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# Irreducible modules for $W_3$-algebra

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## 1 Introduction

Recently a $W_3$-algebra of central charge 6/5 was studied as an orbifold of an order three automorphism of the lattice vertex operator algebra $V_{\sqrt{2}A_2}$ by Dong, Lam, Tanabe, Yokoyama and the author [5]. The paper consists of four parts: (1) realization of a $W_3$-algebra $\mathcal{W}$ of central charge 6/5, (2) classification of irreducible $\mathcal{W}$-modules, (3) $C_2$-cofiniteness and rationality of $\mathcal{W}$, and (4) determination of the character of each irreducible $\mathcal{W}$-module. In this note we will give an outline of the second part of the paper, namely, classification of irreducible $\mathcal{W}$-modules. Details can be found in [5]. Basic references to $\mathcal{W}_3$-algebras are [1] and [7].

## 2 The $W_3$-algebra $\mathcal{W}$

For definitions of the materials discussed here we refer to [2, 8]. We also use certain properties of the vertex operator algebra $V_{\sqrt{2}A_2}$ (cf. [12]).

Let $\alpha_1, \alpha_2$ be the simple roots of type $A_2$ and set $\alpha_0 = -(\alpha_1 + \alpha_2)$. We denote the inner product by $\langle\,\cdot\,,\,\cdot\,\rangle$. Then $\langle \alpha_i, \alpha_i\rangle = 2$ and $\langle \alpha_i, \alpha_j \rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 = \sqrt{2}A_2$ be the lattice spanned by $\beta_1$ and $\beta_2$. We follow Sections 2 and 3 of [2] with $L = \sqrt{2}A_2$, $p = 3$, and $q = 6$. In our case $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$, so that the alternating $\mathbb{Z}$-bilinear map $c_0 : L \times L \to \mathbb{Z}/6\mathbb{Z}$ defined by [2, (2.9)] is trivial. Thus the central extension

$$1 \longrightarrow \langle \kappa_6 \rangle \longrightarrow \hat{L} \xrightarrow{\ -\ } L \longrightarrow 1 \tag{2.1}$$

determined by the commutator condition $aba^{-1}b^{-1} = \kappa_6^{c_0(\bar{a},\bar{b})}$ splits. Then for each $\alpha \in L$, we can choose an element $e^\alpha$ of $\hat{L}$ so that $e^\alpha e^\beta = e^{\alpha+\beta}$. The twisted group algebra $\mathbb{C}\{L\}$ is isomorphic to the ordinary group algebra $\mathbb{C}[L]$.

We adopt the same notation as in [9] to denote cosets of $L$ in the dual lattice $L^\perp = \{\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} L \,|\, \langle \alpha, L \rangle \subset \mathbb{Z}\}$, namely,

$$L^0 = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L,$$

$$L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,$$

$$L^{(i,j)} = L_i + L^j$$

for $i = 0, a, b, c$ and $j = 0, 1, 2$, where $\{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, $L^{(i,j)}, i \in \{0, a, b, c\}, j \in \{0, 1, 2\}$ are all the cosets of $L$ in $L^\perp$ and $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Our notation for the vertex operator algebra $(V_L, Y(\cdot, z))$ associated with $L$ is standard [8]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an abelian Lie algebra, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is the corresponding affine Lie algebra, $M(1) = \mathbb{C}[\alpha(n)\,;\, \alpha \in \mathfrak{h}, n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n > 0$, and $c = 1$. As a vector space $V_L = M(1) \otimes \mathbb{C}[L]$ and for each $v \in V_L$, a vertex operator $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in \mathrm{End}(V_L)[[z, z^{-1}]]$ is defined. The vector $\mathbf{1} = 1 \otimes 1$ is called the vacuum vector.

There are exactly 12 isomorphism classes of irreducible $V_L$-modules, which are represented by $V_{L^{(i,j)}}$, $i = 0, a, b, c$ and $j = 0, 1, 2$. We use the symbol $e^\alpha, \alpha \in L^\perp$ to denote a basis of $\mathbb{C}\{L^\perp\}$.

Let

$$x(\alpha) = e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \qquad y(\alpha) = e^{\sqrt{2}\alpha} - e^{-\sqrt{2}\alpha}, \qquad w(\alpha) = \frac{1}{2}\alpha(-1)^2 - x(\alpha)$$

for $\alpha \in \{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2\}$ and set

$$\begin{aligned} \omega &= \frac{1}{5}\big(w(\alpha_1) + w(\alpha_2) + w(\alpha_0)\big), \\ \widetilde{\omega} &= \frac{1}{6}\big(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2\big), \\ \omega^1 &= \frac{1}{4}w(\alpha_1), \qquad \omega^2 = \omega - \omega^1, \qquad \omega^3 = \widetilde{\omega} - \omega. \end{aligned}$$

Then $\widetilde{\omega}$ is the Virasoro element of $V_L$ and $\omega^1, \omega^2, \omega^3$ are mutually orthogonal conformal vectors of central charge $1/2, 7/10, 4/5$ respectively (cf.[4]). The subalgebra $\mathrm{Vir}(\omega^i)$ generated by $\omega^i$ is isomorphic to the Virasoro vertex operator algebra of given central charge.

