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Kyoto University
LATTICE VERTEX OPERATOR ALGEBRA $V_{\sqrt{2}E_8}$ AND AN ALGEBRA OF MIYAMOTO OF CENTRAL CHARGE $\frac{1}{2} + \frac{21}{22}$

CHING HUNG LAM*

ABSTRACT. Motivated by a work of Miyamoto [17], we construct a vertex operator algebra $U$ of central charge $\frac{1}{2} + \frac{21}{22}$ which has the full automorphism group isomorphic to the symmetry group $S_3$. Actually, we show that the lattice vertex operator algebra $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to a tensor product of unitary Virasoro vertex operator algebras $\mathcal{T} = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{11}{12}, 0) \otimes L(\frac{14}{15}, 0) \otimes L(\frac{21}{22}, 0) \otimes L(\frac{1}{2}, 0)$ and $U$ is a certain coset subalgebra of $V_{\sqrt{2}E_8}$. We also show that $U$ contains exactly 3 conformal vectors of central charge $1/2$ and the inner product between any two of them is $1/2^8$.

1. INTRODUCTION

This work is motivated by a recent article of Miyamoto [17]. In [17], Miyamoto studied a class of vertex operator algebra (VOA) generated by two rational conformal vectors $e$ and $f$ of central charge $1/2$. Among other things, he showed that if the inner product $\langle e, f \rangle$ is equal to $\frac{1}{2^8}$, then the vertex operator algebra $U$ generated by $e$ and $f$ is of central charge $16/11$ and $U$ contains a subalgebra isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$. Moreover, $\dim U_2 = 3$ and the full automorphism group of $U$ is isomorphic to the symmetry group $S_3$. In this paper, we shall construct explicitly a VOA

$$U \cong L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 8) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 45) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{175}{16}) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{175}{16})$$

in the lattice VOA $V_{\sqrt{2}E_8}$ and show that $U$ satisfies all the properties mentioned in [17]. In fact, we shall show that the lattice VOA $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to a tensor product of the unitary Virasoro VOAs

$$\mathcal{T} = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{11}{12}, 0) \otimes L(\frac{14}{15}, 0) \otimes L(\frac{21}{22}, 0) \otimes L(\frac{1}{2}, 0),$$

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and obtain a complete decomposition of $V_{\sqrt{2}E_8}$ into a direct sum of irreducible $\mathfrak{z}$-modules. The VOA $U$ is actually a certain commutant (or coset) subalgebra associated with the above decomposition. We also notice that an automorphism of order $3$ obtained from the abelian group $\sqrt{2}E_8/\sqrt{2}A_8$ induces a natural $\mathbb{Z}_3$-action on $U$. This action together with the usual involution $\theta$ induced by $-1$ will form a group $S_3$ inside the automorphism group of $U$. In addition, we determine all conformal vectors of central charge $1/2$ inside $U$ and show that the inner of any two of them is $1/2^8$ as mentioned by Miyamoto.

2. Lattice vertex operator algebra $V_{\sqrt{2}E_8}$

2.1. The lattice $\sqrt{2}E_8$. Let $\alpha^0, \ldots, \alpha^8$ be vectors in $\mathbb{R}^9$ such that $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$ for any $i, j = 0, \ldots, 8$ and $L = \mathbb{Z}\alpha^0 \oplus \mathbb{Z}\alpha^1 \oplus \cdots \oplus \mathbb{Z}\alpha^8$. Then $L$ is isomorphic to the orthogonal sum of $9$ copies of the root lattice $A_1$. Let $\beta_i = -\alpha_{i-1} + \alpha_i$, $i = 1, \ldots, 8$. Then $N = \text{span}_\mathbb{Z}\{\beta_1, \ldots, \beta_8\}$ is isomorphic to the lattice $\sqrt{2}A_8$. Let

$$\gamma = \frac{1}{3} (2\alpha^0 + 2\alpha^1 + 2\alpha^2 - \alpha^3 - \alpha^4 - \alpha^5 - \alpha^6 - \alpha^7 - \alpha^8).$$

