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Variations of conjectures on counting irreducible characters of finite groups

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1 Introduction

Let G be a finite group, p a prime, R the ring of algebraic integers in some finite Galois extension field K of $\mathbf{Q}$ which contains enough roots of unity, $\mathcal{P}$ a prime ideal of R lying over $p\mathbf{Z}$, $R_{\mathcal{P}}$ the localization of R at $\mathcal{P}$, and k be the residue class field $R_{\mathcal{P}}/\mathcal{P}R_{\mathcal{P}}$ of characteristic p.

For terminology used in modular representation theory, see [10].

Let $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G, e a primitive idempotent of the center $Z(R_{\mathcal{P}}G)$ of $R_{\mathcal{P}}G$, i.e., a block idempotent of G. Let B be the p-block of G corresponding to e.

-We say that $\chi \in \mathrm{Irr}(G)$ belongs to B and write $\chi \in B$, if $\chi(e) \neq 0$.

-We also say that an indecomposable right kG-module M belongs to B, if $M\overline{e} \neq 0$, where $\overline{e}$ is the image of e via the canonical epimorphism from $R_{\mathcal{P}}G$ to kG. We also write $M \in B$.

Definition 1.1 *For $\chi \in \mathrm{Irr}(G)$, let $d(\chi)$ be the exponent of the highest power of p in $|G|/\chi(1)$. Let $d(B) = \max\{d(\chi) |\ \chi \in B\}$.*

-For a block B, there exists a p-subgroup D of G such that every irreducible kG-module belonging to B is isomorphic to a direct summand of a kG-module induced from a kD-module and that $|D| = p^{d(B)}$.

The above D is unique up to G-conjugate and called a defect group of B.

-For any $\chi \in \mathrm{Irr}(G)$ and a conjugacy class C of G, the value

$$\sum_{g\in C}\chi(g)/\chi(1) = \chi(g)|G|/\chi(1)|C_G(g)|$$

lies in R. The map from $Z(kG)$ to k sending any $\widehat{C} = \sum_{g\in C} g$ in $Z(kG)$ to $\sum_{g\in C}\chi(g)/\chi(1) \mod \mathcal{P}$ gives a k-algebra homomorphism. It does not depend on the choice of $\chi \in B$ and is denoted by ω_B.

Definition 1.2 *Let B be a block of G and H a subgroup of G. If a block b of H satisfies $\omega_B(\widehat{C}) = \omega_b(\widehat{C \cap H})$ for all conjugacy class C of G, then we write $b^G = B$. If this is the case, we call b^G the induced block and that the induced block can be*

Theorem 1.3 (The first main theorem of Brauer) *Let D be a p-subgroup of G. Then, there exists a bijection between the set of blocks of G with defect group D and the set of blocks of $N_G(D)$ with defect group D. Moreover, for a block B of G with defect group D, the corresponding block b is the unique block of $N_G(D)$ with defect group D such that $b^G = B$.*

2 Conjectures in the 20th century

For $H \leq G$, a block B of G and $d \in \mathbb{Z}$, let $\mathrm{Irr}(H, B, d)$ be the set of ordinary irreducible characters χ in $\mathrm{Irr}(H)$ belonging to a block b of H with $b^G = B$ and $d(\chi) = d$, and we denote by $k(H, B, d)$ its cardinality. Note that if a block B of G with defect group D corresponds to a block b of $N_G(D)$ via Brauer's first main theorem, then we have $\mathrm{Irr}(N_G(D), B, d) = \mathrm{Irr}(N_G(D), b, d)$ for all d.