We study certain subalgebras, and also submodules for them in $V_{L_i}, i = 0, a, b, c$ and in $V_{L^j}, j = 0, 1, 2$. Set

$$\begin{aligned} M_k^i &= \{v \in V_{L_i} \mid (\omega^3)_1 v = 0\}, \\ W_k^i &= \{v \in V_{L_i} \mid (\omega^3)_1 v = \frac{2}{5}v\}, \qquad \text{for } i = 0, a, b, c, \end{aligned}$$

$$\begin{aligned} M_t^j &= \{v \in V_{L^j} \mid (\omega^1)_1 v = (\omega^2)_1 v = 0\}, \\ W_t^j &= \{v \in V_{L^j} \mid (\omega^1)_1 v = 0, \quad (\omega^2)_1 v = \frac{3}{5}v\}, \qquad \text{for } j = 0, 1, 2. \end{aligned}$$

Then $M_k^0$ and $M_t^0$ are simple vertex operator algebras. Furthermore, $\{M_k^i, W_k^i, i = 0, a, b, c\}$ and $\{M_t^j, W_t^j, j = 0, 1, 2\}$ are the sets of all inequivalent irreducible modules for $M_k^0$ and $M_t^0$, respectively [9, 11, 12]. We have

$$M_k^0 \cong L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}), \qquad M_t^0 \cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3).$$

We also note that

$$V_{L(i,j)} \cong \left(M_k^i \otimes M_t^j\right) \oplus \left(W_k^i \otimes W_t^j\right)$$

as an $M_k^0 \otimes M_t^0$-module.

We consider the following three isometries of $(L, \langle\cdot,\cdot\rangle)$:

$$\begin{aligned}
&\tau : \beta_1 \to \beta_2 \to \beta_0 \to \beta_1, \\
&\sigma : \beta_1 \to \beta_2, \qquad \beta_2 \to \beta_1, \\
&\theta : \beta_i \to -\beta_i, \quad i = 1, 2.
\end{aligned}$$

Set $M = M_k^0$, which is invariant under $\tau$ and $\sigma$. Moreover, $\theta$ acts as the identity on $M$. We are interested in the subalgebra $M^\tau$ of the fixed points of $\tau$ in $M$. The weight 2 subspace of $M^\tau$ is spanned by $\omega$, which is the Virasoro element of $M$ with central charge $6/5$. There are nontrivial relations among $w(\alpha_i)_0 w(\alpha_j)$, $i, j \in \{0, 1, 2\}$. For example,

$$\begin{aligned}
&w(\alpha_1)_0 w(\alpha_2) - w(\alpha_2)_0 w(\alpha_1) \\
&\quad = w(\alpha_2)_0 w(\alpha_0) - w(\alpha_0)_0 w(\alpha_2) \\
&\quad = w(\alpha_0)_0 w(\alpha_1) - w(\alpha_1)_0 w(\alpha_0).
\end{aligned}$$

Set $J = w(\alpha_1)_0 w(\alpha_2) - w(\alpha_2)_0 w(\alpha_1)$. Then $\tau J = J$, $\sigma J = -J$ and $\theta J = J$. The weight 3 subspace of $M^\tau$ is of dimension 2 and it is spanned by $\omega_0\omega$ and $J$. We have $\omega_1 J = 3J$ and $\omega_n J = 0$ for $n \geq 2$. Thus $J$ is a highest weight vector for $\mathrm{Vir}(\omega)$. Let $L(n) = \omega_{n+1}$ and $J(n) = J_{n+2}$. By a direct calculation, we have

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} \cdot \frac{6}{5} \cdot \delta_{m+n,0}, \tag{2.2}$$

$$[L(m), J(n)] = (2m - n)J(m+n), \tag{2.3}$$

$$\begin{aligned}
[J(m), J(n)] &= (m-n)\Big(22(m+n+2)(m+n+3) + 35(m+2)(n+2)\Big)L(m+n) \\
&\quad - 120(m-n)\Big(\sum_{k\leq -2} L(k)L(m+n-k) + \sum_{k\geq -1} L(m+n-k)L(k)\Big) \\
&\quad - \frac{7}{10}m(m^2-1)(m^2-4)\delta_{m+n,0}.
\end{aligned} \tag{2.4}$$

Let $L_n = L(n)$ and $W_n = \sqrt{-1/210}J(n)$. Then the commutation relations in the above theorem coincide with the commutation relations (2.1) and (2.2) of [1]. Thus

**Theorem 2.1** *$\mathcal{W}$ is a $W_3$ algebra of central charge* $6/5$.

## 3 Irreducible modules for $\mathcal{W}$

First, we consider irreducible modules for $M^\tau$. Let $(U, Y_U)$ be one of the 8 irreducible $M$-modules. Following [3], we consider a new $M$-module $(U \circ \tau, Y_{U\circ\tau})$ such that $U \circ \tau = U$ as vector spaces and $Y_{U\circ\tau}(v, z) = Y_U(\tau v, z)$ for $v \in M$. Then $U \mapsto U \circ \tau$ induces a permutation on the set of irreducible $M$-modules. We can easily verify that

**Lemma 3.1** (1) *$M_k^0 \circ \tau = M_k^0$ and $W_k^0 \circ \tau = W_k^0$.*
(2) *$M_k^a \circ \tau = M_k^c$, $M_k^c \circ \tau = M_k^b$, and $M_k^b \circ \tau = M_k^a$.*
(3) *$W_k^a \circ \tau = W_k^c$, $W_k^c \circ \tau = W_k^b$, and $W_k^b \circ \tau = W_k^a$.*