Then $\gamma$ belongs to the dual lattice $N^* = \{x \in \mathbb{Q} \otimes \mathbb{Z} N | \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in N\}$ of $N$ and the lattice $K$ generated by $\gamma$ and $N$ is of rank $8$. Moreover, we have

Lemma 2.1. $K \cong \sqrt{2}E_8$

Proof. First, we shall note that $\langle \gamma, \gamma \rangle = 4$ and $K = \langle \gamma, N \rangle = N \cup \langle \gamma + N \rangle \cup \langle -\gamma + N \rangle$. Moreover, $K/N \cong \mathbb{Z}_3$ as an abelian group.

Let $\theta_i = \frac{1}{\sqrt{2}} \beta_i = \frac{1}{\sqrt{2}} (-\alpha_{i-1} + \alpha_i)$ for $i = 1, \ldots, 7$ and $\theta_8 = \frac{1}{\sqrt{2}} \gamma$. Then

$$\langle \theta_i, \theta_i \rangle = 2 \quad \text{for } i = 1, \ldots, 8,$$

$$\langle \theta_{i-1}, \theta_i \rangle = -1 \quad \text{for } i = 2, \ldots, 7,$$

$$\langle \theta_3, \theta_8 \rangle = -1,$$

$$\langle \theta_i, \theta_j \rangle = 0 \quad \text{for all other } 1 \leq i, j \leq 8.$$

In other words, $\{\theta_1, \ldots, \theta_8\}$ is a set of simple roots of the root lattice $E_8$ and hence

$$K \supset \text{span}_\mathbb{Z}\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \gamma\} \cong \sqrt{2}E_8.$$

Since $|K/N| = 3 = |\sqrt{2}E_8/\sqrt{2}A_8|$, $K = \text{span}_\mathbb{Z}\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \gamma\} \cong \sqrt{2}E_8$. 

Hence we also know that the vertex operator algebra

$$V_{\sqrt{2}E_8} \cong V_K = V_N \oplus V_{\gamma+N} \oplus V_{-\gamma+N}.$$
2.2. Conformal vectors in $V_{\sqrt{2}E_8}$. In this section, we shall study some conformal vectors in $V_{\sqrt{2}E_8}$. We shall show that the Virasoro element of the VOA $V_{\sqrt{2}E_8}$ can be decomposed into a sum of 10 mutually orthogonal conformal vectors $\tilde{\omega}^1, \ldots, \tilde{\omega}^{10}$ and the central charge of $c(\tilde{\omega}^i)$ of $\tilde{\omega}^i$ are given by

$$c(\tilde{\omega}^i) = \begin{cases} 1 - \frac{6}{(i+2)(i+3)} & \text{for } 1 \leq i \leq 8, \\ \frac{1}{2} & \text{and } c(\tilde{\omega}^{10}) = \frac{21}{22}. \end{cases}$$

First, let us recall a construction of certain conformal vectors in $V_{\sqrt{2}A_l}$ from Dong et al.[4]. Let $\Phi$ be the root system of $A_l$ and $\Phi^+$ and $\Phi^-$ the set of all positive roots and negative roots, respectively. Then

$$\Phi = \Phi^+ \cup \Phi^- = \Phi^+ \cup (-\Phi^+).$$

Consider a chain of root systems

$$\Phi = \Phi_l \supset \Phi_{l-1} \supset \cdots \supset \Phi_1$$

such that $\Phi_i$ is a root system of type $A_i$. For any $i = 1, 2, \ldots, l$, define

$$s^i = \frac{1}{2(i+3)} \sum_{\alpha \in \Phi^+_i} \alpha (-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})$$

and

$$\omega = \frac{1}{2(l+1)} \sum_{\alpha \in \Phi^+_l} \alpha (-1)^2 \cdot 1.$$

It was shown by Dong et al. [4] that the elements

$$\omega^1 = s^1, \quad \omega^i = s^i - s^{i-1}, \quad 2 \leq i \leq l, \quad \omega^{l+1} = \omega - s^l$$

