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The Distribution of the Eigenvalues of the Finite Upper Half Plane Graphs.

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概 要

We can define a finite upper half plane as an analogue of the Poincare upper half plane. And we can define some graphs on this plane. A.Terras [9] gave conjecture on the distribution of the eigenvalues of these graphs. This paper is concerning this conjecture.

1 Introduction

Let F_q be a finite field with q elements (q odd). Fix a non-square element $\delta \in F_q$:

$$H_q := F_q(\sqrt{\delta}) - F_q = \{x + y\sqrt{\delta} \mid x, y \neq 0 \in F_q\}$$

is called a finite upper half plane. It is modeled on the real one. And the matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = GL(2, F_q)$ acts transitively on $z \in H_q$ by fractional linear transformation:

$$gz = \frac{az+b}{cz+d} \in H_q.$$

We can identify G/K,where K is a subgroup of G which fixes $\sqrt{\delta}$. That is,

$$K = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \middle| a, b \in F_q, a^2 - \delta b^2 \neq 0 \right\}, \tag{1}$$

which is isomorphic to the multiplicative group $F_q(\sqrt{\delta})^*$.

We define a G-invariant distance d between points z and $w \in F_q$:

$$d(z, w) = \frac{N(z-w)}{\mathrm{Im} z \cdot \mathrm{Im} w},$$

where $z = x + y\sqrt{\delta} \in H_q$, and we denote $N(z) = x^2 - \delta y^2$ and $\mathrm{Im} z = y$.

Definition 1.1 **([9]).** *Define a graph $X_q(\delta, a)$ for $a \in F_q$,as follows. Let the vertices of the graph be the points of H_q. Draw an edge between two vertices z,w in H_q if and only if $d(z, w) = a$. Then define an adjacency matrix A_a by*

$$(A_a)_{z,w} = \begin{cases} 1, & \textit{if } z \textit{ is adjacency to } w, \\ 0, & \textit{otherwise.} \end{cases}$$

A.Terras [9] gave the conjecture about the distribution of these eigenvalues as follow:

Conjecture 1.2. *The distribution of the eigenvalues of the upper half plane graphs is the semi-circle or Sato-Tate distribution. That is,*

$$\frac{1}{q-1}\sharp\{\lambda \mid \frac{\lambda}{\sqrt{q}} \in E\} \sim \frac{1}{2\pi}\int_E \sqrt{4-x^2}\, dx, \textit{ as } q \to \infty\ ,$$

for any Borel set E of the interval $[-2, 2]$.

As remark, we have to neglect "multiplicities" and look only at the $q-1$ eigenvalues. So in the next section, we will refer "multiplicities" and eigenvalues which R.Evans [3] gave as exponential sum known as Soto-Andrede sum. And we also refer the evidence that we believed the conjecture to be true. In section 3, we with refer the new fact.

2 Preliminary

In this section,we will mention the properties and the fact about the finite upper half plane graphs.

Proposition 2.1 **([9]).** *Assume that $q = p^r$, where p is an odd prime. Suppose that δ as a non-square in F_q. Let $a \in F_q$.*
1)The graph $X_q(\delta, a)$ is a $(q+1)$-regular graph provided that $a \neq 0$ or 4δ.
2)The graph $X_q(\delta, a)$ and $X_q(\delta c^2, ac^2)$ are isomorphic for any $c \in F_q^*$.
3)The graph $X_q(\delta, a)$ is a connected, provided that $a \neq 0, 4\delta$. In fact the graph $X_q(\delta, a)$ is a Cayley graph for the affine group

$$Aff(q) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \middle| x, y \in F_q, y \neq 0 \right\} ,$$

using the generators

$$S_q(\delta, a) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \middle| x, y \in F_q, a^2 - \delta b^2 \neq 0 \right\} .$$

4)The K-double cosets for G are represented by the sets $S_q(\delta, a)$,for $a \in F_q$.

And we give the definition of Ramanujan graph, which was made by Lubotzky, Phillips,and Sarnak [8].

