Variations of conjectures on counting irreducible characters of finite groups

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1 Introduction

Let $G$ be a finite group, $p$ a prime, $R$ the ring of algebraic integers in some finite Galois extension field $K$ of $\mathbb{Q}$ which contains enough roots of unity, $\mathcal{P}$ a prime ideal of $R$ lying over $p\mathbb{Z}$, $R_{\mathcal{P}}$ the localization of $R$ at $\mathcal{P}$, and $k$ be the residue class field $R_{\mathcal{P}}/\mathcal{P}R_{\mathcal{P}}$ of characteristic $p$.

For terminology used in modular representation theory, see [10].

Let $\text{Irr}(G)$ be the set of complex irreducible characters of $G$, $e$ a primitive idempotent of the center $Z(R_{G})$ of $R_{G}$, i.e., a block idempotent of $G$. Let $B$ be the $p$-block of $G$ corresponding to $e$.

- We say that $\chi \in \text{Irr}(G)$ belongs to $B$ and write $\chi \in B$, if $\chi(e) \neq 0$.

- We also say that an indecomposable right $kG$-module $M$ belongs to $B$, if $M\overline{e} \neq 0$, where $\overline{e}$ is the image of $e$ via the canonical epimorphism from $R_{G}$ to $kG$. We also write $M \in B$.

Definition 1.1 For $\chi \in \text{Irr}(G)$, let $d(\chi)$ be the exponent of the highest power of $p$ in $|G|/\chi(1)$. Let $d(B) = \max\{d(\chi) | \chi \in B\}$.

- For a block $B$, there exists a $p$-subgroup $D$ of $G$ such that every irreducible $kG$-module belonging to $B$ is isomorphic to a direct summand of a $kG$-module induced from a $kD$-module and that $|D| = p^{d(B)}$.

The above $D$ is unique up to $G$-conjugate and called a defect group of $B$.

- For any $\chi \in \text{Irr}(G)$ and a conjugacy class $C$ of $G$, the value

$$\sum_{g \in C} \chi(g)/\chi(1) = \chi(g)|G|/\chi(1)|C_{G}(g)|$$

lies in $R$. The map from $Z(kG)$ to $k$ sending any $\overline{C} = \sum_{g \in C} g$ in $Z(kG)$ to $\sum_{g \in C} \chi(g)/\chi(1) \mod \mathcal{P}$ gives a $k$-algebra homomorphism. It does not depend on the choice of $\chi \in B$ and is denoted by $\omega_{B}$.

Definition 1.2 Let $B$ be a block of $G$ and $H$ a subgroup of $G$. If a block $b$ of $H$ satisfies $\omega_{B}(\overline{C}) = \omega_{b}(\overline{C \cap H})$ for all conjugacy class $C$ of $G$, then we write $b^{G} = B$. If this is the case, we call $b^{G}$ the induced block and that the induced block can be
Theorem 1.3 (The first main theorem of Brauer) Let $D$ be a $p$-subgroup of $G$. Then, there exists a bijection between the set of blocks of $G$ with defect group $D$ and the set of blocks of $N_G(D)$ with defect group $D$. Moreover, for a block $B$ of $G$ with defect group $D$, the corresponding block $b$ is the unique block of $N_G(D)$ with defect group $D$ such that $b^G = B$.

2 Conjectures in the 20th century

For $H \leq G$, a block $B$ of $G$ and $d \in \mathbb{Z}$, let $\text{Irr}(H, B, d)$ be the set of ordinary irreducible characters $\chi$ in $\text{Irr}(H)$ belonging to a block $b$ of $H$ with $b^G = B$ and $d(\chi) = d$, and we denote by $k(H, B, d)$ its cardinality. Note that if a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem, then we have $\text{Irr}(N_G(D), B, d) = \text{Irr}(N_G(D), b, d)$ for all $d$.

Conjecture 2.1 (Alperin-McKay Conjecture, 1970's, [9], [1]) Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem. Then,

$$k(G, B, d(B)) = k(N_G(D), b, d(b)).$$

Remark 2.2 For a block $B$ of $G$, Brauer's height zero conjecture asks whether $k(G, B, d) = 0$ for all $d$ with $d \neq d(B)$ if and only if $B$ has an abelian defect group.

