Irreducible modules for $W_3$-algebra

1 Introduction

Recently a $W_3$-algebra of central charge $6/5$ was studied as an orbifold of an order three automorphism of the lattice vertex operator algebra $V_{\sqrt{2}A_2}$ by Dong, Lam, Tanabe, Yokoyama and the author [5]. The paper consists of four parts: (1) realization of a $W_3$-algebra $\mathcal{W}$ of central charge $6/5$, (2) classification of irreducible $\mathcal{W}$-modules, (3) $C_2$-cofiniteness and rationality of $\mathcal{W}$, and (4) determination of the character of each irreducible $\mathcal{W}$-module. In this note we will give an outline of the second part of the paper, namely, classification of irreducible $\mathcal{W}$-modules. Details can be found in [5]. Basic references to $\mathcal{W}_3$-algebras are [1] and [7].

2 The $W_3$-algebra $\mathcal{W}$

For definitions of the materials discussed here we refer to [2, 8]. We also use certain properties of the vertex operator algebra $V_{\sqrt{2}A_2}$ (cf. [12]).

Let $\alpha_1, \alpha_2$ be the simple roots of type $A_2$ and set $\alpha_0 = -(\alpha_1 + \alpha_2)$. We denote the inner product by $\langle \cdot, \cdot \rangle$. Then $\langle \alpha_i, \alpha_i \rangle = 2$ and $\langle \alpha_i, \alpha_j \rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2}\alpha_i$ and let $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 = \sqrt{2}A_2$ be the lattice spanned by $\beta_1$ and $\beta_2$. We follow Sections 2 and 3 of [2] with $L = \sqrt{2}A_2$, $p = 3$, and $q = 6$. In our case $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in L$, so that the alternating $\mathbb{Z}$-bilinear map $c_0 : L \times L \rightarrow \mathbb{Z}/6\mathbb{Z}$ defined by [2, (2.9)] is trivial. Thus the central extension

$$1 \rightarrow \langle \kappa_6 \rangle \rightarrow \hat{L} \rightarrow L \rightarrow 1$$

(2.1)

determined by the commutator condition $aba^{-1}b^{-1} = \kappa_6^{(a,b)}$ splits. Then for each $\alpha \in L$, we can choose an element $e^\alpha$ of $\hat{L}$ so that $e^\alpha e^\beta = e^{\alpha+\beta}$. The twisted group algebra $\mathbb{C}\{L\}$ is isomorphic to the ordinary group algebra $\mathbb{C}[L]$.

We adopt the same notation as in [9] to denote cosets of $L$ in the dual lattice $L^\perp = \{ \alpha \in \mathbb{Q} \otimes \mathbb{Z} \mid \langle \alpha, L \rangle \subset \mathbb{Z} \}$, namely,

$$L^0 = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L,$$

$$L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L,$$
\[ L^{(i,j)} = L_i + L^j \]

for \( i = 0, a, b, c \) and \( j = 0, 1, 2 \), where \( \{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Then, \( L^{(i,j)}, i \in \{0, a, b, c\}, j \in \{0, 1, 2\} \) are all the cosets of \( L \) in \( L^\perp \) and \( L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \).

Our notation for the vertex operator algebra \( (V_L, Y(\cdot, z)) \) associated with \( L \) is standard [8]. In particular, \( \mathfrak{h} = \mathbb{C} \otimes L \) is an abelian Lie algebra, \( \mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \) is the corresponding affine Lie algebra, \( M(1) = \mathbb{C}[\alpha(n); \alpha \in \mathfrak{h}, n < 0] \), where \( \alpha(n) = \alpha \otimes t^n \), is the unique irreducible \( \mathfrak{h} \)-module such that \( \alpha(n)1 = 0 \) for all \( \alpha \in \mathfrak{h} \) and \( n > 0 \), and \( c = 1 \). As a vector space \( V_L = M(1) \otimes \mathbb{C}[L] \) and for each \( v \in V_L \), a vertex operator \( Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in \text{End}(V_L)[[z, z^{-1}]] \) is defined. The vector \( 1 = 1 \otimes 1 \) is called the vacuum vector.

There are exactly 12 isomorphism classes of irreducible \( V_L \)-modules, which are represented by \( V_{L(i,j)}, i = 0, a, b, c \) and \( j = 0, 1, 2 \). We use the symbol \( e^\alpha, \alpha \in L^\perp \) to denote a basis of \( \mathbb{C}\{L^\perp\} \).