are mutually orthogonal conformal vectors in $V_{\sqrt{2}A_l}$. The subalgebra $\text{Vir}(\omega^i)$ of the vertex operator algebra $V_{\sqrt{2}A_l}$ generated by $\omega^i$ is isomorphic to the Virasoro vertex operator algebra $L(c(\omega^i), 0)$ which is the irreducible highest weight module for the Virasoro algebra with central charge $c(\omega^i)$ and highest weight 0 and the central charge $c(\omega^i)$ of $\omega^i$ are given by

$$c(\omega^i) = 1 - \frac{6}{(i+2)(i+3)} \quad \text{for } 1 \leq i \leq l \quad \text{and} \quad c(\omega^{l+1}) = \frac{2l}{l+3}.$$

Since $\omega^1, \omega^2, \ldots, \omega^{l+1}$ are mutually orthogonal, the subalgebra $T$ of $V_{\sqrt{2}A_l}$ generated by these conformal vectors is a tensor product of $\text{Vir}(\omega^i)$’s, namely,

$$T = \text{Vir}(\omega^1) \otimes \cdots \otimes \text{Vir}(\omega^{l+1})$$

$$\cong L(c(\omega^1), 0) \otimes \cdots \otimes L(c(\omega^{l+1}), 0).$$

Moreover, $V_{\sqrt{2}A_l}$ is completely reducible as a $T$-module.
For $l = 8$, there are 9 mutually orthogonal conformal vectors $\omega^1, \ldots, \omega^9$ in $V_{\sqrt{2}A_8}$ and the central charge of $\omega^1, \ldots, \omega^9$ are $\frac{1}{2}, \frac{7}{10}, \frac{6}{7}, \frac{25}{28}, \frac{11}{12}, \frac{14}{15}, \frac{28}{55}, \frac{16}{11}, \frac{6}{7}$, respectively. In other words, $V_{\sqrt{2}A_8}$ contains a subalgebra isomorphic to

$$T = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{3}, 0) \otimes L(\frac{6}{7}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{11}{12}, 0) \otimes L(\frac{14}{15}, 0) \otimes L(\frac{52}{55}, 0) \otimes L(\frac{16}{11}, 0)$$

The following lemma can be obtained by direct calculation.

**Lemma 2.2.** Let $\gamma$ be defined as in (2.1) and let

$$a^1 = \sum_{\alpha \in (\gamma + \sqrt{2}A_8), \langle \alpha, \alpha \rangle = 4} e^\alpha \in V_{\gamma + \sqrt{2}A_8} \quad \text{and}$$

$$a^2 = \sum_{\alpha \in (-\gamma + \sqrt{2}A_8), \langle \alpha, \alpha \rangle = 4} e^\alpha \in V_{-\gamma + \sqrt{2}A_8}.$$

Then $a^1$ and $a^2$ are both highest weight vectors of weight $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2)$ with respect to the action of $T$.

**Lemma 2.3.** Let $u = a^1 + a^2 = \sum_{\alpha \in (\gamma + \sqrt{2}A_8), \langle \alpha, \alpha \rangle = 4} (e^\alpha + e^{-\alpha})$. Then

$$\tilde{\omega}^9 = \frac{11}{32} \omega^9 + \frac{1}{32} u \quad \text{and} \quad \tilde{\omega}^{10} = \frac{21}{32} \omega^9 - \frac{1}{32} u$$

are mutually orthogonal conformal vectors of central charge $1/2$ and $21/22$, respectively. Moreover, they are orthogonal to $\omega^1, \ldots, \omega^8$.