Conjecture 2.3 (Broué, 1980's, [2], [3]) Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem and that $D$ is abelian.

(i) (Perfect Isometry Conjecture) Do there exist a bijection $\varphi : \text{Irr}(G) \cap B \rightarrow \text{Irr}(N_G(D)) \cap b$ and a map $\epsilon : \text{Irr}(G) \cap B \rightarrow \{\pm 1\}$ such that

$$\mu(g, h) = \sum_{\chi \in \text{Irr}(G) \cap B} \epsilon(\chi) \chi(g) \varphi(\chi)(h), \ (g \in G, h \in N_G(D))$$

satisfies;

If $\mu(g, h) \neq 0$, then $g$ and $h$ are both $p$-regular or both $p$-singular.

Both $\mu(g, h)/|C_G(g)|$ and $\mu(g, h)/|C_{N_G(D)}(h)|$ lie in $R_P$?

(ii) (Derived Equivalence Conjecture) Does there exist a bounded complex $$C : \cdots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \cdots$$ of $R_PG$-$R_PN_G(D)$-bimodules with each $C_i$ left $R_PG$-projective and right $R_PN_G(D)$-projective such that $C \otimes_{R_PN_G(D)} C^* \cong eR_PG$ and $C^* \otimes_{R_PG} C \cong fR_PN_G(D)$, where $e$ and $f$ are block idempotents of $B$ and $b$, respectively?
(iii) (Splendid Equivalence Conjecture [13]) Does there exist a bounded complex
\[ C: \cdots \to C_{i+1} \to C_i \to \cdots \]
of $R_{P^G}G$-$R_{P}G(D)$-bimodules as in (ii) such that each $C_i$ is a $\Delta(D)$-projective $p$-permutation module, where $\Delta(D) = \{ (g, g^{-1}) \mid g \in D \}$?

If $\varphi$ and $\varepsilon$ in (i) above exist, then we say that $\varphi$ is a $a$ perfect isometry between $B$ and $b$. If $C$ in (ii) above exists, then we say that $B$ and $b$ are derived equivalent. If $C$ in (iii) above exists, then we say that $B$ and $b$ are splendidly equivalent.

- If the derived equivalence conjecture holds for $B$, then the perfect isometry conjecture is also true for $B$. (See §3 of [2].)

- A $\Delta(D)$-projective $p$-permutation module is by definition a direct summand of a module induced from $\Delta(D)$.

- A complex of $R_{P^G}G$-$R_{P}G(D)$-bimodules with the properties described in (iii) exists if and only if a complex of $kG$-$kG(D)$-bimodules with similar properties exists. (Rickard [13]) This fact is based on a result of Scott [18].

A radical $p$-subgroup $P$ is a $p$-subgroup of $G$ satisfying $O_p(N_G(P)) = P$, where $O_p(H)$ is the maximal normal $p$-subgroup of $H$. A radical $p$-chain
\[ \underline{C}: O_p(G) < P_1 < P_2 < \cdots < P_n \]
is a chain of $p$-subgroups $P_i$ of $G$ starting with $O_p(G)$ such that $O_p(\bigcap_{i=1}^{j} N_G(P_i)) = P_j$ for all $j$ with $1 \leq j \leq n$. Let $\mathcal{R}$ be the set of radical $p$-chains of $G$ and $\mathcal{R}/G$ a set of representatives of $G$-orbits in $\mathcal{R}$. For $\underline{C} \in \mathcal{R}$, let $N_G(\underline{C}) = \bigcap_{i=1}^{n} N_G(P_i)$ and $|\underline{C}| = n$. For chain normalizers, we have the following.

**Lemma 2.4** (Knörr, Robinson) Let $\underline{C} \in \mathcal{R}$. Then, for any block $b$ of $N_G(\underline{C})$, the induced block $b^G$ can be defined.