Let
\[
x(\alpha) = e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}, \quad y(\alpha) = e^{\sqrt{2}\alpha} - e^{-\sqrt{2}\alpha}, \quad w(\alpha) = \frac{1}{2}\alpha(-1)^2 - x(\alpha)
\]

for \( \alpha \in \{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2\} \) and set
\[
\omega = \frac{1}{5}(w(\alpha_1) + w(\alpha_2) + w(\alpha_0)), \\
\tilde{\omega} = \frac{1}{6}(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2), \\
\omega^1 = \frac{1}{4}w(\alpha_1), \quad \omega^2 = \omega - \omega^1, \quad \omega^3 = \tilde{\omega} - \omega.
\]

Then \( \tilde{\omega} \) is the Virasoro element of \( V_L \) and \( \omega^1, \omega^2, \omega^3 \) are mutually orthogonal conformal vectors of central charge \( 1/2, 7/10, 4/5 \) respectively (cf.[4]). The subalgebra \( \text{Vir}(\omega^j) \) generated by \( \omega^i \) is isomorphic to the Virasoro vertex operator algebra of given central charge.

We study certain subalgebras, and also submodules for them in \( V_{L_i}, i = 0, a, b, c \) and in \( V_{L_j}, j = 0, 1, 2 \). Set
\[
M^v_L = \{v \in V_{L_i} | (\omega^3)_1 v = 0\}, \\
W^v_L = \{v \in V_{L_i} | (\omega^3)_1 v = \frac{2}{5} v\}, \quad \text{for } i = 0, a, b, c,
\]
\[
M^i_L = \{v \in V_{L_j} | (\omega^1)_1 v = (\omega^2)_1 v = 0\}, \\
W^i_L = \{v \in V_{L_j} | (\omega^1)_1 v = 0, \quad (\omega^2)_1 v = \frac{3}{5} v\}, \quad \text{for } j = 0, 1, 2.
\]

Then \( M^v_L \) and \( M^i_L \) are simple vertex operator algebras. Furthermore, \( \{M^v_L, W^v_L, i = 0, a, b, c\} \) and \( \{M^j_L, W^j_L, j = 0, 1, 2\} \) are the sets of all inequivalent irreducible modules for \( M^0_L \) and \( M^0_L \), respectively [9, 11, 12]. We have
\[
M^0_L \cong L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, \frac{3}{2}), \quad M^0_L \cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3).
\]
We also note that
\[ V_{L^{(i,j)}} \cong (M_k^i \otimes M_t^j) \oplus (W_k^i \otimes W_t^j) \]
as an \( M_k^0 \otimes M_t^0 \)-module.

We consider the following three isometries of \((L, \langle \cdot, \cdot \rangle)\):
\[ \tau : \beta_1 \rightarrow \beta_2 \rightarrow \beta_0 \rightarrow \beta_1, \]
\[ \sigma : \beta_1 \rightarrow \beta_2, \quad \beta_2 \rightarrow \beta_1, \]
\[ \theta : \beta_i \rightarrow -\beta_i, \quad i = 1, 2. \]

Set \( M = M_k^0 \), which is invariant under \( \tau \) and \( \sigma \). Moreover, \( \theta \) acts as the identity on \( M \). We are interested in the subalgebra \( M^\tau \) of the fixed points of \( \tau \) in \( M \). The weight 2 subspace of \( M^\tau \) is spanned by \( \omega \), which is the Virasoro element of \( M \) with central charge 6/5. There are nontrivial relations among \( w(\alpha_i) \omega(\alpha_j), i, j \in \{0, 1, 2\} \). For example,
\[ w(\alpha_1) \omega(\alpha_2) - w(\alpha_2) \omega(\alpha_1) = w(\alpha_2) \omega(\alpha_0) - w(\alpha_0) \omega(\alpha_2) = w(\alpha_0) \omega(\alpha_1) - w(\alpha_1) \omega(\alpha_0). \]

Set \( J = w(\alpha_1) \omega(\alpha_2) - w(\alpha_2) \omega(\alpha_1) \). Then \( \tau J = J \), \( \sigma J = -J \) and \( \theta J = J \). The weight 3 subspace of \( M^\tau \) is of dimension 2 and it is spanned by \( \omega_0 \omega \) and \( J \). We have \( \omega_1 J = 3J \) and \( \omega_n J = 0 \) for \( n \geq 2 \). Thus \( J \) is a highest weight vector for \( \text{Vir}(\omega) \). Let \( L(n) = \omega_{n+1} \) and \( J(n) = J_{\tau n+2} \). By a direct calculation, we have
\[ [L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \cdot \frac{6}{5} \cdot \delta_{m+n,0}, \quad (2.2) \]
\[ [L(m), J(n)] = (2m - n)J(m + n), \quad (2.3) \]
\[ [J(m), J(n)] = (m - n) \left( 22(m + n + 2)(m + n + 3) + 35(m + 2)(n + 2) \right) L(m + n) \\
- 120(m - n) \left( \sum_{k \leq -2} L(k)L(m + n - k) + \sum_{k \geq -1} L(m + n - k)L(k) \right) \\
- \frac{7}{10} m(m^2 - 1)(m^2 - 4) \delta_{m+n,0}. \quad (2.4) \]