**Proof.** First, we shall note that for any $\alpha, \beta$ with square norm 4,

$$(e^\alpha)_1 e^\beta = \begin{cases} e^{\alpha + \beta} & \text{if } \langle \alpha, \beta \rangle = -2 \\ \alpha(-1)^2 & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

and

$$\langle e^\alpha, e^\beta \rangle = (e^\alpha)_3 e^\beta = \begin{cases} 1 & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

Then by direct computation, we have

$$u_1 u = 2(231 \omega^9 + 10 u), \quad \omega_1^9 \omega^9 = 2 \omega^9 \quad \text{and} \quad \omega_1^9 u = 2 u.$$

Now, it is easy to verify that both $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are conformal vectors.
Since $\sqrt{2}A_8$ has exactly 72 vectors of square norm 4 and $\gamma + \sqrt{2}A_8$ and $-\gamma + \sqrt{2}A_8$ each has 84 vectors of square norm 4, we also have
\[
\langle \omega^9, \omega^9 \rangle = \frac{8}{11}, \quad \langle \omega^9, u \rangle = 0, \quad \text{and} \quad \langle u, u \rangle = 168.
\] (2.5)

Therefore,
\[
\langle \tilde{\omega}^9, \tilde{\omega}^9 \rangle = \frac{1}{4}, \quad \langle \tilde{\omega}^9, \tilde{\omega}^{10} \rangle = 0, \quad \text{and} \quad \langle \tilde{\omega}^{10}, \tilde{\omega}^{10} \rangle = \frac{21}{44}
\]
and hence $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are mutually orthogonal conformal vectors of central charge $1/2$ and $21/22$. By the definition, it is also clear that $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are orthogonal to $\{\omega^1, \ldots, \omega^8\}$ as $\omega^9$ and $u$ are orthogonal to $\{\omega^1, \ldots, \omega^8\}$.

As a corollary, we have

**Corollary 2.4.** The lattice VOA $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to
\[
\mathfrak{T} = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \otimes L(\frac{25}{28}, 0) \\
\otimes L(\frac{11}{12}, 0) \otimes L(\frac{14}{15}, 0) \otimes L(\frac{52}{55}, 0) \otimes L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0),
\]

where $W(0)$ is a simple VOA, known as parafermion algebra or $W$-algebra, of central charge $16/11$ and $W(k), k = 0, 2, 4, 6, 8,$ are irreducible $W(0)$-modules.

Since $V_{\gamma + \sqrt{2}A_8}$ and $V_{-\gamma + \sqrt{2}A_8}$ are irreducible $V_{\sqrt{2}A_8}$-modules and both of them contain highest weight vectors of weight $(0, 0, 0, 0, 0, 0, 0, 0, 2)$ with respect to $T$, we also have
where $P(k_l)$ and $Q(k_l)$ are irreducible $W(0)$-modules whose structure are yet to be determined.

Now, let $U = U(0) = \{V \in V_{\sqrt{2}E_8} | (\overline{w}^{i})_1 v = 0 \text{ for } i = 1, 2, \ldots, 8\}$. Then, $U$ is a VOA of central charge $16/11$ and by combining Corollary 2.4 and (2.6–2.8), we have

**Theorem 2.7.** The lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as

$$V_{\sqrt{2}E_8} \cong \bigoplus_{0 \leq k_l \leq l+1, \atop k_j \equiv 0 \mod 2} L(c_{\frac{1}{2}}, h_{k_0+1,k_1+1}^{1}) \otimes \cdots L(c_{l}, h_{k_{l-7}+1,k_{l-6}+1}^{l}) \otimes U(k_{l-8})$$

(2.9)

where $U(k) = W(k) + P(k) + Q(k)$, $k = 0, 2, 4, 6, 8$, are $U(0)$–modules.

**Remark 2.8.** Let $\sigma$ be an automorphism of $V_{\sqrt{2}E_8}$ defined by

$$\sigma(u) = e^{\frac{2\pi}{3} \langle \gamma, \beta \rangle} \quad \text{for any } u \in M(1) \otimes e^\beta \subset V_{\sqrt{2}E_8}.$$

and let $\theta$ be an automorphism of $V_{\sqrt{2}E_8}$ induces by the isometry $\beta \rightarrow -\beta$ of $\sqrt{2}E_8$. Then the subgroup generated by $\sigma$ and $\theta$ is isomorphic to $S_3$. Moreover, $\sigma$ and $\theta$ induce some nontrivial automorphisms of order 3 and order 2 on the subVOA $U(0)$ respectively. In fact, they induce automorphisms of order 3 and order 2 on the submodules $U(k)$, $k = 0, 2, 4, 6, 8$, also. By abuse of notation, we shall still denote them by $\sigma$ and $\theta$.