**Conjecture 2.5** (Dade, 1990’s, [5], [6]) Let $B$ be a block of $G$ with defect group $D$. Suppose that $D \neq \{1\}$ and $O_p(G) = \{1\}$. Then,
\[
\sum_{\underline{C} \in \mathcal{R}/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d) = 0
\]
for all $d$?

**Remark 2.6** There are several forms of Dade’s conjecture. They involve the number of invariant characters under the automorphism action, that of projective irreducible characters, etc.

- Dade’s conjecture (the projective form) implies the Alperin-McKay conjecture. (See Corollary 17.15 and Theorem 18.5 of [6])

- Suppose that $D$ is abelian. Then, Broué’s perfect isometry conjecture implies Dade’s conjecture. (See §2 of [20].)
3 Conjectures in the 21st century

Definition 3.1  For $\chi \in \text{Irr}(G)$, let $r(\chi)$ be the $p'$-part of $|G|/\chi(1)$ in $(\mathbb{Z}/p\mathbb{Z})^*$ the group of units of the finite field $\mathbb{Z}/p\mathbb{Z}$.

For $H \leq G$, a block $B$ of $G$, an integer $d$, and an element $r$ of $(\mathbb{Z}/p\mathbb{Z})^*$, let $\text{Irr}(H, B, d, [\pm r])$ denote the set of irreducible characters $\chi$ in $\text{Irr}(H, B, d)$ such that $r(\chi) = \pm r$, and let $k(H, B, d, [\pm r])$ denote its cardinality.

Conjecture 3.2  (Alperin-McKay-Isaacs-Navarro Conjecture, 2001, [7]) Let a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem. Then,

$$k(G, B, d(B), [\pm r]) = k(N_G(D), b, d(b), [\pm r])$$

for all $r \in (\mathbb{Z}/p\mathbb{Z})^*$.

Conjecture 3.3  (October 2001, see [17], [19]) Let $B$ be a block of $G$ with defect group $D$. Suppose that $D \neq \{1\}$ and $O_p(G) = \{1\}$. Then,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N_G(C), B, d, [\pm r]) = 0$$

for all $d \in \mathbb{Z}$ and $r \in (\mathbb{Z}/p\mathbb{Z})^*$.

- There are also several forms of Conjecture 3.3.

- The projective form of Conjecture 3.3 implies the Alperin-McKay-Isaacs-Navarro conjecture. The proof is similar to those of Corollary 17.15 and Theorem 18.5 of [6].

Also, the following should be noticed.

Remark 3.4  Suppose that $B$ is principal and $D$ is abelian. Assume further that Broué's perfect isometry conjecture holds for $B$ and the trivial character of $G$ corresponds to that of $N_G(D)$ via the perfect isometry. Then, Conjectures 3.2 and 3.3 holds for $B$.

The reason is sketched as follows. Let $b$ be the Brauer correspondent of $B$. Suppose that there exist a bijection $\varphi : \text{Irr}(G) \cap B \rightarrow \text{Irr}(N_G(D)) \cap b$ and a map $\epsilon : \text{Irr}(G) \cap B \rightarrow \{\pm 1\}$ satisfying the condition. Then, we have an isomorphism

$$\tilde{\varphi} : Z(eR_P G) \rightarrow Z(fR_P N_G(D))$$

of $R_P$-algebras satisfying

$$\tilde{\varphi}(\epsilon) = \sum_{\chi \in \text{Irr}(G) \cap B} \frac{\epsilon(\chi)|G|/\chi(1)}{|N_G(D)|/\varphi(\chi)(1)} e_{\varphi(\chi)}$$.
where $e_{\varphi(\chi)}$ is the central idempotent of $KN_{G}(D)$ corresponding to $\varphi(\chi)$. (See §1 of [2].) In particular
\[
\frac{\varepsilon(\chi)|G|/\chi(1)}{|N_{G}(D)|/\varphi(\chi)(1)}
\]
is a unit in $\mathbb{Z}/p\mathbb{Z}$ independent of $\chi$. Since $\varphi(1_{G}) = 1_{N_{G}(D)}$, we must have
\[
\frac{\varepsilon(\chi)|G|/\chi(1)}{|N_{G}(D)|/\varphi(\chi)(1)} = \pm \frac{|G|}{|N_{G}(D)|} \equiv \pm 1 \mod p
\]
for all $\chi \in \text{Irr}(G) \cap B$. Thus, $r(\chi) = \pm r(\varphi(\chi))$ for all $\chi \in \text{Irr}(G) \cap B$. Hence Conjecture 3.2 follows. This observation is due to Broué. For Conjecture 3.3, we use standard pairings of chains and Brauer’s third main theorem. (§2 of [20].)