Definition 2.2 ([8]). *We say that a k-regular graph is Ramanujan if for all eigenvalues λ of the adjacency matrix of the graph such that $|\lambda| \neq k$, we have*

$$|\lambda| \leq 2\sqrt{k-1}.$$

The eigenvalues of our graphs is given by R.Evans [3] as follow. Let χ be a multiplicative character of F_q^* of order $q-1$ and let ω be a multiplicative character of $F_{q^2}^*$ of order q^2-1. Denote $\varepsilon = \chi^{\frac{q-1}{2}}$, or a quadratic character of F_q^*. Then the eigenvalues are

$$\sum_{v \in F_q^*} \chi^j(v)\varepsilon(\delta(v-1)^2 + va), \tag{2}$$

and

$$\sum_{\substack{\alpha \in F_{q^2} \\ N\alpha = 1}} \omega^j(\alpha)\varepsilon(\alpha + \frac{1}{\alpha} - 2 + \frac{a}{\delta}). \tag{3}$$

The sum (3) is called Soto-Andrade sum. Suppose that $\lambda_0(a)_q = q+1$. Let $\lambda_j(a)_q, j = 1, \cdots, \frac{q-1}{2}$ be (2) for $1 \leq j \leq \frac{q-1}{2}$, and $\lambda_j(a)_q, j = \frac{q+1}{2}, \cdots, q-1$ be (3) for $1 \leq j \leq \frac{q-1}{2}$. Then $\lambda_0(a)_q$ has multiplicity 1. $\lambda_{\frac{q-1}{2}}$ has multiplicity q. For $j = 1, \cdots, \frac{q-3}{2}$, $\lambda_j(a)_q$ has multiplicity $q+1$. For $j = \frac{q+1}{2}, \cdots, q-1$, $\lambda_j(a)_q$ has multiplicity $q-1$.

Terras [9] also conjectured that the upper half plane graph is Ramanujan graph, or above nontrivial eigenvalues is bounded by $2\sqrt{q}$ because the valency is $q+1$ by Proposition2.1. Katz [4] and Li [6] [7] independently has been proved this to be true, using different methods.

So the eigenvalues distribute over the closed interval $[-2, 2]$ as in Conjecture1.2. Moreover in Conjecture1.2, we mean that "multiplicity" is not as usual, but the above choice of j. For instance, for the graph $X_5(2,2)$ or $q = 5, \delta = 2, a = 2$, the eigenvalues whose multiplicities are neglected in the view of Conjecture1.2 are 4 eigenvalues $\lambda_1(2)_5$, $\lambda_2(2)_5$, $\lambda_3(2)_5$, $\lambda_4(2)_5$, and $\lambda_5(2)_5$. Actually, we have $\lambda_1(2)_5 = \lambda_2(2)_5 = -2$, $\lambda_3(2)_5 = 3$, and $\lambda_4(2)_5 = 1$. Then the eigenvalues without multipicities are 3 numbers -2, 3, and 1.

Kuang [5] proved that the first moment and the second moment, or average and variance of the distribution asymptotically match those of the semi-circle distribution. It is as follows.

Theorem 2.3 ([5]). *The first moment and the second moment of the distribution of the set $\{\lambda_i(a)_q/\sqrt{q} \mid i = 1, \cdots, q-1\}$ asymptotically match with those of the Wiger or Sato-Tate semi-circle distribution. That is,*

$$\lim_{q \to \infty} \frac{1}{q-1} \sum_{i=1}^{q-1} \frac{\lambda_i(a)_q}{\sqrt{q}} = 0$$

$$\lim_{q\to\infty}\frac{1}{q-1}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^2=1.$$

And the limits are uniform and independent of $a \neq 0$ and δ as long as $a \neq 4\delta$.

Kuang proved [5] this by using that eigenfunctions of adjacency matrix of $X_q(\delta, a)$ are corresponded with the orthonormal basis of $L^2(K\backslash G/K)$, where $L^2(K\backslash G/K) = \{ f : G \to \mathbb{C} \mid f(kxh) = f(x)\forall k, h \in K, \forall x \in G\}$. They are also called spherical functions. See Terras [9] for spherical functions. Since the limits in Theorem2.3 are independent of the choice of $a \neq 0, 4\delta$, Kuang also stated the following which is the evidence for one of the modifications of Conjecture1.2.