For any $\tau$-invariant space $U$, set $U(\epsilon) = \{u \in U \,|\, \tau u = \xi^\epsilon u\}$, $\epsilon = 0, 1, 2$, where $\xi = \exp(2\pi\sqrt{-1}/3)$. Thus $U(0) = U^\tau$ and $M(\epsilon) = \{v \in M_k^0 \,|\, \tau v = \xi^\epsilon v\}$. Likewise, set $W(\epsilon) = \{v \in W_k^0 \,|\, \tau v = \xi^\epsilon v\}$. From Lemma 3.1 and [6, Theorem 6.14], we see that $M(\epsilon)$ and $W(\epsilon)$ are inequivalent irreducible $M^\tau$-modules for $\epsilon = 0, 1, 2$. Moreover, $M_k^i$, $i = a, b, c$ are equivalent irreducible $M^\tau$-modules and that $W_k^i$, $i = a, b, c$ are also equivalent irreducible $M^\tau$-modules by [6, Theorem 6.14]. Hence we obtain 8 inequivalent irreducible $M^\tau$-modules. Those irreducible modules with their top levels and the action of $L(0)$ and $J(0)$ are collected in Table 1.

Table 1: irreducible $M^\tau$-modules in $M_k^i$ and $W_k^i$

| irred. module | top level | $L(0)$ | $J(0)$ |
|---|---|---|---|
| $M(0)$ | $\mathbb{C}1$ | 0 | 0 |
| $M(1)$ | $\mathbb{C}u^1$ | 2 | $-12\sqrt{-3}$ |
| $M(2)$ | $\mathbb{C}u^2$ | 2 | $12\sqrt{-3}$ |
| $W(0)$ | $\mathbb{C}(y(\alpha_1) + y(\alpha_2) + y(\alpha_0))$ | $\frac{8}{5}$ | 0 |
| $W(1)$ | $\mathbb{C}(\alpha_1(-1) - \xi\alpha_2(-1))$ | $\frac{3}{5}$ | $2\sqrt{-3}$ |
| $W(2)$ | $\mathbb{C}(\alpha_1(-1) - \xi^2\alpha_2(-1))$ | $\frac{3}{5}$ | $-2\sqrt{-3}$ |
| $M_k^c$ | $\mathbb{C}(e^{\beta_1/2} - e^{-\beta_1/2})$ | $\frac{1}{2}$ | 0 |
| $W_k^c$ | $\mathbb{C}(e^{\beta_1/2} + e^{-\beta_1/2})$ | $\frac{1}{10}$ | 0 |

We now study irreducible $\tau$-twisted (resp. $\tau^2$-twisted) $M$-modules. The argument here is similar to that in [10, Section 6]. Basic references to twisted modules for lattice vertex operator algebras are [2, 13]. We follow [2] with $L = \sqrt{2}A_2$, $p = 3$, $q = 6$, and $\nu = \tau$. Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ on $L$ to $\mathfrak{h}$ linearly. Set

$$h_1 = \frac{1}{3}(\beta_1 + \xi^2\beta_2 + \xi\beta_0), \qquad h_2 = \frac{1}{3}(\beta_1 + \xi\beta_2 + \xi^2\beta_0).$$

For $n \in \mathbb{Z}$, set $\mathfrak{h}_{(n)} = \{\alpha \in \mathfrak{h} \,|\, \tau\alpha = \xi^n\alpha\}$. Since $\tau$ is fixed-point-free on $L$, it follows that $\mathfrak{h}_{(0)} = 0$. Furthermore, $\mathfrak{h}_{(1)} = \mathbb{C}h_1$ and $\mathfrak{h}_{(2)} = \mathbb{C}h_2$. For $\alpha \in \mathfrak{h}$, we denote by $\alpha_{(n)}$ the component of $\alpha$ in $\mathfrak{h}_{(n)}$. Thus $(\beta_i)_{(1)} = \xi^{i-1}h_1$ and $(\beta_i)_{(2)} = \xi^{2(i-1)}h_2$ for $i = 0, 1, 2$.

Define the $\tau$-twisted affine Lie algebra to be

$$\hat{\mathfrak{h}}[\tau] = \Big(\bigoplus_{n\in\mathbb{Z}} \mathfrak{h}_{(n)} \otimes t^{n/3}\Big) \oplus \mathbb{C}c$$

with the bracket

$$[x \otimes t^m, y \otimes t^n] = m\langle x, y\rangle \delta_{m+n,0} c$$

for $x \in \mathfrak{h}_{(3m)}$, $y \in \mathfrak{h}_{(3n)}$, $m, n \in (1/3)\mathbb{Z}$, and $[c, \hat{\mathfrak{h}}[g]] = 0$. The isometry $\tau$ acts on $\hat{\mathfrak{h}}[\tau]$ by $\tau(x \otimes t^m) = \xi^m x \otimes t^m$ and $\tau(c) = c$. Set

$$\hat{\mathfrak{h}}[\tau]^+ = \bigoplus_{n>0} \mathfrak{h}_{(n)} \otimes t^{n/3}, \quad \hat{\mathfrak{h}}[\tau]^- = \bigoplus_{n<0} \mathfrak{h}_{(n)} \otimes t^{n/3}, \quad \text{and} \quad \hat{\mathfrak{h}}[\tau]^0 = \mathbb{C}c$$