Let \( L_n = L(n) \) and \( W_n = \sqrt{-1/210}J(n) \). Then the commutation relations in the above theorem coincide with the commutation relations (2.1) and (2.2) of [1]. Thus

**Theorem 2.1** \( \mathcal{W} \) is a \( W_3 \) algebra of central charge 6/5.
3 Irreducible modules for $\mathcal{W}$

First, we consider irreducible modules for $M^\tau$. Let $(U, Y_U)$ be one of the 8 irreducible $M$-modules. Following [3], we consider a new $M$-module $(U \circ \tau, Y_{U \circ \tau})$ such that $U \circ \tau = U$ as vector spaces and $Y_{U \circ \tau}(v, z) = Y_U(\tau v, z)$ for $v \in M$. Then $U \mapsto U \circ \tau$ induces a permutation on the set of irreducible $M$-modules. We can easily verify that

**Lemma 3.1**

1. $M_k^0 \circ \tau = M_k^0$ and $W_k^0 \circ \tau = W_k^0$.
2. $M_k^a \circ \tau = M_k^a$, $M_k^c \circ \tau = M_k^c$, and $M_k^0 \circ \tau = M_k^0$.
3. $W_k^0 \circ \tau = W_k^0$, $W_k^c \circ \tau = W_k^c$, and $W_k^0 \circ \tau = W_k^0$.

For any $\tau$-invariant space $U$, set $U(\epsilon) = \{u \in U | \tau u = \xi^\epsilon u\}$, $\epsilon = 0, 1, 2$, where $\xi = \exp(2\pi\sqrt{-1}/3)$. Thus $U(0) = U^\tau$ and $M(\epsilon) = \{v \in M_k^\epsilon | \tau v = \xi^\epsilon v\}$. Likewise, set $W(\epsilon) = \{v \in W_k^\epsilon | \tau v = \xi^\epsilon v\}$. From Lemma 3.1 and [6, Theorem 6.14], we see that $M(\epsilon)$ and $W(\epsilon)$ are inequivalent irreducible $M^\tau$-modules for $\epsilon = 0, 1, 2$. Moreover, $M_k^i$, $i = a, b, c$ are equivalent irreducible $M^\tau$-modules and that $W_k^i$, $i = a, b, c$ are also equivalent irreducible $M^\tau$-modules by [6, Theorem 6.14]. Hence we obtain 8 inequivalent irreducible $M^\tau$-modules. Those irreducible modules with their top levels and the action of $L(0)$ and $J(0)$ are collected in Table 1.

| Table 1: irreducible $M^\tau$-modules in $M^\tau_k$ and $W^\tau_k$ |
|-----------------|-----------------|-----|-----|
| irreducible module | top level | $L(0)$ | $J(0)$ |
| $M(0)$ | C1 | 0 | 0 |
| $M(1)$ | Cu | 2 | $-12\sqrt{-3}$ |
| $M(2)$ | Cu | 2 | $12\sqrt{-3}$ |
| $W(0)$ | C($y(\alpha_1) + y(\alpha_2) + y(\alpha_0)$) | 0 | |
| $W(1)$ | C($\alpha_1(-1) - \xi\alpha_2(-1)$) | 2 | $\sqrt{-3}$ |
| $W(2)$ | C($\alpha_1(-1) - \xi^2\alpha_2(-1)$) | -2 | $\sqrt{-3}$ |
| $M^a_k$ | C($\beta_1/2 - e^{-\beta_1/2}$) | 0 | |
| $W^c_k$ | C($\beta_1/2 + e^{-\beta_1/2}$) | $1/10$ | 0 |

We now study irreducible $\tau$-twisted (resp. $\tau^2$-twisted) $M$-modules. The argument here is similar to that in [10, Section 6]. Basic references to twisted modules for lattice vertex operator algebras are [2, 13]. We follow [2] with $L = \sqrt{2}A_2$, $p = 3$, $q = 6$, and $\nu = \tau$. Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ on $L$ to $\mathfrak{h}$ linearly. Set

$$h_1 = \frac{1}{3}(\beta_1 + \xi \beta_2 + \xi \beta_3), \quad h_2 = \frac{1}{3}(\beta_1 + \xi \beta_2 + \xi^2 \beta_3).$$

For $n \in \mathbb{Z}$, set $\mathfrak{h}(n) = \{\alpha \in \mathfrak{h} | \tau \alpha = \xi^n \alpha\}$. Since $\tau$ is fixed-point-free on $L$, it follows that $\mathfrak{h}(0) = 0$. Furthermore, $\mathfrak{h}(1) = \mathfrak{h} \mathbb{C}_1$ and $\mathfrak{h}(2) = \mathfrak{h} \mathbb{C}_2$. For $\alpha \in \mathfrak{h}$, we denote by $\alpha(n)$ the component of $\alpha$ in $\mathfrak{h}(n)$. Thus $(\beta_1)(1) = \xi^{i-1} h_1$ and $(\beta_1)(2) = \xi^{2(i-1)} h_2$ for $i = 0, 1, 2$.