Conjecture 2.4 ([5]). *Given q, we fix δ. Let Λ be the multi-set of all eigenvalues of the $q-2$ graphs $X_q(\delta, a)$, where a runs through F_q^* with $a \neq 4\delta$. Λ have asymptotic semi-circle distribution.*

Corollary 2.5 ([5]). *The first and second moments of distribution of the set*

$$\left\{\frac{\lambda_i(a)_q}{\sqrt{q}}\Big| i=1,\cdots,q-1, a\neq 4\delta\in F_q^*\right\}$$

asymptotically match with those of the Wigner or Sato-Tate semi-circle distribution. That is,

$$\lim_{q\to\infty}\frac{1}{(q-1)(q-2)}\sum_{\substack{a\in F_q^*\\ a\neq 4\delta}}\sum_{i=1}^{q-1}\frac{\lambda_i(a)_q}{\sqrt{q}}=0,$$

$$\lim_{q\to\infty}\frac{1}{(q-1)(q-2)}\sum_{\substack{a\in F_q^*\\ a\neq 4\delta}}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^2=1.$$

There are some modifications of Conjecture1.2. For example, the multi-set of all eigenvalues of the $q-2$ graphs $X_q(\delta, a)$, where q is given, δ is fixed, and a runs through F_q^* with $a \neq 4\delta$, is conjectured to have asymptotic semi-circle distribution, while the evidence for "Yes" is given by Corollary2.5. By Proposition2.1, it was proved that there are only $q-2$ $(q+1)$-regular graphs on H_q, and that for different non-square elements δ_1 and δ_2 in F_q, there is a unique pair a_1 and a_2 in F_q such that $X_q(\delta_1, a_1)$ and $X_q(\delta_2, a_2)$ are isotropic.

3 Main Results

In this section, we will give the new results and prepare for it. Theorem3.1 and Theorem3.2 are for the Conjecture1.2. Moreover, in next section, we will refer the proof of these theorems, and consider how about Conjecture2.4.

Theorem 3.1. *The third moment moment of the distribution of the set* $\{\lambda_i(a)_q/\sqrt{q} \mid i = 1, \cdots, q-1\}$ *asymptotically matches with those of the Sato-Tate semi-circle distribution. That is,*

$$\lim_{q\to\infty} \frac{1}{q-1} \sum_{i=1}^{q-1} \left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^3 = 0$$

And the limits are uniform and independent of $a \neq 0$ *and* δ *as long as* $a \neq 4\delta$.

Theorem 3.2. *For* $a \neq 0, 2\delta, 4\delta$, *the forth moment moment of the distribution of the set* $\{\lambda_i(a)_q/\sqrt{q} \mid i = 1, \cdots, q-1\}$ *asymptotically matches with those of the Sato-Tate semi-circle distribution. That is,*

$$\lim_{q\to\infty} \frac{1}{q-1} \sum_{i=1}^{q-1} \left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^4 = 2.$$

And the limits are uniform and independent of $a \neq 0, 2\delta, 4\delta$ *and* δ.
For $a = 2\delta$, *the forth moment of the above set doesn't asymptotically match with those of semi-circle distribution.*

The third and forth moments of the semi-circle distribution are, respectively, 0 and 2. Now, we give one definition.

Definition 3.3 ([1, 9]). *A connected graph* $X(V, E)$ *is highly regular with collapsed adjacency matrix* $C = (c_{ij})$ *if only if for every vertex* $v \in V$, *there is partition of* V *into sets* V_i, $i = 1, \cdots, n$, *with* $V_1 = \{v\}$, *such that each vertex* $y \in V_i$ *is adjacent to exactly* c_{ij} *vertices in* V_j.

Proposition 3.4 ([9]). *The graph* $X_q(\delta, a)$ *is highly regular. Also, the entries of the collapsed adjacency matrix of* $X_q(\delta, a)$ *are as follow:*

$$c_{ij} = \begin{cases} q+1, & \text{if } (i,j) = (0,a), (4\delta, a-4\delta), \\ 2, & \text{if } \Delta_{ij} \text{ is square}, \\ 1, & \text{if } \Delta_{ij} = 0, \\ 0, & \text{if } \Delta_{ij} \text{ is non-square}, \end{cases} \tag{4}$$

where $\Delta_{ij} = \delta(i-j)^2 + a\delta(a - 2i - 2j) + aij$.