Conjecture 2.1 (Alperin-McKay Conjecture, 1970's, [9], [1]) *Suppose that a block B of G with defect group D corresponds to a block b of $N_G(D)$ via Brauer's first main theorem. Then,*

$$k(G, B, d(B)) = k(N_G(D), b, d(b)) \quad ?$$

Remark 2.2 *For a block B of G, Brauer's height zero conjecture asks whether $k(G, B, d) = 0$ for all d with $d \neq d(B)$ if and only if B has an abelian defect group.*

Conjecture 2.3 (Broué, 1980's, [2], [3]) *Suppose that a block B of G with defect group D corresponds to a block b of $N_G(D)$ via Brauer's first main theorem and that D is abelian.*

(i) (Perfect Isometry Conjecture) *Do there exist a bijection $\varphi : \mathrm{Irr}(G) \cap B \to \mathrm{Irr}(N_G(D)) \cap b$ and a map $\varepsilon : \mathrm{Irr}(G) \cap B \to \{\pm 1\}$ such that*

$$\mu(g, h) = \sum_{\chi \in \mathrm{Irr}(G) \cap B} \varepsilon(\chi)\chi(g)\varphi(\chi)(h), \quad (g \in G, h \in N_G(D))$$

satisfies;

If $\mu(g, h) \neq 0$, then g and h are both p-regular or both p-singular.
Both $\mu(g, h)/|C_G(g)|$ and $\mu(g, h)/|C_{N_G(D)}(h)|$ lie in $R_{\mathcal{P}}$?

(ii) (Derived Equivalence Conjecture) *Does there exist a bounded complex*

$$\mathbf{C} : \cdots \to C_{i+1} \to C_i \to \cdots$$

of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$-bimodules with each C_i left $R_{\mathcal{P}}G$-projective and right $R_{\mathcal{P}}N_G(D)$-projective such that $\mathbf{C} \otimes_{R_{\mathcal{P}}N_G(D)} \mathbf{C}^* \cong eR_{\mathcal{P}}G$ and $\mathbf{C}^* \otimes_{R_{\mathcal{P}}G} \mathbf{C} \cong fR_{\mathcal{P}}N_G(D)$, where e and f are block idempotents of B and b, respectively ?

(iii) (Splendid Equivalence Conjecture [13]) *Does there exist a bounded complex*

$$\mathbf{C} : \cdots \to C_{i+1} \to C_i \to \cdots$$

of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$*-bimodules as in* (ii) *such that each* C_i *is a* $\Delta(D)$*-projective p-permutation module, where* $\Delta(D) = \{(g, g^{-1}) \mid g \in D\}$ *?*

If φ and ε in (i) above exist, then we say that φ is a *a perfect isometry* between B and b. If $\mathbf{C}$ in (ii) above exists, then we say that B and b are *derived equivalent.* If $\mathbf{C}$ in (iii) above exists, then we say that B and b are *splendidly equivalent.*

-If the derived equivalence conjecture holds for B, then the perfect isometry conjecture is also true for B. (See §3 of [2].)

-A $\Delta(D)$-projective p-permutation module is by definition a direct summand of a module induced from $\Delta(D)$.

-A complex of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$-bimodules with the properties described in (iii) exists if and only if a complex of kG-$kN_G(D)$-bimodules with similar properties exists. (Rickard [13]) This fact is based on a result of Scott [18].

A radical p-subgroup P is a p-subgroup of G satisfying $O_p(N_G(P)) = P$, where $O_p(H)$ is the maximal normal p-subgroup of H for a finite group H. *A radical p-chain*

$$\underline{C} : O_p(G) < P_1 < P_2 < \cdots < P_n$$

is a chain of p-subgroups P_i of G starting with $O_p(G)$ such that $O_p(\cap_{i=1}^{j} N_G(P_i)) = P_j$ for all j with $1 \le j \le n$. Let $\mathcal{R}$ be the set of radical p-chains of G and $\mathcal{R}/G$ a set of representatives of G-orbits in $\mathcal{R}$. For $\underline{C} \in \mathcal{R}$, let $N_G(\underline{C}) = \cap_{i=1}^{n} N_G(P_i)$ and $|\underline{C}| = n$. For chain normalizers, we have the following.