Note also that the automorphism $\sigma$ is in fact induced from the order 3 symmetry among the 3 cosets of $\sqrt{2}A_8$ in $\sqrt{2}E_8$.

Next let us determine the structure of $U(0)$. Since $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$ is rational and contained in $U(0)$, $U(0)$ and $U(k)$, $k = 2, 4, 6, 8$, are direct sum of irreducible $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$-modules. On the other hand,

$$L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0), \quad L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 8), \quad L(\frac{1}{2}, \frac{7}{2}) \otimes L(\frac{21}{22}, \frac{7}{2}),$$

$$L(\frac{1}{2}, \frac{45}{2}) \otimes L(\frac{21}{22}, 2), \quad L(\frac{1}{2}, \frac{3}{16}) \otimes L(\frac{21}{22}, \frac{3}{16}), \quad \text{and } L(\frac{1}{2}, \frac{175}{16}) \otimes L(\frac{21}{22}, 175)$$
are the only irreducible modules of $L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 0\right)$ which have integral weights. Hence,

\[
U(0) = A_1 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 0\right) \oplus A_2 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 8\right) \oplus A_3 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{7}{2}\right) \oplus A_4 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{45}{2}\right) \oplus A_5 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{31}{16}\right) \oplus A_6 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{175}{16}\right),
\]

where $A_1, \ldots, A_6$ are the multiplicities of the irreducible summands. Similarly, we also have

\[
U(2) = B_1 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{13}{11}\right) \oplus B_2 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{35}{11}\right) \oplus B_3 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{15}{22}\right) \oplus B_4 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{91}{22}\right) \oplus B_5 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{21}{176}\right) \oplus B_6 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{175}{176}\right),
\]

\[
U(4) = C_1 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{6}{11}\right) \oplus C_2 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{5}{11}\right) \oplus C_3 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{155}{22}\right) \oplus C_4 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{261}{22}\right) \oplus C_5 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{85}{176}\right) \oplus C_6 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{261}{176}\right),
\]

\[
U(6) = D_1 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{11}{11}\right) \oplus D_2 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{11}{11}\right) \oplus D_3 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{35}{22}\right) \oplus D_4 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{57}{22}\right) \oplus D_5 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{533}{176}\right) \oplus D_6 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{5}{176}\right),
\]

and

\[
U(8) = E_1 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{196}{11}\right) \oplus E_2 L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{20}{11}\right) \oplus E_3 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{117}{22}\right) \oplus E_4 L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{7}{22}\right) \oplus E_5 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{165}{176}\right) \oplus E_6 L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{133}{176}\right),
\]

for some suitable $B_i, C_i, D_i$ and $E_i$. Note that the weights of $U(2), U(4), U(6),$ and $U(8)$ are $2/11 + \mathbb{Z}, 6/11 + \mathbb{Z}, 1/11 + \mathbb{Z},$ and $9/11 + \mathbb{Z},$ respectively.

Now by comparing the characters of the left and the right hand sides of (2.9), we find that all $A_i$'s, $B_i$'s, $C_i$'s, $D_i$'s, and $E_i$'s are equal to 1.
Hence we have

\[
U(0) \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 0\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 8\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{7}{2}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{45}{2}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{31}{16}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{175}{16}\right),
\]

\[
U(2) \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{13}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{35}{11}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{15}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{301}{22}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{155}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{901}{16}\right).
\]

\[
U(4) \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{50}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{6}{11}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{1}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{155}{22}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{117}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{5}{22}\right),
\]

\[
U(6) \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{111}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{1}{11}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{35}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{57}{22}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{533}{176}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{5}{176}\right),
\]

and

\[
U(8) \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{196}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{20}{11}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{117}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{7}{22}\right) \\
\oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{1365}{176}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{133}{176}\right),
\]