Let $\mathcal{H}$ be the subgroup of $\text{Gal}(K)$ defined by
\[
\mathcal{H} = \{ \sigma \in \text{Gal}(K) \mid P^\sigma = P \},
\]
namely, $\mathcal{H}$ is the decomposition group. For a block $B$ of $G$, let $\mathcal{H}_{B}$ denote the set of elements $\sigma \in \mathcal{H}$ which $\sigma$ stabilize $\text{Irr}(G) \cap B$ as a set. This time, for $\sigma \in \mathcal{H}$, let $\text{Irr}(H, B, d, [\pm r], \sigma)$ denote the set of $\sigma$-invariant irreducible characters in $\text{Irr}(H, B, d, [\pm r], \sigma)$ and $k(H, B, d, [\pm r], \sigma)$ its cardinality. Note that, if a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_{G}(D)$ via Brauer’s first main theorem, then $\mathcal{H}_{B} = \mathcal{H}_{b}$.

**Conjecture 3.5** (Alperin-McKay-Isaacs-Navarro Conjecture, July 2002, [11]) Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_{G}(D)$ via Brauer’s first main theorem. Then,
\[
\text{Irr}(G, B, d(B), [\pm r]) \text{ and } \text{Irr}(N_{G}(D), b, d(b), [\pm r])
\]
are isomorphic as $\mathcal{H}_{B}(= \mathcal{H}_{b})$-sets for all $r \in (\mathbb{Z}/p\mathbb{Z})^*$.

**Remark 3.6** The original form of the above states that
\[
k(G, B, d(B), [\pm r], \sigma) = k(N_{G}(D), b, d(b), [\pm r], \sigma)
\]
for all $r$ and $\sigma \in \mathcal{H}$. However, it can be seen that this is equivalent to Conjecture 3.5. One can see it by using the theory of Burnside rings.

Of course, we have one more conjecture.

**Conjecture 3.7** (August 2002) Let $B$ be a block of $G$ with defect group $D$. Suppose that $D \neq \{1\}$ and $O_{p}(G) = \{1\}$. Then,
\[
\sum_{Q \in \mathcal{R}/G} (-1)^{|Q|} k(N_{G}(Q), B, d, [\pm r], \sigma) = 0
\]
for all $d \in \mathbb{Z}$, $r \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\sigma \in \mathcal{H}_{B}$?

- There are also several forms of Conjecture 3.7.
Remark 3.8 Suppose that $B$ is principal and $D$ is abelian. Of course, $B$ is $\mathcal{H}$-invariant. Then, if $B$ and its Brauer correspondent $b$ are splendidly equivalent with an $\mathcal{H}$-invariant complex, then Conjecture 3.5 holds for $B$.

The reason is almost the same as before, since if the complex which gives a splendid equivalence is $\mathcal{H}$-invariant, then the function $\mu$ in Broué's conjecture is $\mathcal{H}$-invariant. Thus $\sigma$-invariant characters must correspond to $\sigma$-invariant characters by the perfect isometry $\varphi$.

This gives rise to the following version of Broué's conjecture.

Conjecture 3.9 (Galois Invariant Splendid Equivalence Conjecture) Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem and that $D$ is abelian. Does there exist an $\mathcal{H}_B$-invariant bounded complex

$$C : \cdots \rightarrow C_{i+1} \rightarrow C_i \rightarrow \cdots$$

of $R_P G - R_P N_G(D)$-bimodules such that each $C_i$ is a $\Delta(D)$-projective $p$-permutation module, where $\Delta(D) = \{(g, g^{-1}) | g \in D\}$?