Lemma 2.4 (Knörr, Robinson) *Let* $\underline{C} \in \mathcal{R}$*. Then, for any block* b *of* $N_G(\underline{C})$*, the induced block* b^G *can be defined.*

Conjecture 2.5 (Dade, 1990's, [5], [6]) *Let* B *be a block of* G *with defect group* D*. Suppose that* $D \ne \{1\}$ *and* $O_p(G) = \{1\}$*. Then,*

$$\sum_{\underline{C} \in \mathcal{R}/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d) = 0$$

for all d *?*

Remark 2.6 *There are several forms of Dade's conjecture. They involve the number of invariant characters under the automorphism action, that of projective irreducible characters, etc.*

-Dade's conjecture (the projective form) implies the Alperin-McKay conjecture. (See Corollary 17.15 and Theorem 18.5 of [6])

-Suppose that D is abelian. Then, Broué's perfect isometry conjecture implies Dade's conjecture. (See §2 of [20].)

3 Conjectures in the 21st century

Definition 3.1 *For $\chi \in \mathrm{Irr}(G)$, let $r(\chi)$ be the p'-part of $|G|/\chi(1)$ in $(\mathbb{Z}/p\mathbb{Z})^*$ the group of units of the finite field $\mathbb{Z}/p\mathbb{Z}$.*

For $H \leq G$, a block B of G, an integer d and an element r of $(\mathbb{Z}/p\mathbb{Z})^*$, let $\mathrm{Irr}(H, B, d, [\pm r])$ denote the set of irreducible characters χ in $\mathrm{Irr}(H, B, d)$ such that $r(\chi) = \pm r$, and let $k(H, B, d, [\pm r])$ denote its cardinality.

Conjecture 3.2 (Alperin-McKay-Isaacs-Navarro Conjecture, 2001, [7]) *Let a block B of G with defect group D corresponds to a block b of $N_G(D)$ via Brauer's first main theorem. Then,*

$$k(G, B, d(B), [\pm r]) = k(N_G(D), b, d(b), [\pm r])$$

for all $r \in (\mathbb{Z}/p\mathbb{Z})^*$?

Conjecture 3.3 (October 2001, see [17], [19]) *Let B be a block of G with defect group D. Suppose that $D \neq \{1\}$ and $O_p(G) = \{1\}$. Then,*

$$\sum_{\underline{C} \in \mathcal{R}/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d, [\pm r]) = 0$$

for all $d \in \mathbb{Z}$ and $r \in (\mathbb{Z}/p\mathbb{Z})^*$?

-There are also several forms of Conjecture 3.3.

-The projective form of Conjecture 3.3 implies the Alperin-McKay-Isaacs-Navarro conjecture. The proof is similar to those of Corollary 17.15 and Theorem 18.5 of [6].

Also, the following should be noticed.

Remark 3.4 *Suppose that B is principal and D is abelian. Assume further that Broué's perfect isometry conjecture holds for B and the trivial character of G corresponds to that of $N_G(D)$ via the perfect isometry. Then, Conjectures 3.2 and 3.3 holds for B.*

The reason is sketched as follows. Let b be the Brauer correspondent of B. Suppose that there exist a bijection $\varphi : \mathrm{Irr}(G) \cap B \to \mathrm{Irr}(N_G(D)) \cap b$ and a map $\varepsilon : \mathrm{Irr}(G) \cap B \to \{\pm 1\}$ satisfying the condition. Then, we have an isomorphism

$$\tilde{\varphi} : Z(eR_{\mathcal{P}}G) \to Z(fR_{\mathcal{P}}N_G(D))$$

of $R_{\mathcal{P}}$-algebras satisfying

$$\tilde{\varphi}(e) = \sum_{\chi \in \mathrm{Irr}(G) \cap B} \frac{\varepsilon(\chi)|G|/\chi(1)}{|N_G(D)|/\varphi(\chi)(1)} e_{\varphi(\chi)},$$