and consider the $\hat{\mathfrak{h}}[\tau]$-module

$$S[\tau] = U(\hat{\mathfrak{h}}[\tau]) \otimes_{U(\hat{\mathfrak{h}}[\tau]^+ \oplus \hat{\mathfrak{h}}[\tau]^0)} \mathbb{C}$$

induced from the $\hat{\mathfrak{h}}[\tau]^+ \oplus \hat{\mathfrak{h}}[\tau]^0$-module $\mathbb{C}$, where $\hat{\mathfrak{h}}[\tau]^+$ acts trivially on $\mathbb{C}$ and $c$ acts as 1 on $\mathbb{C}$. We define the weight in $S[\tau]$ by

$$\mathrm{wt}(x \otimes t^n) = -n \quad \text{and} \quad \mathrm{wt}\, 1 = \frac{1}{9},$$

where $n \in (1/3)\mathbb{Z}$ and $x \in \mathfrak{h}_{(3n)}$ (cf. [2, (4.6), (4.10)]).

For $\alpha \in \mathfrak{h}$ and $n \in (1/3)\mathbb{Z}$, denote by $\alpha(n)$ the operator on $S[\tau]$ induced by $\alpha_{(3n)} \otimes t^n$. Then, as a vector space $S[\tau]$ can be identified with a polynomial algebra with variables $h_1(1/3+n)$ and $h_2(2/3+n)$, $n \in \mathbb{Z}$. The weight of the operator $h_j(j/3+n)$ is $-j/3-n$.

The alternating $\mathbb{Z}$-bilinear map $c_0^\tau : L \times L \to \mathbb{Z}/6\mathbb{Z}$ defined by [2, (2.10)] is such that

$$c_0^\tau(\alpha, \beta) = \sum_{r=0}^{2} (3 + 2r)\langle \tau^r \alpha, \beta\rangle + 6\mathbb{Z}.$$

In our case $\sum_{r=0}^{2} \tau^r \alpha = 0$, since $\tau$ is fixed-point-free on $L$. Moreover,

$$\sum_{r=0}^{2} r\langle \tau^r \beta_i, \beta_j\rangle = \begin{cases} \pm 6 & \text{if } \tau\beta_i \neq \beta_j \\ 0 & \text{if } \tau\beta_i = \beta_j. \end{cases}$$

Hence $c_0^\tau(\alpha, \beta) = 0$ for all $\alpha, \beta \in L$. This means that the central extension

$$1 \longrightarrow \langle \kappa_6 \rangle \longrightarrow \hat{L}_\tau \stackrel{-}{\longrightarrow} L \longrightarrow 1 \tag{3.1}$$

determined by the commutator condition $aba^{-1}b^{-1} = \kappa_6^{c_0^\tau(\bar{a}, \bar{b})}$ splits.

We consider the relation between two central extensions $\hat{L}$ of (2.1) and $\hat{L}_\tau$ of (3.1). Since both of $\hat{L}$ and $\hat{L}_\tau$ are split extensions, we use the same symbol $e^\alpha$ to denote both of an element in $\hat{L}$ and an element in $\hat{L}_\tau$ which correspond naturally to $\alpha \in L$. Actually, in Section 2 we choose $e^\alpha \in \hat{L}$ so that the multiplication in $\hat{L}$ is $e^\alpha \times e^\beta = e^{\alpha+\beta}$. Also we can choose $e^\alpha \in \hat{L}_\tau$ such that the multiplication $e^\alpha \times_\tau e^\beta$ in $\hat{L}_\tau$ is related to the multiplication in $\hat{L}$ by (cf. [2, (2.4)])

$$e^\alpha \times e^\beta = \kappa_6^{\varepsilon_0(\alpha,\beta)} e^\alpha \times_\tau e^\beta, \tag{3.2}$$

where the $\mathbb{Z}$-linear map $\varepsilon_0 : L \times L \to \mathbb{Z}/6\mathbb{Z}$ is defined by [2, (2.13)]. In our case

$$\varepsilon_0(\alpha, \beta) = -\langle \tau^{-1}\alpha, \beta \rangle + 6\mathbb{Z}. \tag{3.3}$$

As in Section 2, we usually write $e^{\alpha}e^{\beta} = e^{\alpha+\beta}$ to denote the product of $e^{\alpha}$ and $e^{\beta}$ in $\hat{L}$. The automorphism $\tau$ of $L$ lifts to an automorphism $\hat{\tau}$ of $\hat{L}$ such that $\hat{\tau}(e^{\alpha}) = e^{\tau\alpha}$ and $\hat{\tau}(\kappa_6) = \kappa_6$. Since $\varepsilon_0$ is $\tau$-invariant, we can also think $\hat{\tau}$ to be an automorphism of $\hat{L}_\tau$ in a similar way. By abuse of notation we denote $\hat{\tau}$ by simply $\tau$ also.

