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A bound for the number of columns $\ell_{(c,a,b)}$ in the intersection array of a distance-regular graph

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Abstract

In this paper we give a bound for the number $\ell_{(c,a,b)}$ of columns $(c,a,b)^T$ in the intersection array of a distance-regular graph. We also show that this bound is intimately related to the Bannai-Ito Conjecture.

1 Introduction

Suppose that Γ is a finite connected graph with vertex set $V\Gamma$. As usual, we define the distance between any two vertices u and v of Γ to be the length of any shortest path in Γ between u and v, and the diameter d of Γ to be the largest distance between any pair of vertices in $V\Gamma$. For $u \in V\Gamma$ and i any non-negative integer not exceeding d, let $\Gamma_i(u)$ denote the set of vertices in $V\Gamma$ that are at distance i from u and put $\Gamma_{-1}(v) = \Gamma_{d+1}(v) := \emptyset$. The graph Γ is called *distance-regular* if there are integers b_i, c_i, $0 \le i \le d$, so that for any two vertices u and v in $V\Gamma$ at distance i, there are precisely c_i neighbors of v in $\Gamma_{i-1}(u)$ and b_i neighbors of v in $\Gamma_{i+1}(u)$. Clearly such a graph is regular with valency $k := b_0$. The numbers c_i, b_i, and a_i, where

$$a_i := k - b_i - c_i \quad (i = 0, \ldots, d)$$

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is the number of neighbors of v in $\Gamma_i(u)$ for $u, v \in V\Gamma$ at distance i, are called the *intersection numbers* of Γ, and

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_j & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_j & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_j & \cdots & b_{d-1} & b_d \end{bmatrix}$$

the *intersection array* of Γ.

Now, suppose that Γ is a distance-regular graph with valency $k \geq 2$, diameter $d \geq 2$ and intersection numbers c_i, a_i, b_i, $0 \leq i \leq d$. Given integers $a \geq 0$ and $b, c \geq 1$ with $a+b+c=k$, we define

$$\ell_{(c,a,b)} = \ell_{(c,a,b)}(\Gamma) := |\{i \mid 1 \leq i \leq d-1 \text{ and } (c_i, a_i, b_i) = (c,a,b)\}|,$$

that is, the number of columns $(c,a,b)^T$ in the intersection array of Γ, and put

$$\mathrm{h} = \mathrm{h}(\Gamma) := \ell_{(1,a_1,b_1)}.$$

Note that since $d \geq 2$ and $c_1 = 1$ we have $\mathrm{h} \geq 1$.

Finding good bounds for $\ell_{(c,a,b)}$ is a powerful technique for understanding distance-regular graphs. For example, in [1] Bannai and Ito showed that, for a distance-regular graph with valency $k \geq 3$, if c is an integer with $0 \leq 2c \leq k$ then $\ell_{(c,k-2c,c)} \leq 10k2^k$, from which they deduced that there are finitely many distance-regular graphs with valency 3. Also, in [4] Biggs et al. used circuit chasing to considerably improve this bound, which enabled them to classify the distance-regular graphs with valency 3.

In this paper we prove the following theorem.

Theorem 1.1 *There exists a function* $\mathbf{k} : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ *so that for all positive integers* b, c, C, $\mathbf{k}(b,c,C) \geq \max\{b+c, 3\}$ *and for all distance-regular graphs* Γ *with valency* $k \geq \mathbf{k}(b,c,C)$, *diameter* $d \geq 2$ *and* $\mathrm{h} \geq 2$,

$$\ell_{(c,k-b-c,b)} \leq C.$$

As might be expected from the previously mentioned results for valency 3 distance-regular graphs, this theorem is closely related to the so-called Bannai-Ito Conjecture. Bannai and Ito conjectured that given an integer $k \geq 3$ there are finitely many distance-regular graphs with valency k. In a series of papers [1, 2, 3] they showed that their conjecture was true for valency 3 and 4 and also that, for $k \geq 3$ an integer, there are finitely many bipartite distance-regular graphs with valency k [2]. In addition, it was recently shown that the Bannai-Ito Conjecture is true for valencies 5, 6 and 7 [12] and also that there are finitely many triangle-free (i.e. containing no 3-cycles) distance-regular graphs with valency 8, 9 or 10 [13].

Using Theorem 1.1, we now prove that the Bannai-Ito Conjecture is basically equivalent to bounding $\ell_{(c,k-b-c,b)}$ by a function of b and c.