Define the $\tau$-twisted affine Lie algebra to be

$$\hat{\mathfrak{h}}[\tau] = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h}(n) \otimes \tau^n/3 \oplus \mathbb{C} c$$
with the bracket

\[ [x \otimes t^m, y \otimes t^n] = m(x, y) \delta_{m+n,0}c \]

for \( x, y \in \mathfrak{h}(3n) \), \( m, n \in (1/3)\mathbb{Z} \), and \( [c, \hat{h}[g]] = 0 \). The isometry \( \tau \) acts on \( \hat{h}[\tau] \) by

\[ \tau(x \otimes t^m) = \xi^m x \otimes t^m \text{ and } \tau(c) = c. \]

Set

\[ \hat{h}[\tau]^+ = \bigoplus_{n>0} \mathfrak{h}(n) \otimes t^{n/3}, \quad \hat{h}[\tau]^0 = \mathfrak{h}(n) \otimes t^{n/3}, \quad \text{and} \quad \hat{h}[\tau]^0 = \mathbb{C}c \]

and consider the \( \hat{h}[\tau] \)-module

\[ S[\tau] = U(\hat{h}[\tau]) \otimes \mathfrak{h}(3n) \]

induced from the \( \hat{h}[\tau]^+ \oplus \hat{h}[\tau]^0 \)-module \( \mathbb{C} \), where \( \hat{h}[\tau]^+ \) acts trivially on \( \mathbb{C} \) and \( c \) acts as \( 1 \) on \( \mathbb{C} \). We define the weight in \( S[\tau] \) by

\[ \text{wt}(x \otimes t^n) = -n \quad \text{and} \quad \text{wt}1 = \frac{1}{9}, \]

where \( n \in (1/3)\mathbb{Z} \) and \( x \in \mathfrak{h}(3n) \) (cf. [2, (4.6), (4.10)]).

For \( \alpha \in \mathfrak{h} \) and \( n \in (1/3)\mathbb{Z} \), denote by \( \alpha(n) \) the operator on \( S[\tau] \) induced by \( \alpha(3n) \otimes t^n \).

Then, as a vector space \( S[\tau] \) can be identified with a polynomial algebra with variables \( h_1(1/3+n) \) and \( h_2(2/3+n), n \in \mathbb{Z} \). The weight of the operator \( h_j(j/3+n) \) is \(-j/3-n\).

The alternating \( \mathbb{Z} \)-bilinear map \( c_0^\tau : L \times L \to \mathbb{Z}/6\mathbb{Z} \) defined by [2, (2.10)] is such that

\[ c_0^\tau(\alpha, \beta) = \sum_{r=0}^{2} (3+2r) \langle \tau^r \alpha, \beta \rangle + 6\mathbb{Z}. \]

In our case \( \sum_{r=0}^{2} \tau^r \alpha = 0 \), since \( \tau \) is fixed-point-free on \( L \). Moreover,

\[ \sum_{r=0}^{2} r \langle \tau^r \beta_i, \beta_j \rangle = \begin{cases} \pm 6 & \text{if } \tau \beta_i \neq \beta_j \\ 0 & \text{if } \tau \beta_i = \beta_j. \end{cases} \]

Hence \( c_0^\tau(\alpha, \beta) = 0 \) for all \( \alpha, \beta \in L \). This means that the central extension

\[ 1 \rightarrow \langle \kappa_6 \rangle \rightarrow \hat{L}_\tau \rightarrow L \rightarrow 1 \quad (3.1) \]

determined by the commutator condition \( aba^{-1}b^{-1} = \kappa_6^{c_0^\tau(a,b)} \) splits.

We consider the relation between two central extensions \( \hat{L} \) of (2.1) and \( \hat{L}_\tau \) of (3.1). Since both of \( \hat{L} \) and \( \hat{L}_\tau \) are split extensions, we use the same symbol \( e^\alpha \) to denote both of an element in \( \hat{L} \) and an element in \( \hat{L}_\tau \) which correspond naturally to \( \alpha \in L \). Actually, in Section 2 we choose \( e^\alpha \in \hat{L} \) so that the multiplication in \( \hat{L} \) is \( e^\alpha \times e^\beta = e^{\alpha+\beta} \). Also we can choose \( e^\alpha \in \hat{L}_\tau \) such that the multiplication \( e^\alpha \times e^\beta \in \hat{L}_\tau \) is related to the multiplication in \( \hat{L} \) by (cf. [2, (2.4)])

\[ e^\alpha \times e^\beta = \kappa_6^{c_0(\alpha,\beta)} e^\alpha \times e^\beta, \quad (3.2) \]
where the $\mathbb{Z}$-linear map $\varepsilon_0 : L \times L \to \mathbb{Z}/6\mathbb{Z}$ is defined by [2, (2.13)]. In our case
\[
\varepsilon_0(\alpha, \beta) = -\langle \tau^{-1}\alpha, \beta \rangle + 6\mathbb{Z}. \tag{3.3}
\]