where $e_{\varphi(\chi)}$ is the central idempotent of $KN_G(D)$ corresponding to $\varphi(\chi)$. (See §1 of [2].) In particular

$$\frac{\varepsilon(\chi)|G|/\chi(1)}{|N_G(D)|/\varphi(\chi)(1)}$$

is a unit in $\mathbb{Z}/p\mathbb{Z}$ independent of χ. Since $\varphi(1_G) = 1_{N_G(D)}$, we must have

$$\frac{\varepsilon(\chi)|G|/\chi(1)}{|N_G(D)|/\varphi(\chi)(1)} = \pm\frac{|G|}{|N_G(D)|} \equiv \pm 1 \mod p$$

for all $\chi \in \mathrm{Irr}(G) \cap B$. Thus, $r(\chi) = \pm r(\varphi(\chi))$ for all $\chi \in \mathrm{Irr}(G) \cap B$. Hence Conjecture 3.2 follows. This observation is due to Broué. For Conjecture 3.3, we use standard pairings of chains and Brauer's third main theorem. (§2 of [20].)

Let $\mathcal{H}$ be the subgroup of $\mathrm{Gal}(K)$ defined by

$$\mathcal{H} = \{\sigma \in \mathrm{Gal}(K) \mid \mathcal{P}^\sigma = \mathcal{P}\},$$

namely, $\mathcal{H}$ is the decomposition group. For a block B of G, let $\mathcal{H}_B$ denote the set of elements $\sigma \in \mathcal{H}$ which σ stabilize $\mathrm{Irr}(G) \cap B$ as a set. This time, for $\sigma \in \mathcal{H}$, let $\mathrm{Irr}(H, B, d, [\pm r], \sigma)$ denote the set of σ-invariant irreducible characters in $\mathrm{Irr}(H, B, d, [\pm r])$, and $k(H, B, d, [\pm r], \sigma)$ its cardinality. Note that, if a block B of G with defect group D corresponds to a block b of $N_G(D)$ via Brauer's first main theorem, then $\mathcal{H}_B = \mathcal{H}_b$.

Conjecture 3.5 (Alperin-McKay-Isaacs-Navarro Conjecture, July 2002, [11]) *Suppose that a block B of G with defect group D corresponds to a block b of $N_G(D)$ via Brauer's first main theorem. Then,*

$$\mathrm{Irr}(G, B, d(B), [\pm r]) \text{ and } \mathrm{Irr}(N_G(D), b, d(b), [\pm r])$$

are isomorphic as $\mathcal{H}_B(= \mathcal{H}_b)$-sets for all $r \in (\mathbb{Z}/p\mathbb{Z})^*$?

Remark 3.6 *The original form of the above states that*

$$k(G, B, d(B), [\pm r], \sigma) = k(N_G(D), b, d(b), [\pm r], \sigma)$$

for all r and $\sigma \in \mathcal{H}$. However, it can be seen that this is equivalent to Conjecture 3.5. One can see it by using the theory of Burnside rings.

Of course, we have one more conjecture.

Conjecture 3.7 (August 2002) *Let B be a block of G with defect group D. Suppose that $D \neq \{1\}$ and $O_p(G) = \{1\}$. Then,*

$$\sum_{\underline{C} \in \mathcal{R}/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d, [\pm r], \sigma) = 0$$

for all $d \in \mathbb{Z}$, $r \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\sigma \in \mathcal{H}_B$?

-There are also several forms of Conjecture 3.7.

Remark 3.8 *Suppose that B is principal and D is abelian. Of course, B is $\mathcal{H}$-invariant. Then, if B and its Brauer correspondent b are splendidly equivalent with an $\mathcal{H}$-invariant complex, then Conjecture 3.5 holds for B.*

The reason is almost the same as before, since if the complex which gives a splendid equivalence is $\mathcal{H}$-invariant, then the function μ in Broué's conjecture is $\mathcal{H}$-invariant. Thus σ-invariant characters must correspond to σ-invariant characters by the perfect isometry φ.