We have $(1-\tau)L = \mathrm{span}_{\mathbb{Z}}\{\beta_1 - \beta_2, \beta_1 + 2\beta_2\}$. The quotient group $L/(1-\tau)L$ is of order 3 and generated by $\beta_1 + (1-\tau)L$. Now $K = \{a^{-1}\tau(a) \mid a \in \hat{L}_\tau\}$ is a central subgroup of $\hat{L}_\tau$ with $\overline{K} = (1-\tau)L$ and $K \cap \langle \kappa_6 \rangle = 1$. Here note that $a^{-1}$ is the inverse of $a$ in $\hat{L}_\tau$ and $a^{-1}\tau(a)$ is the product $a^{-1} \times_\tau \tau(a)$ in $\hat{L}_\tau$. In $\hat{L}_\tau$ we can verify that

$$e^{3\beta_1} = (e^{\beta_0 - \beta_1})^{-1} \times_\tau \tau(e^{\beta_0 - \beta_1}) \in K.$$

Since $\kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{\beta_1} = e^{3\beta_1}$ and $\kappa_3 e^{\beta_1} \times_\tau \kappa_3 e^{-\beta_1} = 1$, it follows that

$$\hat{L}_\tau / K = \{K, \kappa_3 e^{\beta_1} K, \kappa_3 e^{-\beta_1} K\} \times \langle \kappa_6 \rangle K/K \cong \mathbb{Z}_3 \times \mathbb{Z}_6.$$

For $j = 0, 1, 2$, define a linear character $\chi_j : \hat{L}_\tau / K \to \mathbb{C}^{\times}$ by

$$\chi_j(\kappa_6) = \xi_6, \quad \chi_j(\kappa_3 e^{\beta_1} K) = \xi^j, \quad \text{and} \quad \chi_j(\kappa_3 e^{-\beta_1} K) = \xi^{-j},$$

where $\xi_6 = \exp(2\pi\sqrt{-1}/6)$. Let $T_{\chi_j}$ be the one dimensional $\hat{L}_\tau/K$-module affording the character $\chi_j$. As an $\hat{L}_\tau$-module, $K$ acts trivially on $T_{\chi_j}$. Since $\sum_{r=0}^{2} \tau^r \alpha = 0$ for $\alpha \in L$, those $T_{\chi_j}$, $j = 0, 1, 2$, are the irreducible $\hat{L}_\tau$-modules constructed in [13, Section 6].

Let

$$V_L^{T_{\chi_j}} = V_L^{T_{\chi_j}}(\tau) = S[\tau] \otimes T_{\chi_j}$$

and define the $\tau$-twisted vertex operator $Y^{\tau}(\cdot, z) : V_L \to \mathrm{End}(V_L^{T_{\chi_j}}(\tau))\{z\}$ as in [2]. We extend the action of $\tau$ to $V_L^{T_{\chi_j}}(\tau)$ so that $\tau$ is the identity on $T_{\chi_j}$. The weight of every element in $T_{\chi_j}$ is defined to be 0.

By [2, Theorem 7.1], $(V_L^{T_{\chi_j}}(\tau), Y^{\tau}(\cdot, z))$, $j = 0, 1, 2$ are inequivalent irreducible $\tau$-twisted $V_L$-modules. Now among the 12 irreducible $V_L$-modules $V_{L^{(i,j)}}$, $i \in \{0, a, b, c\}$ and $j \in \{0, 1, 2\}$, the $\tau$-stable irreducible modules are $V_{L^{(0,j)}}$, $j \in \{0, 1, 2\}$. Hence by [3, Theorem 10.2], $(V_L^{T_{\chi_j}}(\tau), Y^{\tau}(\cdot, z))$, $j = 0, 1, 2$, are all the inequivalent irreducible $\tau$-twisted $V_L$-modules. The isometry $\theta$ of $(L, \langle \cdot, \cdot \rangle)$ induces a permutation on $V_L^{T_{\chi_j}}(\tau)$, $j = 0, 1, 2$. In fact, the permutation leaves $V_L^{T_{\chi_0}}(\tau)$ invariant and interchanges $V_L^{T_{\chi_1}}(\tau)$ and $V_L^{T_{\chi_2}}(\tau)$. Since $M^{\tau} \otimes M_t^0$ is contained in the subalgebra $(V_L)^{\tau}$ of fixed points of $\tau$ in $V_L$, we can deal with $(V_L^{T_{\chi_j}}(\tau), Y^{\tau}(\cdot, z))$ as an $M^{\tau} \otimes M_t^0$-module.

The decomposition of $V_L^{T_{\chi_j}}(\tau)$ as a $\tau$-twisted $M \otimes M_t^0$-module was studied in [10]. Set

$$
\begin{aligned}
M_T^0(\tau) &= \{u \in V_L^{T_{\chi_0}}(\tau) \mid (\omega^3)_1 u = 0\},\\
W_T^0(\tau) &= \{u \in V_L^{T_{\chi_0}}(\tau) \mid (\omega^3)_1 u = \frac{2}{5}u\},\\
M_T^j(\tau) &= \{u \in V_L^{T_{\chi_j}}(\tau) \mid (\omega^3)_1 u = \frac{2}{3}u\}, \quad j = 1, 2,\\
W_T^j(\tau) &= \{u \in V_L^{T_{\chi_j}}(\tau) \mid (\omega^3)_1 u = \frac{1}{15}u\}, \quad j = 1, 2.
\end{aligned}
$$

Then, by [10, Proposition 6.8], $M_T^j(\tau)$ and $W_T^j(\tau)$, $j = 0, 1, 2$, are irreducible $\tau$-twisted $M$-modules. Furthermore, for $j = 0, 1, 2$,

$$V_L^{T_{\chi_j}}(\tau) \cong M_T^j(\tau) \otimes M_t^j \oplus W_T^j(\tau) \otimes W_t^j$$

as $\tau$-twisted $M \otimes M_t^0$-modules.