As in Section 2, we usually write $e^\alpha e^\beta = e^{\alpha + \beta}$ to denote the product of $e^\alpha$ and $e^\beta$ in $\hat{L}$. The automorphism $\tau$ of $L$ lifts to an automorphism $\hat{\tau}$ of $\hat{L}$ such that $\hat{\tau}(e^\alpha) = e^{\tau\alpha}$ and $\hat{\tau}(\kappa_6) = \kappa_6$. Since $\varepsilon_0$ is $\tau$-invariant, we can also think $\hat{\tau}$ to be an automorphism of $\hat{L}_r$ in a similar way. By abuse of notation we denote $\tilde{\tau}$ by simply $\tau$ also.

We have $(1 - \tau)L = \text{span}_\mathbb{Z}\{\beta_1 - \beta_2, \beta_1 + 2\beta_2\}$. The quotient group $L/(1 - \tau)L$ is of order 3 and generated by $\beta_1 + (1 - \tau)L$. Now $K = \{a^{-1}\tau(a) \mid a \in \hat{L}_r\}$ is a central subgroup of $\hat{L}_r$ with $\overline{K} = (1 - \tau)L$ and $K \cap (\kappa_6) = 1$. Here note that $a^{-1}$ is the inverse of $a$ in $\hat{L}_r$ and $a^{-1}\tau(a)$ is the product $a^{-1} \times_{\tau} \tau(a)$ in $\hat{L}_r$. In $\hat{L}_r$ we can verify that
\[
eq e^{3\beta_1} = (e^{\beta_0 - \beta_1})^{-1} \times_{\tau} \tau(e^{\beta_0 - \beta_1}) \in K.
\]

Since $\kappa_3 e^{\beta_1} \times_{\tau} \kappa_3 e^{\beta_1} \times_{\tau} \kappa_3 e^{-\beta_1} = e^{3\beta_1}$ and $\kappa_3 e^{\beta_1} \times_{\tau} \kappa_3 e^{-\beta_1} = 1$, it follows that
\[
\hat{L}_r/K = \{K, \kappa_3 e^{\beta_1}K, \kappa_3 e^{-\beta_1}K\} \times (\kappa_6)/K \cong \mathbb{Z}_3 \times \mathbb{Z}_6.
\]

For $j = 0, 1, 2$, define a linear character $\chi_j : \hat{L}_r/K \to \mathbb{C}^\times$ by
\[
\chi_j(\kappa_6) = \xi_6, \quad \chi_j(\kappa_3 e^{\beta_1}K) = \xi_j, \quad \text{and} \quad \chi_j(\kappa_3 e^{-\beta_1}K) = \xi^{-j},
\]

where $\xi_6 = \exp(2\pi \sqrt{-1}/6)$. Let $T_{x_j}$ be the one dimensional $\hat{L}_r/K$-module affording the character $\chi_j$. As an $\hat{L}_r$-module, $K$ acts trivially on $T_{x_j}$. Since $\sum_{r=0}^{2} \tau^r \alpha = 0$ for $\alpha \in L$, those $T_{x_j}$, $j = 0, 1, 2$, are the irreducible $\hat{L}_r$-modules constructed in [13, Section 6].

Let
\[
V_L^{T_{x_j}} = V_L^{T_{x_j}}(\tau) = S[\tau] \otimes T_{x_j}
\]
and define the $\tau$-twisted vertex operator $Y^\tau(\cdot, z) : V_L \to \text{End}(V_L^{T_{x_j}}(\tau))\{z\}$ as in [2]. We extend the action of $\tau$ to $V_L^{T_{x_j}}(\tau)$ so that $\tau$ is the identity on $T_{x_j}$. The weight of every element in $T_{x_j}$ is defined to be 0.