Theorem 1.2 *The following statements are equivalent:*

(1) *For each integer* $k \geq 3$, *there are finitely many distance-regular graphs with valency* k.

(2) *There exists a function* $\mathbf{f} : \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ *such that for all* $k, b, c \in \mathbb{N}^+$ *and for all distance-regular graphs* Γ *with valency* $k \geq \max\{b+c, 3\}$, *diameter* $d \geq 2$ *and* $\mathrm{h} \geq 2$

$$\ell_{(c,k-b-c,b)} \leq \mathbf{f}(b, c).$$

Proof: (1) $\Rightarrow$ (2) : By (1) there is a function $\mathbf{g} : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ such that, for all distance-regular graphs Γ with valency $k \geq 3$, and diameter $d \geq 2$,

$$d \leq \mathbf{g}(k).$$

For $b, c \in \mathbb{N}^+$ put

$$\mathbf{f}(b, c) := \max\{\mathbf{g}(k) \mid \max\{b+c, 3\} \leq k < \mathbf{k}(b, c, 1)\},$$

where $\mathbf{k} : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ is a function with the properties given in Theorem 1.1.

Now suppose $b, c \in \mathbb{N}^+$ and that Γ is a distance-regular graph with valency $k \geq \max\{b+c, 3\}$, diameter $d \geq 2$, and $\mathrm{h} \geq 2$. Then

$$\ell_{(c,k-b-c,b)}(\Gamma) \leq d \leq \mathbf{g}(k)$$

and, by Theorem 1.1 applied with $C = 1$, if $k \geq \mathbf{k}(b, c, 1)$ then

$$\ell_{(c,k-b-c,b)}(\Gamma) \leq 1.$$

Hence $\ell_{(c,k-b-c,b)}(\Gamma) \leq \mathbf{f}(b, c)$ and so (2) holds.
(2) $\Rightarrow$ (1) : Put

$$F(k) := \max\{\mathbf{f}(b, 1) \mid 1 \leq b \leq k-1\}.$$

Suppose that Γ is a distance-regular graph with valency $k \geq 3$ and diameter $d \geq 2$. Note that $k \geq 1 + b_1$ since otherwise $k < b_1 + 1 = k - a_1$ which is a contradiction. By (2)

$$\mathrm{h} = \ell_{(1,k-b_1-1,b_1)} \leq F(k)$$

and so, since $d < \frac{1}{2}k^3\mathrm{h}$ [10, Theorem 1.1],

$$d < \frac{1}{2}k^3 F(k).$$

It is now straight-forward to check that (1) holds. ■

In view of results and examples contained in [6] and [8], it is plausible, for a distance-regular graph with $\mathrm{h} = 1$ and diameter $d \geq 4$, that $c_4 \geq 2$. If this were indeed the case, then Theorem 1.1 would also hold for $\mathrm{h} = 1$ and so the condition $\mathrm{h} \geq 2$ in Theorem 1.2 (2) could be removed. Bearing this in mind, we make the following conjecture.

Conjecture 1.3 *There exists a function* $\mathbf{f} : \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ *such that for all* $b, c \in \mathbb{N}^+$ *satisfying* $b + c \leq k$ *and for all distance-regular graph* Γ *with valency* $k \geq \max\{b+c, 3\}$

$$\ell_{(c,k-b-c,b)} \leq \mathbf{f}(b, c).$$

In [7] Hiraki proved $\ell_{(1,k-2,1)} \leq 20$ for every distance-regular graph with valency $k \geq 3$, and hence this conjecture is true in case $b = c = 1$. Using Theorem 1.1 we now prove a theorem that generalizes Hiraki's result in case $\mathbf{h} \neq 1$.

Theorem 1.4 *There exists a function* $\mathbf{f} : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ *such that for all* $c \in \mathbb{N}^+$ *and all distance-regular graphs* Γ *with valency* $k \geq \max\{2c, 3\}$*, diameter* $d \geq 2$ *and* $\mathbf{h} \geq 2$*,*

$$\ell_{(c,k-2c,c)}(\Gamma) \leq \mathbf{f}(c).$$

Proof: Suppose that $\mathbf{k} : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ is a function with the properties given in Theorem 1.1. Given $c \in \mathbb{N}^+$, put $\mathbf{k}_c := \mathbf{k}(c, c, 1) - 1$ and define

$$\mathbf{f}(c) := 10\, \mathbf{k}_c\, 2^{\mathbf{k}_c}.$$

Note that if $k \geq \max\{2c, 3\}$, then $\mathbf{k}(c, c, 1) \geq \max\{2c, 3\}$, and hence $\mathbf{f}(c) > 1$.