Proof. Let $F_q = \{0, a_2, a_3, \cdots, a_q\}$. Given an arbitrary vertex $v \in V$, let $S_1 = \{v\}$, and for $i \leq 2$, let $S_i = \{\tau(g) \mid g \in S_q(\delta, a_i)\}$, where τ is an element of $GL(2, F_q)$ such that $\tau(\sqrt(\delta) = v)$. Notice that for $w_1 \in S_i$ and $w_2 \in S_j$, $w_1 = \tau(g_1)$,and $w_2 = \tau(g_2)$ for some $g_1 \in S_q(\delta, a_i)$ and $S_q(\delta, a_j)$. Then $d(w_1, w_2) = d(g_1, g_2)$ by the invariant of the distance. Hence,the number of vertices in S_i to which a vertex in S_j is adjacent is equal to the number of vertices in $S_q(\delta, a_j)$ to which a vertex in $S_q(\delta, a_i)$ is adjacent. So, we may assume that τ is the identity, $v = \sqrt{\delta}$, and that $S_i = S_q(\delta, a_i)$. Now take z_1, $z_2 \in S_q(\delta, a_i)$. We want to show that the number of vertices in $S_q(\delta, a_i)$ which are adjacent to z_1 is the same as the number of vertices which are adjacent to z_2.

From Theorem2.1, $z_1 = h(z_2)$ for some $h \in K$. Since $d(z_1, w) = d(h(z_1), h(w)) = d(z_2, h(w))$ for $w \in S_q(\delta, a_j)$, and $h(w)$ runs through $S_q(\delta, a_j)$ as w runs through $S_q(\delta, a_j)$, w is distance a form z_1 if and only if $h(w)$ is distance a form z_2. Hence, Z_1 and z_2 are adjacent to the same number of vertices in $S_q(\delta, a_j)$ and the graph $X_q(\delta, a)$ is highly regular.

First, let $i \neq 0, 4\delta$. Then, there exists $x + y\sqrt{\delta} \in S_q(\delta, i)$ such that $x \neq 0$ since $|S_q(\delta, i)| = q + 1$. The reason for those choices of x will became apparent later. Now, assume this element is adjacent to $x_1 + y_1\sqrt{\delta} \in S_q(\delta, j)$. This occurs if and only if

$$d(z_1, z) = a,$$

which is equivalent to

$$(x - x_1)^2 - \delta(y - y_1)^2 = ayy_1,$$

and can be expanded out by

$$(x^2 - \delta y^2) + (x_1^2 - \delta y_1^2) - 2xx_1 + 2\delta yy_1 = ayy_1. \tag{5}$$

Since $x + y\sqrt{\delta} \in S_q(\delta, i)$ and $x_1 + y_1\sqrt{\delta} \in S_q(\delta, j)$, respectively, are equivalent to

$$x^2 - \delta y^2 = (i - 2\delta)y + \delta \tag{6}$$

and

$$x_1^2 - \delta y_1^2 = (i - 2\delta)y_1 + \delta. \tag{7}$$

Substituting these into the equation(5) yields

$$2xx_1 = \{\, (j - 2\delta) + (2\delta - a)y\}\; y_1 + (i - 2\delta)y + 2\delta.$$

Now, by the choice of the element $x + y\sqrt{\delta}$, we can divide the equation by x. To simplify the equations, let

$$A = \frac{\{\, (j - 2\delta) + (2\delta - a)y\}}{2x} \text{ and } B = \frac{(i - 2\delta)y + 2\delta}{2x}.$$