If such a $C$ exists, then we say that a splendid equivalence between $B$ and $b$ is Galois invariant.

4 Cyclic defect case

If a defect group of a block is cyclic, then, we have the following.

Theorem 4.1 Suppose that a block $B$ of $G$ has a cyclic defect group. Then, all conjectures appearing in the previous section hold for $B$.

It suffices to show that Conjectures 3.7 and 3.9 are true. In this situation, Rouquier proved that the splendid equivalence conjecture is true. ([15]) The complex he constructed is $\mathcal{H}_B$-invariant. Thus Conjecture 3.9 is true. For Conjecture 3.7 we use the standard reduction through the normalizer of the unique subgroup of $D$ of order $p$. ([39 of [5]])

5 A reduction theorem

In [6] Theorem 16.4, Dade proved a reduction theorem. This can be generalized easily to the situation involving $r$. Instead of giving a general result, we consider the following situation.

5.1 Let $P$ be a cyclic normal $p$-subgroup of $G$ with $|P| = p^s$. Assume that a Sylow $p$-subgroup $S$ of $G$ is $P \times Q$ for a cyclic subgroup $Q$ of $G$. 
Proposition 5.2 (i) Assume 5.1 and suppose that $O_p(G) = P$. (Note that then $P$ is radical.) Then, we have the following.

$$\sum_{C \in RG/G} (-1)^{|C|} k(N_G(C), B, d, [\pm r]) = 0$$

for all $p$-blocks $B$ of $G$ with $d(B) > s$, and for all $d$ and $r$.

(ii) Assume 5.1. For any block $B$ of $G$ with defect group $S$, there exists a Galois invariant splendid equivalence between $B$ and its Brauer correspondent.

For (i), an argument similar to that found in the proof of Theorem 16.4 in [6] gives the result. It can be used when obtaining cancellation results for Dade's conjecture. (Conjecture 3.3.) For (ii) the proof uses Rouquier's construction [15] of complex for cyclic defect case and the argument of Marcus [8] for the existence of extensions of complexes. As an application of (ii) above, we have the following.

Corollary 5.3 Assume that a Sylow $p$-subgroup $S$ of $G$ is an elementary abelian of order $p^2$. Then, for a block $B$ with defect group $S$, Conjecture 3.9 implies Conjecture 3.7.

For the proof of the above, consider a radical $p$-chain starting with $1 < P$ for some $P$ with $|P| = p$. If such a chain exists, then $N_G(P)$ satisfies 5.1 and thus

$$k(N_G(1 < P), B, d, [\pm r], \sigma) = k(N_G(1 < P < S), B, d, [\pm r], \sigma)$$

for all $d$, $r$ and $\sigma \in \mathcal{H}_B$ by Proposition 5.2 (ii). Now, the remaining radical $p$-chains are the trivial one and $1 < S$. Thus, the result holds.

6 Examples

Let us give examples of blocks whose defect groups are not abelian. Let $G$ be the sporadic simple Conway's group $Co_2$ or $Co_3$, and let $p = 5$. We verify Conjecture 3.7 in this case. A Sylow 5-subgroup $S$ of $G$ is an extra special group of order $5^3$ and exponent 5. It follows from the Atlas [4] that groups of order $5^2$ are not radical 5-subgroups of $G$. Moreover, among subgroups of order 5, one generated by a 5-$B$-element is a radical 5-subgroup, while the center of $S$, which is generated by a 5-$A$-element is not a radical 5-subgroup of $G$. Let $P$ denote a subgroup generated by a 5-$B$-element and $S'$ a Sylow 5-subgroup of $N_G(P)$. Note that $S'$ is an elementary abelian of order $5^2$. Then the following give representatives of radical 5-chains of $G$.