Now suppose that Γ is a distance-regular graph with valency $k \geq \max\{2c, 3\}$ and $\mathbf{h} \geq 2$. In view of Bannai and Ito's bound, $\ell_{(c,k-2c,c)} \leq 10\, k\, 2^k$, mentioned above and since $10\, k\, 2^k$ is an increasing function on $[\max\{2c, 3\}, \infty)$, for all k with $\max\{2c, 3\} \leq k \leq \mathbf{k}(c, c, 1)$,

$$\ell_{(c,k-2c,c)} \leq 10\, k\, 2^k \leq \mathbf{f}(c).$$

The theorem now follows since by Theorem 1.1, for $k \geq \mathbf{k}(c, c, 1)$,

$$\ell_{(c,k-2c,c)} \leq 1 < \mathbf{f}(c).$$

■

This rest of this paper is organized as follows. In Section 2 we present some definitions and results concerning distance-regular graphs. We also present a partial solution to a problem posed on [5, p.189] that is of independent interest and follows from Theorem 1.1. In Section 3 we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph. Finally, in Section 4 we use these bounds to prove Theorem 1.1.

2 Distance-Regular Graphs

We begin this section by presenting some basic facts concerning distance-regular graphs (for more details see [5]). Suppose that Γ is a distance-regular graph with valency $k \geq 2$, diameter $d \geq 2$ and intersection numbers c_i, a_i, b_i, $0 \leq i \leq d$. Clearly, $b_d = c_0 = a_0 = 0$ and $c_1 = 1$. In [5, Section 4.1], it is shown that $\Gamma_i(u)$ contains k_i elements, where

$$k_0 := 1, \;\; k_1 := k, \;\; k_{i+1} := k_i b_i / c_{i+1}, \quad i = 0, \ldots, d-1, \tag{1}$$

and in [5, Proposition 4.1.6] that

$$k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-1} > b_d = 0 \text{ and } 1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k. \tag{2}$$

Recall that the eigenvalues of Γ are the eigenvalues of the adjacency matrix of Γ. In particular, if θ is an eigenvalue of Γ then $\theta \in [-k, k]$. We now state a result concerning the second largest eigenvalue of a distance regular graph.

Lemma 2.1 [12, Theorem 6.2] *Suppose $b, c \in \mathbb{N}^+$ and $k \geq \max\{b+c, 3\}$ is a positive integer. Let Γ be a distance-regular graph with valency k and put $\ell := \ell_{(c,k-b-c,b)}$. The second largest eigenvalue θ_1 of Γ satisfies*

$$\theta_1 \geq k - b - c + 2\sqrt{bc}\cos\left(\frac{2\pi}{\ell+1}\right).$$

The *standard sequence* $(u_i = u_i(\theta) \mid 0 \leq i \leq d)$ associated to an eigenvalue θ of Γ is defined recursively by the equations

$$u_0 = 1,\ u_1 = \theta/k,\ b_i u_{i+1} - (\theta - a_i)u_i + c_i u_{i-1} = 0 \text{ for } i = 1, 2, \ldots, d-1.$$

As is well-known, see e.g. [5, Theorem 4.1.4], if $v := |V\Gamma|$, then the multiplicity $m(\theta)$ of θ is given by

$$m(\theta) = \frac{v}{M(\theta)}, \tag{3}$$

where

$$M(\theta) = \sum_{i=0}^{d} k_i u_i(\theta)^2.$$

Now given a positive integer c, define

$$\begin{aligned} \xi_c &:= \min\{i \mid 1 \leq i \leq d \text{ and } c_i = c\}, \quad \text{and} \\ \eta_c &:= |\{i \mid 1 \leq i \leq d \text{ and } c_i = c\}|. \end{aligned}$$