**Theorem 2.9.** \(U\) is a simple VOA and \(U(k)\) for \(k = 0, 2, 4, 6, 8\) are irreducible \(U\)-modules.
Proof. Since

\[ U(0) = L\left( \frac{1}{2}, 0 \right) \otimes L\left( \frac{1}{2}, 0 \right) + L\left( \frac{1}{2}, 0 \right) \otimes L\left( \frac{1}{2}, \frac{7}{2} \right) + L\left( \frac{1}{2}, 0 \right) \otimes L\left( \frac{1}{2}, \frac{175}{16} \right) \]

as an \( L\left( \frac{1}{2}, 0 \right) \otimes L\left( \frac{1}{2}, 0 \right) \)-module, by the fusion rules, \( U \) is clearly simple.

Now, by the fusion rules and the decomposition, it is also clear that \( U(k) \) for \( k = 0, 2, 4, 6, 8 \), are irreducible as \( U \)-modules.

\[ \square \]

3. Conformal vectors in \( U \)

In this section, we shall compute all the conformal vectors in \( U \). First, we shall note that \( \dim U_2 = 3 \) and \( \{ \tilde{\omega} = \omega^9, u, v \} \) forms a basis of \( U_2 \).

**Theorem 3.1.** There are exactly 7 conformal vectors in \( U \), namely, the Virasoro element \( \tilde{\omega} \) of \( U \), 3 conformal vectors of central charge 1/2 and 3 conformal vectors of central charge 21/22.

**Proof.** First we shall note that \( U_2 \) is spanned by \( \{ \tilde{\omega}, u, v \} \). Let \( x = a\tilde{\omega} + bu + cv \) be a conformal vector in \( U_2 \). Then \( x_1x = 2x \). Since \( \tilde{\omega}_1\tilde{\omega} = 2\tilde{\omega}, \tilde{\omega}_1u = 2u, \tilde{\omega}_1v = 2v, u_1u = 2(231\tilde{\omega} + 10u), u_1v = -20v, \) and \( v_1v = 2(-231\tilde{\omega} + 10u) \), by direct computation, we know that

\[ a^2 + 231b^2 - 231c^2 = a, \]
\[ 2ab + 10b^2 + 10c^2 = b, \]
\[ 2ac - 20b^2 = c, \] (3.1)

Solving the above equations, we obtain 7 non-trivial solutions, namely,

\[ \{a = 1, b = 0, c = 0\}, \]
\[ \{a = \frac{11}{32}, b = \frac{1}{32}, c = 0\}, \]
\[ \{a = \frac{11}{32}, b = -\frac{1}{64}, c = \frac{\sqrt{3}}{64}\}, \]
\[ \{a = \frac{21}{32}, b = -\frac{1}{64}, c = \frac{-\sqrt{3}}{64}\}, \]
\[ \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{\sqrt{3}}{64}\}, \]
\[ \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{-\sqrt{3}}{64}\}. \]

When \( \{a = 1, b = 0, c = 0\}, x = \tilde{\omega} \) is the Virasoro element of \( U \).

When \( \{a = \frac{11}{32}, b = \frac{1}{32}, c = 0\}, \{a = \frac{11}{32}, b = \frac{1}{64}, c = \frac{\sqrt{3}}{64}\}, \) or \( \{a = \frac{1}{32}, b = \frac{1}{64}, c = \frac{-\sqrt{3}}{64}\} \), \( \langle x, x \rangle = 1/4 \) and \( x \) is a conformal vector of central charge 1/2.
When \( \{a = \frac{21}{32}, b = \frac{-1}{32}, c = 0\} \), \( \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{\sqrt{-3}}{64}\} \), or \( \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{-\sqrt{-3}}{64}\} \), \( \langle x, x\rangle = 21/44 \) and \( x \) is a conformal vector of central charge 21/22.