There are at most two inequivalent irreducible $\tau$-twisted $M$-modules by Lemma 4.1 and [3, Theorem 10.2]. Then, looking at the smallest weight of $M_T^j(\tau)$ and $W_T^j(\tau)$, we have that $M_T^0(\tau) \cong M_T^1(\tau) \cong M_T^2(\tau)$ and $W_T^0(\tau) \cong W_T^1(\tau) \cong W_T^2(\tau)$ and that $M_T^0(\tau) \not\cong W_T^0(\tau)$ as $\tau$-twisted $M$-modules. We denote $M_T^0(\tau)$ by $M_T(\tau)$ and $W_T^0(\tau)$ by $W_T(\tau)$. We conclude that there are exactly two inequivalent irreducible $\tau$-twisted $M$-modules, which are represented by $M_T(\tau)$ and $W_T(\tau)$. As $\tau$-twisted $M \otimes \mathrm{Vir}(\omega^3)$-modules, we have

$$V_L^{T_{\chi_0}}(\tau) \cong M_T(\tau) \otimes \left(L(\frac{4}{5}, 0) + L(\frac{4}{5}, 3)\right) \oplus W_T(\tau) \otimes \left(L(\frac{4}{5}, \frac{2}{5}) + L(\frac{4}{5}, \frac{7}{5})\right), \tag{3.4}$$

$$V_L^{T_{\chi_1}}(\tau) \cong V_L^{T_{\chi_2}}(\tau) \cong M_T(\tau) \otimes L(\frac{4}{5}, \frac{2}{3}) \oplus W_T(\tau) \otimes L(\frac{4}{5}, \frac{1}{15}). \tag{3.5}$$

For $\epsilon = 0, 1, 2$, let

$$
\begin{aligned}
M_T(\tau)(\epsilon) &= \{u \in M_T(\tau) \mid \tau u = \xi^\epsilon u\},\\
W_T(\tau)(\epsilon) &= \{u \in W_T(\tau) \mid \tau u = \xi^\epsilon u\}.
\end{aligned}
$$

Those 6 modules for $M^\tau$ are inequivalent irreducible modules by [14, Theorem 2]. Their top levels are of dimension one. Those top levels and the eigenvalues for the action of $L^\tau(0) = \omega_1$ and $J^\tau(0) = J_2$ are collected in Table 2.

By a similar argument we obtain 6 inequivalent irreducible $M^\tau$-modules inside $\tau^2$-twisted $M$-modules. The results are collected in Table 3.

We have obtained 20 irreducible $M^\tau$-modules. Now we turn to the vertex operator algebra $\mathcal{W}$ generated by $\omega$ and $J$ in $M^\tau$. Note that

$$L(n)\mathbf{1} = 0 \quad \text{for} \quad n \geq -1, \qquad L(-2)\mathbf{1} = \omega, \tag{3.6}$$

$$J(n)\mathbf{1} = 0 \quad \text{for} \quad n \geq -2, \qquad J(-3)\mathbf{1} = J, \tag{3.7}$$

Then by using commutation relations (2.2), (2.3), and (2.4), we see that

Table 2: irreducible $M^\tau$-modules in $M_T(\tau)$ and $W_T(\tau)$

| irred. module | top level | $L^\tau(0)$ | $J^\tau(0)$ |
|---|---|---|---|
| $M_T(\tau)(0)$ | $\mathbb{C}1 \otimes v$ | $\frac{1}{9}$ | $\frac{14}{81}\sqrt{-3}$ |
| $M_T(\tau)(1)$ | $\mathbb{C}h_2(-\frac{1}{3})^2 \otimes v$ | $\frac{1}{9}+\frac{2}{3}$ | $-\frac{238}{81}\sqrt{-3}$ |
| $M_T(\tau)(2)$ | $\mathbb{C}(\frac{4}{3}h_1(-\frac{2}{3})^2 \otimes v + h_2(-\frac{1}{3})^4 \otimes v)$ | $\frac{1}{9}+\frac{4}{3}$ | $\frac{374}{81}\sqrt{-3}$ |
| $W_T(\tau)(0)$ | $\mathbb{C}h_2(-\frac{1}{3}) \otimes v$ | $\frac{2}{45}$ | $-\frac{4}{81}\sqrt{-3}$ |
| $W_T(\tau)(1)$ | $\mathbb{C}h_1(-\frac{2}{3}) \otimes v$ | $\frac{2}{45}+\frac{1}{3}$ | $-\frac{22}{81}\sqrt{-3}$ |
| $W_T(\tau)(2)$ | $\mathbb{C}h_2(-\frac{1}{3})^3 \otimes v$ | $\frac{2}{45}+\frac{2}{3}$ | $\frac{176}{81}\sqrt{-3}$ |