To prove the next lemma we will use the following relationships between these numbers that were given in [10] (Lemma 2.1 and Proposition 3.2, respectively). If $c > 1$ is an integer, then

$$\eta_c \leq 2\xi_c - 3, \tag{4}$$

and if c is a positive integer and $\eta_c \neq 0$, then

$$\xi_c \leq \frac{c^2}{4}\eta_1 + 1. \tag{5}$$

Put

$$e := \max\{i \mid 1 \leq i \leq d-1 \text{ and } c_i \leq b_i\}.$$

Lemma 2.2 *Suppose that Γ is a distance-regular graph with valency $k \geq 3$ and diameter $d \geq 2$, and that b, c are positive integers with $k \geq b + c$. If $\ell_{(c,k-b-c,b)} \geq 1$, then*

$$d < \begin{cases} 2(\eta_1 + 1) & \text{if } c_e = 1, \\ \frac{3}{2}\max\{b,c\}^2\eta_1 & \text{if } c_e \geq 2. \end{cases}$$

Proof: Since $c_{e+1} > b_{e+1}$, by [5, Proposition 4.1.6 (ii)]

$$d < 2(e+1). \tag{6}$$

Thus, if $c_e = 1$, then since $e \leq \eta_1$ it follows that $d \leq 2\eta_1 + 1$ holds.

Now suppose $c_e \geq 2$. Since $\{i \mid c_i = c_e\} = \{\xi_{c_e}, \xi_{c_e} + 1, \ldots, \xi_{c_e} + \eta_{c_e} - 1\}$,

$$e \leq \xi_{c_e} + \eta_{c_e} - 1.$$

By applying (4) and then (5) to the righthand side of this inequality, we have

$$e \leq \frac{3}{4} c_e^2 \eta_1 - 1. \tag{7}$$

But $c_e \leq \max\{b, c\}$, since $1 \leq \ell_{(c,k-b-c,b)}$. Thus, in view of (6) and (7) we have $d < \frac{3}{2}\max\{b, c\}^2 \eta_1$. This completes the proof. ∎

3 Bounding Terms of the Standard Sequence

In this section we derive some bounds for terms in the standard sequence associated to an eigenvalue of a distance-regular graph that we use in the proof of Theorem 1.1. We begin with some definitions.

Suppose that Γ is a distance-regular graph with valency $k \geq 3$ and diameter $d \geq 2$, and that θ is an eigenvalue of Γ with $a_1 + 2\sqrt{b_1} < \theta < k$. Let $1 \leq p < d$ be the largest integer for which $c_p \leq b_p$ and $a_p + 2\sqrt{b_p c_p} < \theta$ both hold. Define

$$T := T(\theta) = \{i \mid 0 \leq i \leq p \text{ and } (c_i, a_i, b_i) \neq (c_{i+1}, a_{i+1}, b_{i+1})\}.$$

Put $s := |T| - 1$ and let $t_0, t_1, \ldots, t_s$ be the ordering of T with $0 = t_0 < t_1 < \cdots < t_s = p$.

Now, if $(u_i = u_i(\theta) \mid 0 \leq i \leq d)$ is the standard sequence associated to θ and, for $1 \leq i \leq s$, the largest and smallest roots of the equation

$$b_{t_i} u_{t_i+1} + (a_{t_i} - \theta) u_{t_i} + c_{t_i} u_{t_i-1} = 0$$

are $\rho_i := \rho_i(\theta)$ and $\sigma_i := \sigma_i(\theta)$, respectively, then by the theory of three-term recurrences there are numbers γ_i and δ_i with

$$u_j = \gamma_i \rho_i^{j-t_{i-1}} + \delta_i \sigma_i^{j-t_{i-1}} \quad (t_{i-1} \leq j \leq t_i + 1). \tag{8}$$

Note that since $a_i + 2\sqrt{b_i c_i} < \theta < k$, we have $0 < \sigma_i < \rho_i < 1$, $1 \leq i \leq s$.

We now list some inequalities that will be used in the proof of Theorem 1.1.

Proposition 3.1 *Suppose $1 \leq i \leq s$ and u_i, γ_i and ρ_i are as defined just above. Then the following inequalities hold*

(i) $\rho_{i+1} < \rho_i$, $i \neq s$,

(ii) $u_{t_{i-1}+1} > \rho_i u_{t_{i-1}}$,

(iii) $\gamma_i > u_{t_{i-1}}$,

(iv) $u_{t_i} > \prod_{j=1}^{i} \rho_j^{t_j - t_{j-1}}$.