This reduces the above equation to

$$x_1 = Ay_1 + B.$$

Substituting this into (7) and expanding it out, we get

$$(A^2 - 2\delta)y_1 + (2AB + 2\delta - j)y_1 + B^2 - \delta = 0, \tag{8}$$

which is a quadratic equation in y_1 since we let δ be non-square. We have the discriminant D , which is equivalent to

$$D = (2AB + 2\delta - j)^2 - 4(A^2 - \delta)(B^2 - \delta),$$

and substituting above A and B into D, we have

$$D = \left(\frac{y}{x}\right)^2 \{\delta(i-j)^2 + a\delta(a-2i-2j) + aij\}\ .$$

We set $\Delta_{ij} = \delta(i-j)^2 + a\delta(a-2i-2j) + aij$. So, there are 2 solutions to the equation(8) when Δ_{ij} is square in F_q, and since $x_1 = Ay_1 + B$, there are two elements in $S_q(\delta, i)$ which are adjacent to $x + y\sqrt{\delta}$. Hence, that entry of the collapsed adjacency matrix is equal to 2. By the same way, we find that $c_{ij} = 1$ if $\Delta = 0$, and that $c_{ij} = 0$ if Δ is non-square in F_q.

For the case where $i = 0$, the only element in $S_q(\delta, 0)$ is $\sqrt{\delta}$. Since $\sqrt{\delta}$ is adjacent to all vertices in $S_q(\delta, a)$, $c_{0,a} = q+1$. For the case where $i = 4\delta$, the only element in $S_q(\delta, 4\delta)$ is $-\sqrt{\delta}$. It is only adjacent to the vertices in $S_q(\delta, 4\delta - a)$ because $d(x - y\sqrt{\delta}, \sqrt{\delta}) = 4\delta - a$ if $d(x + y\sqrt{\delta}, \sqrt{\delta}) = a$. So, we also have $c_{4\delta, 4\delta - a} = q+1$. □

In view of association scheme, Definition3.3 is not important, and Proposition3.4 is trivial. It is known that the upper half plane is symmetric association scheme with relation of distance. And the entry of collapsed adjacency matrix c_{ij} is corresponding to intersection number of this association scheme. But here we used above definition, as well as Terras[9] and J.Angel[1].

4 Proof of the Main Result

We will give the proofs of Theorem3.1 and Theorem3.2 using different way from Kuang[5]. We use the idea in N.Biggs[2] that the number of walks of length l in graph is equal to the sum of l powers of each eigenvalue of the adjacency matrix. To get the third and forth moments, we will count up all the walks of length three and four.

Lemma 4.1. *For $a \neq 0, 4\delta$, let N_3 be the number of the walks of the length 3 in the graph $X_q(\delta, a)$. Then, N_3 is given by*

$$N_3 = \begin{cases} 2q(q+1)(q-1), & \text{if } a - 3\delta \text{ is square,} \\ q(q+1)(q-1), & \text{if } a - 3\delta = 0, \\ 0, & \text{if } a - 3\delta \text{ is non-square.} \end{cases} \tag{9}$$

Proof. Since G acts transitively on H_q, we consider the closed walks whose origin and terminal are $\sqrt{\delta}$. If two vertices z_1 and z_2 in $S_q(\delta, a)$ are adjacent, we get the such walk. In other words, if the entry in position (a, a) of the collapsed adjacency matrix is one or two, we have one or two triangles for one vertex in $S_q(\delta, a)$.

Since $\Lambda_{a,a} = a^2(a - 3\delta)$, when $a - 3\delta = 0$, we have the one walk $\sqrt{\delta}$, z_1, z_2, $\sqrt{\delta}$ for all $z_1 \in S_q(\delta, a)$, where $z_1 \in S_q(\delta, a)$ is adjacent to z_1. By Proposition2.1(5),

$S_q(\delta, a) = Kz_a$ for $z_a \in S_q(\delta, a)$. So, we have such $|K|$ walks, then $N_3 = q(q+1)(q-1)$.

For a such that $a - 3\delta$ is square, as well as above, we have $N_3 = 2q(q+1)(q-1)$. The factor 2 cause form that since $\Lambda_{a,a}$ is square, we have the two closed walks whose origin and terminal are $\sqrt{\delta}$ for all $z_1 \in S_q(\delta, a)$. □

Before we refer lemma4.4 which give the number of the walks of length 4, we give some lemmas as preparation.