By [2, Theorem 7.1], $(V_L^{T_{x_j}}(\tau), Y^\tau(\cdot, z))$, $j = 0, 1, 2$ are inequivalent irreducible $\tau$-twisted $V_L$-modules. Now among the 12 irreducible $V_L$-modules $V_{L_{(i,j)}}$, $i \in \{0, a, b, c\}$ and $j \in \{0, 1, 2\}$, the $\tau$-stable irreducible modules are $V_{L_{(0,j)}}$, $j \in \{0, 1, 2\}$. Hence by [3, Theorem 10.2], $(V_L^{T_{x_j}}(\tau), Y^\tau(\cdot, z))$, $j = 0, 1, 2$, are all the inequivalent irreducible $\tau$-twisted $V_L$-modules. The isometry $\theta$ of $(L, \langle \cdot, \cdot \rangle)$ induces a permutation on $V_L^{T_{x_j}}(\tau)$, $j = 0, 1, 2$. In fact, the permutation leaves $V_L^{T_{x_0}}(\tau)$ invariant and interchanges $V_L^{T_{x_1}}(\tau)$ and $V_L^{T_{x_2}}(\tau)$. Since $M^\tau \otimes M_0^\tau$ is contained in the subalgebra $(V_L^\tau)^\tau$ of fixed points of $\tau$ in $V_L$, we can deal with $(V_L^{T_{x_j}}(\tau), Y^\tau(\cdot, z))$ as an $M^\tau \otimes M_0^\tau$-module.
The decomposition of $V_{L}^{T_{X_{0}}}(\tau)$ as a $\tau$-twisted $M \otimes M_{t}^{0}$-module was studied in [10]. Set

\begin{align*}
M_{T_{0}}^{0}(\tau) &= \{ u \in V_{L}^{T_{X_{0}}}(\tau) \mid (\omega^{3})_{1}u = 0 \}, \\
W_{T_{0}}^{0}(\tau) &= \{ u \in V_{L}^{T_{X_{0}}}(\tau) \mid (\omega^{3})_{1}u = \frac{2}{5}u \}, \\
M_{T_{j}}^{j}(\tau) &= \{ u \in V_{L}^{T_{X_{j}}}(\tau) \mid (\omega^{3})_{1}u = \frac{2}{3}u \}, \quad j = 1, 2, \\
W_{T_{j}}^{j}(\tau) &= \{ u \in V_{L}^{T_{X_{j}}}(\tau) \mid (\omega^{3})_{1}u = \frac{1}{15}u \}, \quad j = 1, 2.
\end{align*}

Then, by [10, Proposition 6.8], $M_{T_{j}}^{j}(\tau)$ and $W_{T_{j}}^{j}(\tau)$, $j = 0, 1, 2$, are irreducible $\tau$-twisted $M$-modules. Furthermore, for $j = 0, 1, 2$,

\begin{align*}
V_{L}^{T_{X_{0}}}(\tau) &\cong M_{T_{0}}^{0}(\tau) \oplus W_{T_{0}}^{0}(\tau), \\
V_{L}^{T_{X_{1}}}(\tau) &\cong M_{T_{1}}^{1}(\tau) \oplus W_{T_{1}}^{1}(\tau), \\
V_{L}^{T_{X_{2}}}(\tau) &\cong M_{T_{2}}^{2}(\tau) \oplus W_{T_{2}}^{2}(\tau).
\end{align*}

There are at most two inequivalent irreducible $\tau$-twisted $M$-modules by Lemma 4.1 and [3, Theorem 10.2]. Then, looking at the smallest weight of $M_{T_{j}}^{j}(\tau)$ and $W_{T_{j}}^{j}(\tau)$, we have that $M_{T_{0}}^{0}(\tau) \cong M_{T_{1}}^{1}(\tau) \cong M_{T_{2}}^{2}(\tau)$ and $W_{T_{0}}^{0}(\tau) \cong W_{T_{1}}^{1}(\tau) \cong W_{T_{2}}^{2}(\tau)$ and that $M_{T_{j}}^{j}(\tau) \not\cong W_{T_{j}}^{j}(\tau)$ as $\tau$-twisted $M$-modules. We denote $M_{T_{j}}^{j}(\tau)$ by $M_{T}(\tau)$ and $W_{T_{j}}^{j}(\tau)$ by $W_{T}(\tau)$. We conclude that there are exactly two inequivalent irreducible $\tau$-twisted $M$-modules, which are represented by $M_{T}(\tau)$ and $W_{T}(\tau)$. As $\tau$-twisted $M \otimes \text{Vir}(\omega^{3})$-modules, we have

\begin{align*}
V_{L}^{T_{X_{0}}}(\tau) &\cong M_{T}(\tau) \otimes \left( L(\frac{4}{5}, 0) + L(\frac{4}{5}, 3) \right) \oplus W_{T}(\tau) \otimes \left( L(\frac{4}{5}, \frac{2}{5}) + L(\frac{4}{5}, \frac{7}{5}) \right), \\
V_{L}^{T_{X_{1}}}(\tau) &\cong M_{T}(\tau) \otimes L(\frac{4}{5}, \frac{2}{3}) \oplus W_{T}(\tau) \otimes L(\frac{4}{5}, \frac{1}{15}).
\end{align*}

For $\epsilon = 0, 1, 2$, let

\begin{align*}
M_{T}(\tau)(\epsilon) &= \{ u \in M_{T}(\tau) \mid \tau u = \xi^{\epsilon}u \}, \\
W_{T}(\tau)(\epsilon) &= \{ u \in W_{T}(\tau) \mid \tau u = \xi^{\epsilon}u \}.
\end{align*}

Those 6 modules for $M_{T}$ are inequivalent irreducible modules by [14, Theorem 2]. Their top levels are of dimension one. Those top levels and the eigenvalues for the action of $L^{\tau}(0) = \omega_{1}$ and $J^{\tau}(0) = J_{2}$ are collected in Table 2.