$$1, 1 < P, 1 < P < S', 1 < S$$

Now, by the argument in the paragraph following Corollary 5.3, in order to verify Conjecture 3.7, it suffices to consider only the trivial chain and $1 < S$. The character
tables of $G$ and $N_G(S)$ are found in [4] and [12], respectively. The principal 5-block $B$ of $G$ is the only 5-block of $G$ with defect group $S$ and $N_G(S)$ has of course only the principal block. The numbers of relevant characters are shown as follows.

\[
\begin{array}{cccccc}
(d, \pm r) & (3, \pm 1) & (3, \pm 2) & (2, \pm 1) & (2, \pm 2) \\
\hline
k(Co_2, B, d, [\pm r]) & 10 & 10 & 3 & 4 \\
k(N_{Co_2}(S), B, d, [\pm r]) & 10 & 10 & 3 & 4 \\
k(Co_3, B, d, [\pm r]) & 10 & 10 & 2 & 4 \\
k(N_{Co_3}(S), B, d, [\pm r]) & 10 & 10 & 2 & 4
\end{array}
\]

For $Co_2$, irreducible characters in $B$ with defect 3 are all $\mathcal{H}$-invariant. Let us consider those with defect 2. In the notation of the Atlas [4], we have

\[
\begin{align*}
\text{Irr}(Co_2, B, 2, [\pm 1]) &= \{\chi_{12}, \chi_{13}, \chi_{23}\}, \\
\text{Irr}(Co_2, B, 2, [\pm 2]) &= \{\chi_{31}, \chi_{32}, \chi_{45}, \chi_{55}\},
\end{align*}
\]

with $\chi_{12}(1) = \chi_{13}(1) = 10395 = 2079 \cdot 5$, $\chi_{28}(1) = 212520 = 42504 \cdot 5$, $\chi_{31}(1) = \chi_{32}(1) = 239085 = 47817 \cdot 5$, $\chi_{45}(1) = 637560 = 127512 \cdot 5$, $\chi_{55}(1) = 1943040 = 388608 \cdot 5$. Among those, only $\chi_{12}, \chi_{13}, \chi_{31}, \chi_{32}$ have irrational values involving $\sqrt{-15}$. It follows that, if $\sigma \in \mathcal{H}$ sends $\sqrt{-15}$ to $-\sqrt{-15}$, then $\chi_{12}^\sigma = \chi_{13}$ and $\chi_{31}^\sigma = \chi_{32}$.

On the other hand, characters of $N_{Co_2}(S)$ with defect 3 are all $\mathcal{H}$-invariant, and in the notation of [12], we have

\[
\begin{align*}
\text{Irr}(N_{Co_2}(S), B, 2, [\pm 1]) &= \{\chi_{17}, \chi_{18}, \chi_{19}\}, \\
\text{Irr}(N_{Co_2}(S), B, 2, [\pm 2]) &= \{\chi_{24}, \chi_{26}, \chi_{27}\},
\end{align*}
\]

with $\chi_{17}(1) = \chi_{18}(1) = \chi_{19}(1) = 20 = 4 \cdot 5$, $\chi_{24}(1) = \chi_{25}(1) = \chi_{26}(1) = 40 = 8 \cdot 5$, $\chi_{27}(1) = 60 = 12 \cdot 5$. Among those, only $\chi_{17}, \chi_{18}, \chi_{24}, \chi_{25}$ have irrational values involving $\sqrt{-15}$, and if $\sigma \in \mathcal{H}$ sends $\sqrt{-15}$ to $-\sqrt{-15}$, then $\chi_{17}^\sigma = \chi_{18}$ and $\chi_{24}^\sigma = \chi_{25}$. Hence, we have

\[
k(Co_2, B, d, [\pm r], \sigma) = k(N_{Co_2}(S), B, d, [\pm r], \sigma)
\]

for all $d$, $r$ and $\sigma$.