Proof: (i): For positive integers b, c satisfying $b + c \leq k$, $c \leq b$ and $k - b - c + 2\sqrt{bc} < \theta$ we define

$$f_{b,c}(x) := bx^2 + (k - b - c - \theta)x + c.$$

Let $\rho_{b,c}$ be the largest root of $f_{b,c}(x) = 0$. Since $b \geq c$,

$$\theta > k - b - c + 2\sqrt{bc} > k - (b+1) - c + 2\sqrt{(b+1)c},$$

and hence both $\rho_{b,c}$ and $\rho_{b+1,c}$ are positive. Moreover, $0 < \rho_{b,c} < 1$ since $k - b - c + 2\sqrt{bc} < \theta < k$. Hence

$$f_{b+1,c}(\rho_{b,c}) = \rho_{b,c}^2 - \rho_{b,c} = \rho_{b,c}(\rho_{b,c} - 1) < 0$$

and therefore $\rho_{b,c} < \rho_{b+1,c}$. It is straight-forward to show in a similar fashion that $\rho_{b,c} < \rho_{b,c-1}$ holds. It now follows in view of (2) that (i) must hold.

(ii) and (iii): We will prove that these hold using induction on i. Suppose $i = 1$. Then $u_{t_0} = u_0 = 1$ and $u_{t_0+1} = u_1 = \frac{\theta}{k}$. Since $a_1 + 2\sqrt{b_1} < \theta < k$ and ρ_1 is the largest root of

$$b_1 x^2 + (a_1 - \theta)x + 1 = 0,$$

we have

$$b_1\left(\frac{\theta}{k}\right)^2 + (a_1 - \theta)\frac{\theta}{k} + 1 = \left(1 - \frac{\theta}{k}\right)\left(1 + (a_1 + 1)\frac{\theta}{k}\right) > 0.$$

Hence $\frac{\theta}{k} > \rho_1$. Thus $\gamma_1 > 1$ since $\gamma_1\rho_1 + \delta_1\sigma_1 = u_1 = \frac{\theta}{k} > \rho_1$, $\gamma_1 + \delta_1 = u_0 = 1$ and $\rho_1 > \sigma_1 > 0$. Therefore (ii) and (iii) hold for $i = 1$.

Now suppose $2 \leq i < s$ and suppose $u_{t_{i-1}+1} > \rho_i u_{t_{i-1}}$ and $\gamma_i > u_{t_{i-1}}$ both hold. Then $\delta_i < 0$ since $\gamma_i + \delta_i = u_{t_{i-1}}$. Thus, using equations

$$u_{t_i} = \gamma_i \rho_i^{t_i - t_{i-1}} + \delta_i \sigma_i^{t_i - t_{i-1}} \text{ and } u_{t_i+1} = \gamma_i \rho_i^{t_i - t_{i-1}+1} + \delta_i \sigma_i^{t_i - t_{i-1}+1},$$

we obtain

$$\rho_i u_{t_i} < u_{t_i+1}. \tag{9}$$

Hence $\rho_{i+1} u_{t_i} < \rho_i u_{t_i} < u_{t_i+1}$ by (i) and (9) and so (ii) holds.

Now, in view of

$$u_{t_i} = \gamma_{i+1} + \delta_{i+1} \text{ and } u_{t_i+1} = \gamma_{i+1}\rho_{i+1} + \delta_{i+1}\sigma_{i+1},$$

it follows that

$$\gamma_{i+1} = \frac{u_{t_i+1} - \sigma_{i+1} u_{t_i}}{\rho_{i+1} - \sigma_{i+1}}$$

holds, and hence by (i) and (9)

$$\gamma_{i+1} > \frac{\rho_i - \sigma_{i+1}}{\rho_{i+1} - \sigma_{i+1}} u_{t_i} > u_{t_i}$$

holds. Thus (iii) holds.

(iv) We prove this by using induction on i. Suppose $i = 1$. Then by (8), (ii) and (iii) we have

$$\begin{aligned} u_{t_1} - \rho_1^{t_1} &= (\gamma_1 - 1)\rho_1^{t_1} + \delta_1\sigma_1^{t_1} \\ &= (\gamma_1 - 1)\rho_1^{t_1} + \sigma_1^{t_1-1}(u_1 - \gamma_1\rho_1) \\ &> \rho_1(\gamma_1 - 1)(\rho_1^{t_1-1} - \sigma_1^{t_1-1}) > 0. \end{aligned}$$

Therefore (iv) holds for $i = 1$.