Table 3: irreducible $M^\tau$-modules in $M_T(\tau^2)$ and $W_T(\tau^2)$

| irred. module | top level | $L^{\tau^2}(0)$ | $J^{\tau^2}(0)$ |
|---|---|---|---|
| $M_T(\tau^2)(0)$ | $\mathbb{C}1 \otimes v$ | $\frac{1}{9}$ | $-\frac{14}{81}\sqrt{-3}$ |
| $M_T(\tau^2)(1)$ | $\mathbb{C}h'_2(-\frac{1}{3})^2 \otimes v$ | $\frac{1}{9}+\frac{2}{3}$ | $\frac{238}{81}\sqrt{-3}$ |
| $M_T(\tau^2)(2)$ | $\mathbb{C}(\frac{4}{3}h'_1(-\frac{2}{3})^2 \otimes v + h'_2(-\frac{1}{3})^4 \otimes v)$ | $\frac{1}{9}+\frac{4}{3}$ | $-\frac{374}{81}\sqrt{-3}$ |
| $W_T(\tau^2)(0)$ | $\mathbb{C}h'_2(-\frac{1}{3}) \otimes v$ | $\frac{2}{45}$ | $\frac{4}{81}\sqrt{-3}$ |
| $W_T(\tau^2)(1)$ | $\mathbb{C}h'_1(-\frac{2}{3}) \otimes v$ | $\frac{2}{45}+\frac{1}{3}$ | $\frac{22}{81}\sqrt{-3}$ |
| $W_T(\tau^2)(2)$ | $\mathbb{C}h'_2(-\frac{1}{3})^3 \otimes v$ | $\frac{2}{45}+\frac{2}{3}$ | $-\frac{176}{81}\sqrt{-3}$ |

**Lemma 3.2** *$\mathcal{W}$ is spanned by the vectors of the form*

$$L(-m_1)\cdots L(-m_p)J(-n_1)\cdots J(-n_q)\mathbf{1} \tag{3.8}$$

*with $m_1 \geq \cdots \geq m_p \geq 2$, $n_1 \geq \cdots \geq n_q \geq 3$, $p = 0, 1, 2, \ldots$, and $q = 0, 1, 2, \ldots$.*

A vector $v \in \mathcal{W}$ of weight $h$ is called a singular vector for $\mathcal{W}$ if it satisfies

(1) $L(0)v = hv$,

(2) $L(n)v = 0$ and $J(n)v = 0$ for $n \geq 1$.

Note that $v$ is not necessarily an eigenvector for $J(0)$. By commutation relations (2.2) and (2.3), it is easy to show that the condition (2) holds if $v$ satisfies

(2)′ $L(1)v = L(2)v = J(1)v = 0$.

We consider $\mathcal{W}$ as a space spanned by the vectors of the form (3.8). The weight of such a vector is $m_1 + \cdots + m_p + n_1 + \cdots + n_q$. Using a computer algebra system Risa/Asir we have the following lemma.

**Lemma 3.3** *Let $v$ be a linear combination of the vectors of the form* (3.8) *of weight $h$. Under the conditions* (3.6) *and* (3.7) *and the commutation relations* (2.2), (2.3), *and* (2.4), *we have $L(1)v = L(2)v = J(1)v = 0$ only if $v = 0$ in the case $h \leq 11$. In the case $h = 12$, there exists a unique, up to scalar multiple, linear combination $\mathbf{v}^{12}$ which satisfies $L(1)\mathbf{v}^{12} = L(2)\mathbf{v}^{12} = J(1)\mathbf{v}^{12} = 0$. The explicit form of $\mathbf{v}^{12}$ is given in Appendix. We also have $J(0)\mathbf{v}^{12} = 0$.*

By a general theory of lattice vertex operator algebras, $V_L$ possesses an invariant positive definite hermitian form. From this it follows that

**Proposition 3.4** *The singular vector* $\mathbf{v}^{12} = 0$.

Using the explicit form of $\mathbf{v}^{12}$, $J(-1)\mathbf{v}^{12}$, $J(-2)\mathbf{v}^{12}$, and $J(-1)\mathbf{v}^{12}$, we can determine the Zhu algebra $Z(\mathcal{W})$ of $\mathcal{W}$. The standard reference to Zhu's theory is [15, Section 2]. The Zhu algebra $Z(\mathcal{W})$ is a quotient vector space $A(\mathcal{W}) = \mathcal{W}/O(\mathcal{W})$ equipped with a commutative associative algebra structure with respect to an operation $*$. We denote the image of a vector $v \in \mathcal{W}$ in $A(\mathcal{W}) = \mathcal{W}/O(\mathcal{W})$ by $[v]$. We can define an algebra homomorphism from $\mathbb{C}[x, y]$ onto $A(\mathcal{W})$ by $x \mapsto [\omega]$ and $y \mapsto [J]$. The primary decomposition of its kernel $\mathcal{I}$ is such that $\mathcal{I} = \cap_{i=1}^{20} \mathcal{P}_i$, where $\mathcal{P}_i$, $1 \le i \le 20$ are