Lemma 4.2. *Let Q be the set of squares in F_q^* or $\{x^2 \mid F_q^*\}$, and let N be the set of non-square in F_q^*. For any $c \in F_q^*$, we have*

$$|(N+c) \cap Q| = \frac{1}{4}\{q - 1 - \varepsilon(c) + \varepsilon(-c)\} ,$$

$$|(N+c) \cap N| = \frac{1}{4}\{q - 3 + \varepsilon(c) + \varepsilon(-c)\} .$$

Here ε is a quadratic character of F_q^*, and $N + c = \{y + c \mid y \in F_q^*\}$.

Lemma 4.3. *The number of the set $\{n \in F_q^* \mid \Delta_{n,a} is\ square\}$ is* $\frac{1}{2}(q + \varepsilon(-1))$.

Lemma4.3 is given by Lemma4.2. Now, we are ready to give the number of the walks of length 4. We will prove this in the same way as lemma4.1.

Lemma 4.4. *For $a \neq 0, 4\delta$, let N_4 be the number of the walks of the length 4 in the graph $X_q(\delta, a)$. Then, N_4 is given by*

$$N_4 = \begin{cases} q(q+1)(q-1)(4q + 2\varepsilon(-1) + 2), & \text{if } a = 2\delta, \\ q(q+1)(q-1)(3q + 2\varepsilon(-1) + 2), & \text{if } a \neq 2\delta, \end{cases} \tag{10}$$

where ε is a quadratic character of F_q^*.

Proof. As well as the proof of lemma4.1, we consider the closed walks of length 4 whose origin and terminal are $\sqrt{\delta}$. If $\Delta_{n,a}$ is square, there exists two edge form one vertex $x + y\sqrt{\delta} \in S_q(\delta, n)$ to two different vertices in $S_q(\delta, a)$. So, for $x + y\sqrt{\delta} \in S_q(\delta, n)$, we have 4-cycle containing $\sqrt{\delta}$ and $x + y\sqrt{\delta}$. Since $S_q(\delta, a) = \{-\sqrt{\delta}\}$ and

$$\Delta_{4\delta,a} = 4\delta(a - 2\delta)^2 = \begin{cases} 0, & \text{if } a = 2\delta, \\ non-square, & \text{if } a \neq 2\delta, \end{cases}$$

we have two cases whether a is 2δ or not.

When $a \neq 2\delta$,for n such that $\Delta_{n,a}$ is square, we get the above 4-cycle, and there is a path of length 2 whose origin is $\sqrt{\delta}$ on that cycle. Also, for n such that $\Delta_{n,a} = 0$, that is, $n = \frac{a(4\delta - a)}{\delta}$, we have $q+1$ paths of length 2 whose origin is $\sqrt{\delta}$. Clearly, the path of length 2 is the walk of length 4. So, we have

$$\begin{aligned} N_4 = \Bigg[& \left\{ (q+1) \times 2 \times \frac{1}{2}(q + \varepsilon(-1)) \right\} \\ & + \left\{ (q+1) \times 2 \times \frac{1}{2}(q + \varepsilon(-1)) + (q+1) + q(q+1) \right\} \\ & + (q+1) \Bigg] \times q(q-1) \end{aligned}$$

When $a = 2\delta$, we get above 4-cycles for n such that $\Delta_{n,a}$ is a square, as well as $a \neq 2\delta$. For n such that $\Delta_{n,a} = 0$, that is, $n = 4\delta$, there are $q+1$ edges from $-\sqrt{\delta}$ in $S_q(\delta, 4\delta)$ to vertices in $S_q(\delta, a)$. So, we have more 4-cycles than $a \neq 4\delta$ by $\binom{q+1}{2}$. Therefore, we have

$$\begin{aligned} N_4 = \Bigg[& \left\{ (q+1) \times 2 \times \frac{1}{2}(q + \varepsilon(-1)) + 2 \times \binom{q+1}{2} \right\} \\ & + \left\{ (q+1) \times 2 \times \frac{1}{2}(q + \varepsilon(-1)) + (q+1) + q(q+1) \right\} \\ & + (q+1) \Bigg] \times q(q-1) \end{aligned}$$