This gives rise to the following version of Broué's conjecture.

Conjecture 3.9 (Galois Invariant Splendid Equivalence Conjecture) *Suppose that a block B of G with defect group D corresponds to a block b of $N_G(D)$ via Brauer's first main theorem and that D is abelian. Does there exist an $\mathcal{H}_B$-invariant bounded complex*

$$\mathbf{C}: \cdots \to C_{i+1} \to C_i \to \cdots$$

of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$-bimodules such that each C_i is a $\Delta(D)$-projective p-permutation module, where $\Delta(D) = \{(g, g^{-1}) \mid g \in D\}$?

If such a $\mathbf{C}$ exists, then we say that a splendid equivalence between B and b is Galois invariant.

4 Cyclic defect case

If a defect group of a block is cyclic, then, we have the following.

Theorem 4.1 *Suppose that a block B of G has a cyclic defect group. Then, all conjectures appearing in the previous section hold for B.*

It suffices to show that Conjectures 3.7 and 3.9 are true. In this situation, Rouquier proved that the splendid equivalence conjecture is true. ([15]) The complex he constructed is $\mathcal{H}_B$-invariant. Thus Conjecture 3.9 is true. For Conjecture 3.7 we use the standard reduction through the normalizer of the unique subgroup of D of order p. (§9 of [5])

5 A reduction theorem

In [6] Theorem 16.4, Dade proved a reduction theorem. This can be generalized easily to the situation involving r. Instead of giving a general result, we consider the following situation.

5.1 *Let P be a cyclic normal p-subgroup of G with $|P| = p^s$. Assume that a Sylow p-subgroup S of G is $P \times Q$ for a cyclic subgroup Q of G.*

Proposition 5.2 (i) *Assume 5.1 and suppose that $O_p(G) = P$. (Note that then P is radical.) Then, we have the following.*

$$\sum_{\underline{C} \in \mathcal{R}G/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d, [\pm r]) = 0$$

for all p-blocks B of G with $d(B) > s$, and for all d and r.

(ii) *Assume 5.1. For any block B of G with defect group S, there exists a Galois invariant splendid equivalence between B and its Brauer correspondent.*

For (i), an argument similar to that found in the proof of Theorem 16.4 in [6] gives the result. It can be used when obtaining cancellation results for Dade's conjecture. (Conjecture 3.3.) For (ii) the proof uses Rouquier's construction [15] of complex for cyclic defect case and the argument of Marcus [8] for the existence of extensions of complexes. As an application of (ii) above, we have the following.

Corollary 5.3 *Assume that a Sylow p-subgroup S of G is an elementary abelian of order p^2. Then, for a block B with defect group S, Conjecture 3.9 implies Conjecture 3.7.*

For the proof of the above, consider a radical p-chain starting with $1 < P$ for some P with $|P| = p$. If such a chain exists, then $N_G(P)$ satisfies 5.1 and thus

$$k(N_G(1 < P), B, d, [\pm r], \sigma) = k(N_G(1 < P < S), B, d, [\pm r], \sigma)$$

for all d, r and $\sigma \in \mathcal{H}_B$ by Proposition 5.2 (ii). Now, the remaining radical p-chains are the trivial one and $1 < S$. Thus, the result holds.

6 Examples

Let us give examples of blocks whose defect groups are not abelian. Let G be the sporadic simple Conway's group Co_2 or Co_3, and let $p = 5$. We verify Conjecture 3.7 in this case. A Sylow 5-subgroup S of G is an extra special group of order 5^3 and exponent 5. It follows from the Atlas [4] that groups of order 5^2 are not radical 5-subgroups of G. Moreover, among subgroups of order 5, one generated by a $5B$-element is a radical 5-subgroup, while the center of S, which is generated by a $5A$-element is not a radical 5-subgroup of G. Let P denote a subgroup generated by a $5B$-element and S' a Sylow 5-subgroup of $N_G(P)$. Note that S' is an elementary abelian of order 5^2. Then the following give representatives of radical 5-chains of G.