$$\begin{array}{ll}
\langle x, y\rangle, & \langle 5x-8, y\rangle, \\
\langle 2x-1, y\rangle, & \langle 10x-1, y\rangle, \\
\langle x-2, y-12\sqrt{-3}\rangle, & \langle x-2, y+12\sqrt{-3}\rangle, \\
\langle 5x-3, y-2\sqrt{-3}\rangle, & \langle 5x-3, y+2\sqrt{-3}\rangle, \\
\langle 9x-1, 81y-14\sqrt{-3}\rangle, & \langle 9x-1, 81y+14\sqrt{-3}\rangle, \\
\langle 9x-7, 81y-238\sqrt{-3}\rangle, & \langle 9x-7, 81y+238\sqrt{-3}\rangle, \\
\langle 9x-13, 81y-374\sqrt{-3}\rangle, & \langle 9x-13, 81y+374\sqrt{-3}\rangle, \\
\langle 45x-2, 81y-4\sqrt{-3}\rangle, & \langle 45x-2, 81y+4\sqrt{-3}\rangle, \\
\langle 45x-17, 81y-22\sqrt{-3}\rangle, & \langle 45x-17, 81y+22\sqrt{-3}\rangle, \\
\langle 45x-32, 81y-176\sqrt{-3}\rangle, & \langle 45x-32, 81y+176\sqrt{-3}\rangle,
\end{array} \tag{3.9}$$

These primary ideals correspond to the 20 irreducible $M^\tau$-modules listed in Tables 1, 2, and 3. The correspondence is given by substituting $x$ and $y$ with the eigenvalues for $L(0)$ and $J(0)$ on the top levels of 20 irreducible modules. The eigenvalues are the zeros of those primary ideals.

By Zhu's theory we obtain the classification of irreducible $\mathcal{W}$-modules, namely,

**Theorem 3.5** (1) $M^\tau = \mathcal{W}$.

(2) $A(M^\tau) \cong \oplus_{i=1}^{20} \mathbb{C}[x, y]/\mathcal{P}_i$ *is a 20-dimensional commutative associative algebra.*

(3) *There are exactly* 20 *inequivalent irreducible* $M^\tau$*-modules. Their representatives are listed in Tables 1, 2, and 3 in Section 2, namely,* $M(\epsilon)$, $W(\epsilon)$, $M_k^c$, $W_k^c$, $M_T(\tau^i)(\epsilon)$, *and* $W_T(\tau^i)(\epsilon)$ *for* $\epsilon = 0, 1, 2$ *and* $i = 1, 2$.

# Appendix

$$\begin{aligned}
\mathbf{v}^{12} = &-(5877264800/3501)L(-12)\mathbf{1} + (3404072000/3501)L(-10)L(-2)\mathbf{1}\\
&- (2653990000/3501)L(-9)L(-3)\mathbf{1} - (266376800/3501)L(-8)L(-4)\mathbf{1}\\
&+ (282988000/1167)L(-8)L(-2)^2\mathbf{1} - (23744800/1167)L(-7)L(-5)\mathbf{1}\\
&- (30824000/1167)L(-7)L(-3)L(-2)\mathbf{1} + (1242377600/1167)L(-6)^2\mathbf{1}\\
&- (61947200/3501)L(-6)L(-4)L(-2)\mathbf{1} - (1313806000/1167)L(-6)L(-3)^2\mathbf{1}\\
&- (45496000/1167)L(-6)L(-2)^3\mathbf{1} - (3046768400/3501)L(-5)^2L(-2)\mathbf{1}\\
&+ (299424800/1167)L(-5)L(-4)L(-3)\mathbf{1} + (2347094000/3501)L(-5)L(-3)L(-2)^2\mathbf{1}\\
&- (17280400/1167)L(-4)^3\mathbf{1} - (2036373200/3501)L(-4)^2L(-2)^2\mathbf{1}\\
&+ (82996000/3501)L(-4)L(-3)^2L(-2)\mathbf{1} + (1074512000/3501)L(-4)L(-2)^4\mathbf{1}\\
&+ (511628125/3501)L(-3)^4\mathbf{1} - (418850000/3501)L(-3)^2L(-2)^3\mathbf{1}\\
&- (59680000/3501)L(-2)^6\mathbf{1} - (505200/389)L(-6)J(-3)^2\mathbf{1}\\
&+ (3380480/1167)L(-4)L(-2)J(-3)^2\mathbf{1} + 1150L(-3)^2J(-3)^2\mathbf{1}\\
&- (184400/1167)L(-2)^3J(-3)^2\mathbf{1} + (3788680/1167)L(-5)J(-4)J(-3)\mathbf{1}\\
&- (8788400/3501)L(-3)L(-2)J(-4)J(-3)\mathbf{1} - (12761440/3501)L(-4)J(-5)J(-3)\mathbf{1}\\
&- (5727500/10503)L(-4)J(-4)^2\mathbf{1} + (352400/389)L(-2)^2J(-5)J(-3)\mathbf{1}\\
&+ (5727500/10503)L(-2)^2J(-4)^2\mathbf{1} + (1593900/389)L(-3)J(-6)J(-3)\mathbf{1}\\
&+ (12935800/10503)L(-3)J(-5)J(-4)\mathbf{1} + (4108000/3501)L(-2)J(-7)J(-3)\mathbf{1}\\
&- (2811800/1167)L(-2)J(-6)J(-4)\mathbf{1} - (3131600/10503)L(-2)J(-5)^2\mathbf{1}\\
&- (14904160/3501)J(-9)J(-3)\mathbf{1} + (32677600/10503)J(-8)J(-4)\mathbf{1}\\
&+ (9423200/10503)J(-7)J(-5)\mathbf{1} + (2432375/1167)J(-6)^2\mathbf{1}\\
&+ J(-3)^4\mathbf{1}.
\end{aligned}$$

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