Thus we obtain Lemma. □

We finished preparation to proof Theorem3.1 and 3.2. Supposed that the eigenvalue $\lambda_i(a)_q$ has the multiplicity m_i, that is, $m_i = q-1$, q, or $q+1$ for $1 \leq i \leq q-1$. First, by Lemma4.1, for $a \neq 0, 4\delta$, we have

$$0 \leq \sum_{i=0}^{q-1} m_i(\lambda_i(a_q))^3 \leq 2q(q-1)(q+1).$$

Since $m_0 = 1$, $\lambda_0(a)_q = q+1$,

$$-\frac{(q+1)^3}{(q-1)q^2\sqrt{q}} \leq \frac{1}{q-1}\sum_{i=1}^{q-1} \frac{m_i}{q}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^3 \leq \frac{(q+1)(q^2-4q-1)}{(q-1)q^2\sqrt{q}}.$$

This equation implies Theorem3.1.

Next, by Lemma4.4, for $a \neq 0, 2\delta, 4\delta$, we have

$$\sum_{i=0}^{q-1} m_i(\lambda_i(a_q))^4 = q(q-1)(q+1)(3q + 2\varepsilon(-1) + 2).$$

So, we get

$$\frac{1}{q-1}\sum_{i=1}^{q-1} \frac{m_i}{q}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^4 = \frac{q+1}{q^3(q-1)}\{2q^3 + (2\varepsilon(-1) - 4)q^2 - (2\varepsilon(-1) + 5) - 1\}.$$

This equation implies Theorem3.2. Moreover, for $a = 2\delta$, we have

$$\frac{1}{q-1}\sum_{i=1}^{q-1}\frac{m_i}{q}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^4 = \frac{q+1}{q^3(q-1)}\{\,3q^3 + (2\varepsilon(-1)-5)q^2 - (2\varepsilon(-1)+5) - 1\}\,.$$

Thus, the limit of above equation is 3, which is not the forth moment of the semi-circle.

Finally, using Lemma4.1 and 4.4, we consider about Conjecture2.4. Using Lemma4.2, we get

$$|\{\, a - 3\delta \in Q \mid a \in F_q^*, a \neq 4\delta\} \mid = \frac{1}{2}(q - 2 + \varepsilon(-3)).$$

So, we have the equation

$$\sum_{\substack{a\in F_q^* \\ a\neq 4\delta}}\sum_{i=1}^{q-1} m_i(\lambda_i(a)_q)^3 = (q+1)\{\,(\varepsilon(-3)-2)q^2 - (4+\varepsilon(-3))q - 2\}\,.$$

Using same way as above, we found that the third moments of distribution of the set Λ asymptotically match with that of the semi-circle distribution. That is,

$$\lim_{q\to\infty}\frac{1}{(q-1)(q-2)}\sum_{\substack{a\in F_q^* \\ a\neq 4\delta}}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^3 = 0.$$

And we have the equation

$$\sum_{\substack{a\in F_q^* \\ a\neq 4\delta}}\sum_{i=1}^{q-1} m_i(\lambda_i(a)_q)^4 = (q+1)\{\,2q^4 + (2\varepsilon(-1)-1)q^3 + (-2\varepsilon(-1)-4)q^2 + (-4\varepsilon(-1)+1)q + 2\}$$

So, the forth moment of distribution of the set Λ asymptotically match with that of the semi-circle distribution. That is,

$$\lim_{q\to\infty}\frac{1}{(q-1)(q-2)}\sum_{\substack{a\in F_q^* \\ a\neq 4\delta}}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^4 = 2.$$

5 Remarks

By Theorem3.2, we have counter-example $a = 2\delta$ for Conjecture1.2. Yet, for any natural number k, we don't have the k-th moment of the distribution of the upper half plane. It is interesting that we determine the k-th moment, and that we know when Conjecture1.2 is true or false.

Though we gave some evidence for truth of Conjecture2.4, we don't know this is true or false. So we should research this as well as Conjecture1.2. Moreover, though we supposed q is odd, for q is even or 2^r, the upper half plane is defined, as well as the upper half plane graphs. There is not conjecture for $q = 2^r$. It is natural to think about $q = 2^r$.

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