$$1, \ 1 < P, \ 1 < P < S', \ 1 < S$$

Now, by the argument in the paragraph following Corollary 5.3, in order to verify Conjecture 3.7, it suffices to consider only the trivial chain and $1 < S$. The character tables of G and $N_G(S)$ are found in [4] and [12], respectively. The principal 5-block B of G is the only 5-block of G with defect group S and $N_G(S)$ has of course only the principal block. The numbers of relevant characters are shown as follows.

$(d, \pm r)$	$(3, \pm 1)$	$(3, \pm 2)$	$(2, \pm 1)$	$(2, \pm 2)$
$k(Co_2, B, d, [\pm r])$	10	10	3	4
$k(N_{Co_2}(S), B, d, [\pm r])$	10	10	3	4
$k(Co_3, B, d, [\pm r])$	10	10	2	4
$k(N_{Co_3}(S), B, d, [\pm r])$	10	10	2	4

For Co_2, irreducible characters in B with defect 3 are all $\mathcal{H}$-invariant. Let us consider those with defect 2. In the notation of the Atlas [4], we have

$$\mathrm{Irr}(Co_2, B, 2, [\pm 1]) = \{\chi_{12}, \chi_{13}, \chi_{28}\},$$
$$\mathrm{Irr}(Co_2, B, 2, [\pm 2]) = \{\chi_{31}, \chi_{32}, \chi_{45}, \chi_{55}\},$$

with $\chi_{12}(1) = \chi_{13}(1) = 10395 = 2079 \cdot 5$, $\chi_{28}(1) = 212520 = 42504 \cdot 5$, $\chi_{31}(1) = \chi_{32}(1) = 239085 = 47817 \cdot 5$, $\chi_{45}(1) = 637560 = 127512 \cdot 5$, $\chi_{55}(1) = 1943040 = 388608 \cdot 5$. Among those, only χ_{12}, χ_{13}, χ_{31}, χ_{32} have irrational values involving $\sqrt{-15}$. It follows that, if $\sigma \in \mathcal{H}$ sends $\sqrt{-15}$ to $-\sqrt{-15}$, then $\chi_{12}^{\sigma} = \chi_{13}$ and $\chi_{31}^{\sigma} = \chi_{32}$.

On the other hand, characters of $N_{Co_2}(S)$ with defect 3 are all $\mathcal{H}$-invariant, and in the notation of [12], we have

$$\mathrm{Irr}(N_{Co_2}(S), B, 2, [\pm 1]) = \{\chi_{17}, \chi_{18}, \chi_{19}\},$$
$$\mathrm{Irr}(N_{Co_2}(S), B, 2, [\pm 2]) = \{\chi_{24}, \chi_{25}, \chi_{26}, \chi_{27}\},$$

with $\chi_{17}(1) = \chi_{18}(1) = \chi_{19}(1) = 20 = 4 \cdot 5$, $\chi_{24}(1) = \chi_{25}(1) = \chi_{26}(1) = 40 = 8 \cdot 5$, $\chi_{27}(1) = 60 = 12 \cdot 5$. Among those, only χ_{17}, χ_{18}, χ_{24}, χ_{25} have irrational values involving $\sqrt{-15}$, and if $\sigma \in \mathcal{H}$ sends $\sqrt{-15}$ to $-\sqrt{-15}$, then $\chi_{17}^{\sigma} = \chi_{18}$ and $\chi_{24}^{\sigma} = \chi_{25}$. Hence, we have

$$k(Co_2, B, d, [\pm r], \sigma) = k(N_{Co_2}(S), B, d, [\pm r], \sigma)$$

for all d, r and σ.