Now, suppose $2 \leq i < s$ and assume

$$u_{t_i} > \prod_{j=1}^{i} \rho_j^{t_j - t_{j-1}}. \tag{10}$$

Then using (iii), $u_{t_i} = \gamma_{i+1} + \delta_{i+1}$ and $u_{t_{i+1}} = \gamma_{i+1}\rho_{i+1}^{t_{i+1}-t_i} + \delta_{i+1}\sigma_{i+1}^{t_{i+1}-t_i}$, we obtain

$$u_{t_{i+1}} - u_{t_i}\rho_{i+1}^{t_{i+1}-t_i} = \delta_{i+1}\left(\sigma_{i+1}^{t_{i+1}-t_i} - \rho_{i+1}^{t_{i+1}-t_i}\right) > 0.$$

But by (10) it then follows that

$$u_{t_{i+1}} > u_{t_i}\rho_{i+1}^{t_{i+1}-t_i} > \prod_{j=1}^{i} \rho_j^{t_j-t_{j-1}} \rho_{i+1}^{t_{i+1}-t_i} = \prod_{j=1}^{i+1} \rho_j^{t_j-t_{j-1}}$$

holds. This completes the proof of (iv) ■

4 Proof of Theorem 1.1

Before proving the theorem, we first present some definitions. Suppose that b, c and C are arbitrary positive integers. Put

$$\begin{aligned} \phi = \phi_{b,c} &:= -b - c - 2\sqrt{bc} \quad \text{and} \\ \phi' = \phi'_{b,c,C} &:= -b - c + 2\sqrt{bc}\cos\Big(\frac{2\pi}{C+2}\Big). \end{aligned}$$

Note

$$\phi < -b - c - \sqrt{bc} \leq \phi'.$$

For each c' with $1 \leq c' \leq c$, let $\beta_{c'}$ be the smallest positive integer satisfying both $\beta_{c'} \geq c'$ and $\phi \geq -\beta_{c'} - c' + 2\sqrt{\beta_{c'}c'}$.

Now, for l, m any positive integers and for any real number $\lambda \geq -l - m - 2\sqrt{lm}$, let $\tau_{l,m}(\lambda)$ denote the largest root of the equation

$$lx^2 - (l + m + \lambda)x + m = 0.$$

Note that since $2\sqrt{\beta_{c'}c'} \leq \phi + \beta_{c'} + c' < \phi' + \beta_{c'} + c'$, it follows that

$$0 < \sqrt{\frac{c'}{\beta_{c'}}} < \tau_{\beta_{c'},c'}(\phi') < 1. \tag{11}$$

Define

$$\begin{aligned} \rho = \rho_{b,c,C} &:= \min\{\tau_{\beta_{c'},c'}(\phi') \mid 1 \leq c' \leq c\} \text{ and} \\ \alpha &:= \max\Big\{\frac{\beta_{c'}}{c'} \mid 1 \leq c' \leq c\Big\}. \end{aligned}$$

By (11) and $\beta_1 \geq 9$, we have

$$\rho < 1 \quad \text{and} \quad 9 \leq \alpha. \tag{12}$$

Proof of Theorem 1.1: We define a function $\mathbf{k}$ and prove that it has the required properties. For b, c and C arbitrary positive integers, put

$$\mathbf{k}(b,c,C) := \max\Big\{\frac{\alpha^{20}}{\rho^{12}},\ 2\Big(\frac{\alpha^{2\max\{b,c\}^2}}{\rho^{c^2}}\Big)^9,\ b+c,\ 3\Big\}.$$

Now suppose that Γ is a distance-regular graph with $\mathrm{h}(\Gamma) \geq 2$, valency $k \geq \max\{b+c, 3\}$, diameter $d \geq 2$ and

$$\ell_{(c,k-b-c,b)} > C.$$

We prove

$$k < \begin{cases} \frac{\alpha^{20}}{\rho^{12}} & \text{if } c = 1, \\ 2\Big(\frac{\alpha^{2\max\{b,c\}^2}}{\rho^{c^2}}\Big)^9 & \text{if } c \geq 2, \end{cases}$$

from which the theorem immediately follows.

Let w be the largest non-negative integer so that $t := t_w$ is the largest element of $T(\theta_1)$ with

$$k - b_t - c_t + 2\sqrt{b_t c_t} < k - b - c + 2\sqrt{bc}. \tag{13}$$

Note that this last equation implies $c_t \leq c$.