By a similar argument we obtain 6 inequivalent irreducible $M_{T}$-modules inside $\tau^{2}$-twisted $M$-modules. The results are collected in Table 3.

We have obtained 20 irreducible $M_{T}$-modules. Now we turn to the vertex operator algebra $\mathcal{W}$ generated by $\omega$ and $J$ in $M_{T}$. Note that

\begin{align*}
L(n)1 &= 0 \quad \text{for} \quad n \geq -1, \\
J(n)1 &= 0 \quad \text{for} \quad n \geq -2,
\end{align*}

\begin{align*}
L(-2)1 &= \omega, \\
J(-3)1 &= J.
\end{align*}

Then by using commutation relations (2.2), (2.3), and (2.4), we see that
Lemma 3.2 \( \mathcal{W} \) is spanned by the vectors of the form

\[
L(-m_1) \cdots L(-m_p)J(-n_1) \cdots J(-n_q)1
\]  

with \( m_1 \geq \cdots \geq m_p \geq 2, n_1 \geq \cdots \geq n_q \geq 3, p = 0, 1, 2, \ldots, \) and \( q = 0, 1, 2, \ldots \).

A vector \( v \in \mathcal{W} \) of weight \( h \) is called a singular vector for \( \mathcal{W} \) if it satisfies

1. \( L(0)v = hv \),
2. \( L(n)v = 0 \) and \( J(n)v = 0 \) for \( n \geq 1 \).

Note that \( v \) is not necessarily an eigenvector for \( J(0) \). By commutation relations (2.2) and (2.3), it is easy to show that the condition (2) holds if \( v \) satisfies

\[
(2)' \quad L(1)v = L(2)v = J(1)v = 0.
\]

We consider \( \mathcal{W} \) as a space spanned by the vectors of the form (3.8). The weight of such a vector is \( m_1 + \cdots + m_p + n_1 + \cdots + n_q \). Using a computer algebra system Risa/Asir we have the following lemma.

Lemma 3.3 Let \( v \) be a linear combination of the vectors of the form (3.8) of weight \( h \). Under the conditions (3.6) and (3.7) and the commutation relations (2.2), (2.3), and (2.4), we have \( L(1)v = L(2)v = J(1)v = 0 \) only if \( v = 0 \) in the case \( h \leq 11 \). In the case \( h = 12 \), there exists a unique, up to scalar multiple, linear combination \( v^{12} \) which satisfies \( L(1)v^{12} = L(2)v^{12} = J(1)v^{12} = 0 \). The explicit form of \( v^{12} \) is given in Appendix. We also have \( J(0)v^{12} = 0 \).
By a general theory of lattice vertex operator algebras, \( V_L \) possesses an invariant positive definite hermitian form. From this it follows that

**Proposition 3.4** The singular vector \( v^{12} = 0 \).

Using the explicit form of \( v^{12} \), \( J(-1)v^{12} \), \( J(-2)v^{12} \), and \( J(-1)v^{12} \), we can determine the Zhu algebra \( Z(W) \) of \( W \). The standard reference to Zhu's theory is [15, Section 2]. The Zhu algebra \( Z(W) \) is a quotient vector space \( A(W) = W/O(W) \) equipped with a commutative associative algebra structure with respect to an operation \(*\). We denote the image of a vector \( v \in W \) in \( A(W) = W/O(W) \) by \( [v] \). We can define an algebra homomorphism from \( \mathbb{C}[x, y] \) onto \( A(W) \) by \( x \mapsto [\omega] \) and \( y \mapsto [J] \). The primary decomposition of its kernel \( I \) is such that \( I = \bigcap_{i=1}^{20} P_i \), where \( P_i \), \( 1 \leq i \leq 20 \) are