For Co_3, in the notation of [4] we have the following.

$$\mathrm{Irr}(Co_3, B, 3, [\pm 1]) = \{\chi_2, \chi_3, \chi_4, \chi_{18}, \chi_{19}, \chi_{21}, \chi_{25}, \chi_{32}, \chi_{36}, \chi_{38}\},$$
$$\mathrm{Irr}(Co_3, B, 3, [\pm 2]) = \{\chi_1, \chi_6, \chi_7, \chi_8, \chi_9, \chi_{13}, \chi_{14}, \chi_{33}, \chi_{34}, \chi_{42}\},$$
$$\mathrm{Irr}(Co_3, B, 2, [\pm 1]) = \{\chi_{27}, \chi_{30}\},$$
$$\mathrm{Irr}(Co_3, B, 2, [\pm 2]) = \{\chi_{10}, \chi_{11}, \chi_{15}, \chi_{40}\},$$

with $\chi_{10}(1) = \chi_{11}(1) = 3520 = 704 \cdot 5$, $\chi_{15}(1) = 8855 = 1771 \cdot 5$, $\chi_{27}(1) = 57960 = 11592 \cdot 5$, $\chi_{30}(1) = 80960 = 16192 \cdot 5$, $\chi_{40}(1) = 249480 = 49896 \cdot 5$. Among those only χ_6, χ_7, χ_{18}, χ_{19} have irrational values involving $\sqrt{-11}$, and χ_{10} and χ_{11} have those involving $\sqrt{-5}$. Any $\sigma \in \mathcal{H}$ leaves $\sqrt{-11}$ invariant, since $\sqrt{-11}$ is expressed as the

Gauss sum and 5 is a square in $\mathbb{Z}/11\mathbb{Z}$. Moreover, it follows that, if $\sigma \in \mathcal{H}$ sends $\sqrt{-5}$ to $-\sqrt{-5}$, then $\chi_{10}^{\sigma} = \chi_{11}$.

On the other hand, for $N_{Co_3}(S)$, in the notation of [12] we have the following.

$$\begin{aligned}
\mathrm{Irr}(N_{Co_3}, B, 3, [\pm 1]) &= \{\chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{16}, \chi_{17}, \chi_{18}\},\\
\mathrm{Irr}(N_{Co_3}, B, 3, [\pm 2]) &= \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_{23}, \chi_{24}\},\\
\mathrm{Irr}(N_{Co_3}, B, 2, [\pm 1]) &= \{\chi_{25}, \chi_{26}\},\\
\mathrm{Irr}(N_{Co_3}, B, 2, [\pm 2]) &= \{\chi_{19}, \chi_{20}, \chi_{21}, \chi_{22}\}.
\end{aligned}$$

There are 14 characters having irrational values. However, they are $\mathcal{H}$-invariant except for χ_{19} and χ_{20}, which have values involving $\sqrt{-5}$. It follows that, if $\sigma \in \mathcal{H}$ sends $\sqrt{-5}$ to $-\sqrt{-5}$, then $\chi_{19}^{\sigma} = \chi_{20}$. Hence, we have

$$k(Co_3, B, d, [\pm r], \sigma) = k(N_{Co_3}(S), B, d, [\pm r], \sigma)$$

for all d, r and σ.

Concerning sporadic simple groups, several results are known. In [14] Rouquier verified Perfect Isometry Conjecture for all principal blocks with abelian defect groups. In [7], Isaacs and Navarro confirmed the 2001 version of Alperin-McKay-Isaacs-Navarro Conjecture in the group form using [21]. Up to now, Conjecture 3.3 was verified for all primes for sporadic simple groups except for J_4, Fi'_{24}, BM, M. Conjectures 3.7 and 3.9 have not been verified in almost all cases. For current situation of Broué's conjecture, see, for example, 5.2 of [16]. In particular, the results on Broué's conjecture for sporadic simple groups include those for J_1 for $p = 2$, M_{11}, M_{22}, M_{23}, ON, HS for $p = 3$ and J_2 for $p = 5$.

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