Now, since $\ell_{(c,k-b-c,b)} \geq C + 1 \geq 2$, by Lemma 2.1 the second largest eigenvalue θ_1 of Γ satisfies

$$\theta_1 \geq k + \phi'.$$

Hence, in view of the definitions of ρ_i and ρ,

$$\rho_w(\theta_1) \geq \rho_w(k + \phi') = \tau_{b_t c_t}(\phi') \geq \rho.$$

Therefore, since $\rho_i(\theta_1) \geq \rho$ for $1 \leq i \leq w$, it follows by Proposition 3.1 (i) and (iv) that

$$u_t > \rho^t. \tag{14}$$

Thus, by (3) and (14) we have
$$m(\theta_1) < \frac{v}{k_t u_t^2} < \frac{v}{k_t \rho^{2t}}. \tag{15}$$
Moreover, since $b_1 \geq \frac{1}{2}k$ and $\mathrm{h} \geq 2$, the Terwilliger Tree bound [11, Proposition 3.3] implies
$$2\Big(\frac{k}{2}\Big)^{\frac{1}{2}\mathrm{h}} \leq 2(b_1)^{\frac{1}{2}\mathrm{h}} \leq m(\theta_1). \tag{16}$$
In addition, by (1) and (2) we have
$$\begin{array}{ll} k_i \leq k_t \leq \alpha^i k_t & 0 \leq i \leq t-1, \\ k_{t+i} \leq \alpha^i k_t \leq \alpha^{t+i} k_t & 0 \leq i \leq d-t, \end{array}$$
and so, as $d \geq 2$ and $\alpha \geq 2$,
$$v \leq k_t \sum_{j=0}^{d} \alpha^j = k_t \left[\frac{\alpha^{d+1}-1}{\alpha-1}\right] < k_t \alpha^{\frac{3}{2}d}. \tag{17}$$
Thus, by (12), (15), (16), (17) and $\mathrm{h} \geq 2$,
$$k < 2\left(\frac{\alpha^{\frac{3}{2}d}}{2\rho^{2t}}\right)^{\frac{2}{\mathrm{h}}}. \tag{18}$$

Now, suppose $c = 1$. Since $c_t \leq c = 1$ we have $t \leq \eta_1$. Hiraki [9, Theorem 2] has shown that if $\mathrm{h} = \mathrm{h}(\Gamma) \geq 2$, then
$$\eta_1 \leq 2(\mathrm{h}+1). \tag{19}$$

Thus Lemma 2.2 implies $d \leq 2\eta_1 + 1 \leq 4\mathrm{h} + 5$ and so
$$\frac{\alpha^{\frac{3}{2}d}}{2\rho^{2t}} < \frac{\alpha^{6\mathrm{h}+8}}{2\rho^{4\mathrm{h}+4}}.$$
So, by (18) and $\mathrm{h} \geq 2$, we obtain
$$k < \frac{2\alpha^{12}}{\rho^8}\Big(\frac{\alpha^{16}}{4\rho^8}\Big)^{\frac{1}{\mathrm{h}}} \leq \frac{\alpha^{20}}{\rho^{12}}.$$

Now, to complete the proof, suppose $c \geq 2$. Since $c_t \leq c$, by (4), (5) and (19), we have
$$t < \xi_c + \eta_c \leq \frac{3}{2}c^2(\mathrm{h}+1).$$
Also, by Lemma 2.2 and (19),
$$d < \frac{3}{2}\max\{b,c\}^2\eta_1 \leq 3\max\{b,c\}^2(\mathrm{h}+1).$$
Thus by (18), $\mathrm{h} \geq 2$ and the last two bounds on t and d,
$$k < 2\Big(\frac{\alpha^{\frac{9}{2}\max\{b,c\}^2(\mathrm{h}+1)}}{2\rho^{3c^2(\mathrm{h}+1)}}\Big)^{\frac{2}{\mathrm{h}}} = 2^{1-\frac{2}{\mathrm{h}}}\Big(\frac{\alpha^{\frac{3}{2}\max\{b,c\}^2}}{\rho^{c^2}}\Big)^{\frac{6(\mathrm{h}+1)}{\mathrm{h}}} < 2\Big(\frac{\alpha^{2\max\{b,c\}^2}}{\rho^{c^2}}\Big)^9.$$
This completes the proof. ∎

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