\[
\begin{align*}
\langle x, y \rangle, & \quad \langle 5x - 8, y \rangle, \\
\langle 2x - 1, y \rangle, & \quad \langle 10x - 1, y \rangle, \\
\langle x - 2, y - 12\sqrt{-3} \rangle, & \quad \langle x - 2, y + 12\sqrt{-3} \rangle, \\
\langle 5x - 3, y - 2\sqrt{-3} \rangle, & \quad \langle 5x - 3, y + 2\sqrt{-3} \rangle, \\
\langle 9x - 1, 81y - 14\sqrt{-3} \rangle, & \quad \langle 9x - 1, 81y + 14\sqrt{-3} \rangle, \\
\langle 9x - 7, 81y - 238\sqrt{-3} \rangle, & \quad \langle 9x - 7, 81y + 238\sqrt{-3} \rangle, \\
\langle 9x - 13, 81y - 374\sqrt{-3} \rangle, & \quad \langle 9x - 13, 81y + 374\sqrt{-3} \rangle, \\
\langle 45x - 2, 81y - 4\sqrt{-3} \rangle, & \quad \langle 45x - 2, 81y + 4\sqrt{-3} \rangle, \\
\langle 45x - 17, 81y - 22\sqrt{-3} \rangle, & \quad \langle 45x - 17, 81y + 22\sqrt{-3} \rangle, \\
\langle 45x - 32, 81y - 176\sqrt{-3} \rangle, & \quad \langle 45x - 32, 81y + 176\sqrt{-3} \rangle.
\end{align*}
\]

These primary ideals correspond to the 20 irreducible \( M^\tau \)-modules listed in Tables 1, 2, and 3. The correspondence is given by substituting \( x \) and \( y \) with the eigenvalues for \( L(0) \) and \( J(0) \) on the top levels of 20 irreducible modules. The eigenvalues are the zeros of those primary ideals.

By Zhu's theory we obtain the classification of irreducible \( W \)-modules, namely,

**Theorem 3.5**

1. \( M^\tau = W \).
2. \( A(M^\tau) \cong \bigoplus_{i=1}^{20} \mathbb{C}[x, y]/P_i \) is a 20-dimensional commutative associative algebra.
3. There are exactly 20 inequivalent irreducible \( M^\tau \)-modules. Their representatives are listed in Tables 1, 2, and 3 in Section 2, namely, \( M(\epsilon), W(\epsilon), M_\epsilon^k, W_\epsilon^k, M_T(\tau^i)(\epsilon), \) and \( W_T(\tau^i)(\epsilon) \) for \( \epsilon = 0, 1, 2 \) and \( i = 1, 2 \).
\[ v^{12} = -(5877264800/3501)L(-12)1 + (3404072000/3501)L(-10)L(-2)1 \\
- (2653990000/3501)L(-9)L(-3)1 - (2663768000/3501)L(-8)L(-4)1 \\
+ (282988000/1167)L(-8)L(-2)^21 - (23744800/1167)L(-7)L(-5)1 \\
- (3082400/1167)L(-7)L(-3)L(-2)1 + (1242377600/1167)L(-6)L(-2)1 \\
+ (61947200/3501)L(-6)L(-4)L(-2)1 - (1313806000/1167)L(-6)L(-3)^21 \\
+ (45496000/1167)L(-6)L(-2)^31 - (304678400/3501)L(-5)L(-2)1 \\
-(299424800/1167)L(-5)L(-4)L(-3)1 + (2347094000/3501)L(-5)L(-3)L(-2)^21 \\
- (17280400/1167)L(-4)^31 - (2036373200/3501)L(-4)L(-2)^21 \\
+ (82996000/3501)L(-4)L(-3)^2L(-2)1 + (1074512000/3501)L(-4)L(-2)^41 \\
+ (511628125/3501)L(-3)^41 - (418850000/3501)L(-3)L(-2)^31 \\
- (59680000/3501)L(-2)^61 - (505200/389)L(-6)L(-3)^21 \\
+ (3380480/1167)L(-4)L(-2)J(-3)^21 + 1150L(-3)^2J(-3)^21 \\
- (164400/1167)L(-2)^3J(-3)^21 + (3788680/1167)L(-5)L(-4)J(-3)1 \\
- (8788400/3501)L(-3)L(-2)L(-4)J(-3)1 - (12761440/3501)L(-4)L(-5)L(-3)1 \\
- (5727500/10503)L(-4)L(-5)L(-3)^21 + (352400/389)L(-2)^2J(-5)L(-3)1 \\
+ (5727500/10503)L(-2)^2J(-4)^21 + (1593900/389)L(-3)L(-6)J(-3)1 \\
+ (12935800/10503)L(-3)L(-5)L(-4)J(-3)1 + (4108000/3501)L(-2)L(-7)L(-3)1 \\
- (2811800/1167)L(-2)L(-6)L(-4)1 - (3131600/10503)L(-2)L(-5)^21 \\
- (14904160/3501)L(-9)L(-3)1 + (32677600/10503)L(-8)L(-4)1 \\
+ (9423200/10503)L(-7)L(-5)L(-4)1 + (2432375/1167)L(-6)^21 \\
+ J(-3)^41. \]
References