Lemma 3.2. Let \( e^1 = \frac{11}{32}w^9 + \frac{1}{32}u \), \( e^2 = \frac{11}{32}w^9 - \frac{1}{64}u + \frac{\sqrt{-3}}{64}v \), and \( e^3 = \frac{11}{32}w^9 - \frac{1}{64}u - \frac{\sqrt{-3}}{64}v \) be the three rational conformal vectors of central charge \( \frac{1}{2} \) in \( U \). Then \( \langle e^i, e^j \rangle = \frac{1}{2^8} \) if \( i \neq j \).

Proof. By (2.4), it is easy to show that
\[
\langle \omega^9, \omega^9 \rangle = \frac{8}{11}, \quad \langle u, u \rangle = 168, \quad \langle v, v \rangle = -168,
\]
and
\[
\langle \omega^9, u \rangle = \langle \omega^9, v \rangle = \langle u, v \rangle = 0.
\]
Thus, we have
\[
\langle e^i, e^j \rangle = \begin{cases} 1/2^8 & \text{if } i \neq j, \\ 1/4 & \text{if } i = j, \end{cases}
\]
as desired.

Theorem 3.3. Let \( U_2 \) be the Griess algebra of \( U \). Then \( \text{Aut } U_2 \cong S_3 \).

Proof. Let \( g \) be an element of \( \text{Aut } U_2 \). Then it will induce a permutation on the three conformal vectors \( e^1, e^2 \) and \( e^3 \). Since \( U_2 \) is generated by \( e^1, e^2 \) and \( e^3 \), \( \text{Aut } U_2 \) must itself a permutation subgroup on \( \{e^1, e^2, e^3\} \). On the other hand, by our construction, \( \text{Aut } U_2 \) already contains elements of order 3 and order 2, namely \( \sigma \) and \( \theta \). Thus \( \text{Aut } U_2 \cong S_3 \).

Theorem 3.4. The full automorphism group of \( U \) is isomorphic to \( S_3 \).

Proof. Let \( g \in \text{Aut } U \) and let \( G \) be the subgroup of \( \text{Aut } U \) generated by \( \sigma \) and \( \theta \). Since
\[
\text{Aut } U_2 = \{h|_{U_2} \mid h \in G\},
\]
there exists an \( h \in G \) such that \( gh^{-1}|_{U_2} = id_{U_2} \). In particular, \( \rho = gh^{-1} \) will fix the conformal vectors \( \tilde{\omega}^9, \tilde{\omega}^{10} \) and thus fixes the subVOA \( L(1/2, 0) \otimes L(21/22, 0) \). Hence \( \rho \) will map highest weight vectors to highest weight vectors of the same type. Moreover in \( U \) highest weight vectors are unique (up to scalar multiple) and \( \rho \) preserves their inner product. Hence \( \rho \) must fix \( U \). Thus \( g = h \in G \) and \( \text{Aut } U = G \cong S_3 \).

Remark 3.5. Recall from Miyamoto [14] that for each conformal vector \( e \) of central charge \( 1/2 \), one can define an automorphism \( \tau_e \) by
\[
\tau_e = \begin{cases} 1 & \text{on the summands isomorphic to } L(1/2, 0) \text{ or } L(1/2, 1/2), \\ -1 & \text{on the summands isomorphic to } L(1/2, 1/16). \end{cases}
\]
In the VOA $U$, $\tau_{e^{1}}$ actually corresponds the permutation $e^{2} \leftrightarrow e^{3}$ and $\tau_{e^{2}}$ corresponds to $e^{1} \leftrightarrow e^{3}$. On the other hand, the order 3 automorphism $\sigma$ corresponds to the cyclic permutation $e^{1} \rightarrow e^{2} \rightarrow e^{3} \rightarrow e^{1}$. Hence we have

$$\sigma = \tau_{e^{2}}\tau_{e^{1}}.$$ 

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