For $Co_3$, in the notation of [4] we have the following.

\[
\begin{align*}
\text{Irr}(Co_3, B, 3, [\pm 1]) &= \{\chi_2, \chi_3, \chi_4, \chi_{18}, \chi_{19}, \chi_{21}, \chi_{25}, \chi_{32}, \chi_{36}, \chi_{38}\}, \\
\text{Irr}(Co_3, B, 3, [\pm 2]) &= \{\chi_1, \chi_6, \chi_7, \chi_8, \chi_9, \chi_{13}, \chi_{14}, \chi_{33}, \chi_{34}, \chi_{42}\}, \\
\text{Irr}(Co_3, B, 2, [\pm 1]) &= \{\chi_{27}, \chi_{30}\}, \\
\text{Irr}(Co_3, B, 2, [\pm 2]) &= \{\chi_{10}, \chi_{11}, \chi_{15}, \chi_{40}\},
\end{align*}
\]

with $\chi_{10}(1) = \chi_{11}(1) = 3520 = 704 \cdot 5$, $\chi_{15}(1) = 8855 = 1771 \cdot 5$, $\chi_{27}(1) = 57960 = 11592 \cdot 5$, $\chi_{30}(1) = 80960 = 16192 \cdot 5$, $\chi_{40}(1) = 249480 = 49896 \cdot 5$. Among those only $\chi_6, \chi_7, \chi_{18}, \chi_{19}$ have irrational values involving $\sqrt{-11}$, and $\chi_{10}$ and $\chi_{11}$ have those involving $\sqrt{-5}$. Any $\sigma \in \mathcal{H}$ leaves $\sqrt{-11}$ invariant, since $\sqrt{-11}$ is expressed as the
Gauss sum and 5 is a square in \( \mathbb{Z}/11\mathbb{Z} \). Moreover, it follows that, if \( \sigma \in \mathcal{H} \) sends \( \sqrt{-5} \) to \(-\sqrt{-5} \), then \( \chi_{\sigma}^{5} = \chi_{11} \).

On the other hand, for \( N_{Co_{3}}(S) \), in the notation of [12] we have the following.

\[
\begin{align*}
\text{Irr}(N_{Co_{3}}, B, 3, [\pm 1]) &= \{\chi_{9}, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{16}, \chi_{17}, \chi_{18}\}, \\
\text{Irr}(N_{Co_{3}}, B, 3, [\pm 2]) &= \{\chi_{1}, \chi_{2}, \chi_{3}, \chi_{4}, \chi_{5}, \chi_{6}, \chi_{7}, \chi_{8}, \chi_{23}, \chi_{24}\}, \\
\text{Irr}(N_{Co_{3}}, B, 2, [\pm 1]) &= \{\chi_{25}, \chi_{26}\}, \\
\text{Irr}(N_{Co_{3}}, B, 2, [\pm 2]) &= \{\chi_{19}, \chi_{20}, \chi_{21}, \chi_{22}\}.
\end{align*}
\]

There are 14 characters having irrational values. However, they are \( \mathcal{H} \)-invariant except for \( \chi_{19} \) and \( \chi_{20} \), which have values involving \( \sqrt{-5} \). It follows that, if \( \sigma \in \mathcal{H} \) sends \( \sqrt{-5} \) to \(-\sqrt{-5} \), then \( \chi_{\sigma}^{5} = \chi_{20} \). Hence, we have

\[
k(Co_{3}, B, d, [\pm r], \sigma) = k(N_{Co_{3}}(S), B, d, [\pm r], \sigma)
\]

for all \( d, r \) and \( \sigma \).

Concerning sporadic simple groups, several results are known. In [14] Rouquier verified Perfect Isometry Conjecture for all principal blocks with abelian defect groups. In [7], Isaacs and Navarro confirmed the 2001 version of Alperin-McKay-Isaacs-Navarro Conjecture in the group form using [21]. Up to now, Conjecture 3.3 was verified for all primes for sporadic simple groups except for \( J_4, \, F_{24}' \), \( BM \), \( M \).

Conjectures 3.7 and 3.9 have not been verified in almost all cases. For current situation of Broué’s conjecture, see, for example, 5.2 of [16]. In particular, the results on Broué’s conjecture for sporadic simple groups include those for \( J_1 \) for \( p = 2 \), \( M_{11} \), \( M_{22} \), \( M_{23} \), \( ON \), \( HS \) for \( p = 3 \) and \( J_2 \) for \( p = 5 \).

References


