alpha - 1} \right] < k_t \alpha^{\frac{3}{2} d}.
\]  
(17)

Thus, by (12), (15), (16), (17) and \( h \geq 2 \),
\[
k < 2 \left( \frac{\alpha^{\frac{3}{2} d}}{2 \rho^{2t}} \right)^{\frac{1}{3} h}.
\]  
(18)

Now, suppose \( c = 1 \). Since \( c_t \leq c = 1 \) we have \( t \leq \eta_1 \). Hiraki [9, Theorem 2] has shown that if \( h = h(\Gamma) \geq 2 \), then
\[
\eta_1 \leq 2 (h + 1). \]  
(19)

Thus Lemma 2.2 implies \( d \leq 2 \eta_1 + 1 \leq 4 h + 5 \) and so
\[
\frac{\alpha^{\frac{3}{2} d}}{2 \rho^{2t}} < \frac{\alpha^{6 h + 8}}{2 \rho^{4 h + 4}}.
\]

So, by (18) and \( h \geq 2 \), we obtain
\[
k < \frac{2 \alpha^{12}}{\rho^{6}} \left( \frac{\alpha^{16}}{4 \rho^{8}} \right)^{\frac{1}{3} h} < \frac{\alpha^{20}}{\rho^{12}}.
\]

Now, to complete the proof, suppose \( c \geq 2 \). Since \( c_t \leq c \), by (4), (5) and (19), we have
\[
t < \xi_c + \eta_c \leq \frac{3}{2} c^2 (h + 1).
\]

Also, by Lemma 2.2 and (19),
\[
d < \frac{3}{2} \max\{b, c\}^2 \eta_1 \leq 3 \max\{b, c\}^2 (h + 1).
\]

Thus by (18), \( h \geq 2 \) and the last two bounds on \( t \) and \( d \),
\[
k < 2 \left( \frac{\alpha^{\frac{3}{2} \max\{b, c\}^2 (h + 1)}}{2 \rho^{3c^2 (h + 1)}} \right)^{\frac{1}{3} h} = 2^{1-\frac{3}{2}} \left( \frac{\alpha^{\frac{3}{2} \max\{b, c\}^2}}{\rho^{2c^2}} \right)^{\frac{6 h + 11}{h}} < 2 \left( \frac{\alpha^{2 \max\{b, c\}^2}}{\rho^{2c^2}} \right)^{\frac{9}{4}}.
\]

This completes the proof.
References


[13] J. H. Koolen and V. Moulton, There are finitely many triangle-free distance-regular graphs with degree 8, 9 or 10, *submitted*