数理解析研究所講究録 1328 巻 2003 年 73-113

Lipschitz Equisingularity Problems

Tadeusz Mostowski

(Warsaw University)

First I would like to recall some notions of equisingularity and of a stratification of a space. Then I want to state Sullivan's questions concerning "Lipschitz equisingularity" and its solution via "Lipschitz stratifications", both in the complex and real cases. I plan to give some ideas of proofs. I want to relate the notion of a Lipschitz stratification with other regularity conditions put on stratifications and to discuss some particularities of Lipschitz stratifications. Then I want to give some low dimensional examples. Finally I want to discuss some rather recent ideas, most of them unpublished: a solution to a question of M. Gromov, relations between the notions of Zariski's equisingularity and Lipschitz equisingularity and existence of moduli for the relation of Lipschitz equivalence of functions. Also I shall state a few open questions.

<u>Classes of sets (or germs at 0) to be considered</u>

$\mathbb{C}$: analytic, algebraic

$\mathbb{R}$: semi-algebraic, semi-analytic, sub-analytic

$k = \mathbb{R}$ or $\mathbb{C}$

$X, Y \subset k^n$ (sets or germs) have the same <u>topological</u> type (resp. <u>Lipschitz</u>) if $\exists$

$$(k^n, X) \xrightarrow{h} (k^n, Y) \qquad \text{— means } h(X) = Y$$

homeomorphism
(resp. bi-Lipschitz homeom.)

(for germs – we ask h to be also a germ)

Corresponding notion for <u>families</u> :

$$\begin{array}{ccccc} X & \subset & k^n_x \times k^m_t & & \\ \pi \downarrow & & \downarrow \pi = pr_2 & & X_t = \pi^{-1}(t) \cap X \\ T & \subset & k^m_t & & \subset k^n_x \end{array}$$

A family is locally <u>topological</u> trivial (or equisingular) if

$\forall\, t_0 \in T$ $\exists$ nbd $T_0 \ni t_0$ and a homeom

$$\begin{tikzcd} k_x^n \times T_0 \arrow[rr, "h"] \arrow[dr, "\pi"'] & & k_x^n \times T_0 \arrow[dl, "\pi"] \\ & T_0 & \end{tikzcd}$$

such that if $h_t(x) = h(x,t)$,
then

$$h_t(X_{t_0}) = X_t$$

$$h_{t_0} = id.$$

In the Lipschitz case we ask h to be bi-Lipschitz and

$$h_t = id + \varphi_t$$

$\varphi_t : k_x^n \to k_x^n$ is Lipschitz with a constant $C|t-t_0|$, $C = C(t_0)$

(if h_t is obtained by integration of a Lipschitz vector field $\dot{x} = v(x)$, then h_t is as required, as follows at once from

$$x(t) = x(0) + \int_0^t v(x(s))\,ds \quad)$$

<u>Question</u> (D. Sullivan): For any family

$\begin{array}{c} X \\ \downarrow \\ T \end{array}$ does there exist a $T' \subset T$, $\dim T' <$ $\dim T$, T' in the same class of sets, such that the family is Lipschitz equisingular outside of T'?

If YES, then the number of distinct Lipschitz types of all fibers X_t ($t \in$ compact set in T) is finite; if NO, then moduli are to be expected.

In the <u>topological</u> case the answer is well known (YES) since a very long time (Whitney, Thom, Mather, Varchenko, ...)

<u>Motivations</u> for studying bi-Lipschitz homeomorphism in the context of analytic sets:

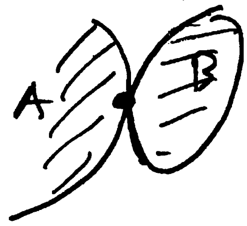
1) they preserve obvious and important properties of analytic sets like order of contact of curves

or, more generally, the exponent σ in Łojasiewicz's inequality

$$d(a, B) \geq C\, d(a, A \cap B)^{\sigma}$$

A, B compact subanalytic

$a \in A$



2) they have a lot of good properties:

- are differentiable a.e.
- preserve sets of measure 0, or, more generally:
- formula for variable change in the integral holds if the change is bi Lipschitz,
- there exists a theory of differential forms, invariant under bi-Lipschitz homeomorphism

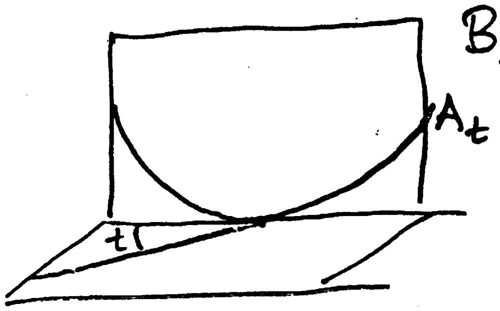
⋮

<u>Trivial example</u> (both for $\mathbb{R}$ and $\mathbb{C}$)

$t \in k = T$, $X_t^1 = A_t^1 \cup B_t^1 \subset k_x^3$

A_t^1 : $z = x^2$, $y = 0$

B_t^1 : $z = 0$, $y = tx$



If $t = 0$, then A_0^1, B_0^1 are tangent;

but A_t^1, B_t^1 are not tangent if $t \neq 0$

Thus X_0^1 has different Lipschitz type than X_t^1 $(t \neq 0)$.

However, this family is obviously topologically trivial, and even $X = \bigsqcup_t \left(X_t^1 \times \{t\}\right)$ is Whitney equisingular along the t-axis.

Stratifications

$\mathfrak{X}: \quad k^n \supset X^d \supset X^{d-1} \supset \cdots$ $\qquad X^j$ skeletons: closed, in one of the classes of sets

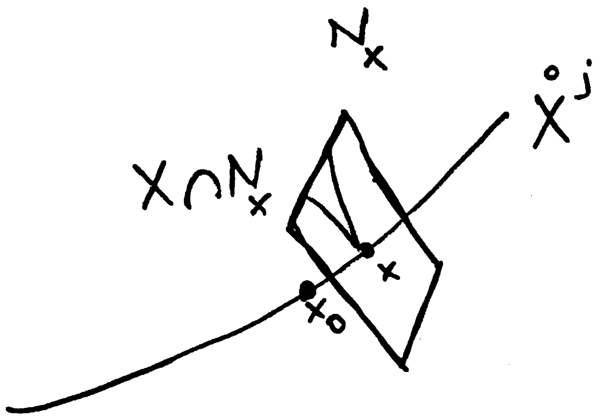
$\overset{\circ}{X}{}^j = X^j \setminus X^{j-1}$ $\qquad$ either $\emptyset$ or smooth, j-dimension

strata

$\mathfrak{X}$ is compatible with a subset Y if Y is a union of components of strata of $\mathfrak{X}$.

Locally Lipschitz trivial stratifications

Take any $x_0 \in \overset{\circ}{X}{}^j$



If $x \in \overset{\circ}{X}{}^j$ is close to x_0, let

$$N_x = T_x^{\perp} \overset{\circ}{X}{}^j$$

normal space to $\overset{\circ}{X}{}^j$

We ask that the family

$X \cap N_x$ $\quad$ (parametrised by $x \in \overset{\circ}{X}{}^j$)

be Lipschitz trivial

If one can prove that every set X has a loc. Lipschitz trivial stratification, compatible with a given Y, then the answer to Sullivan's question is YES (at least for germs of sets).

In fact, if $\begin{matrix} X \\ \downarrow \\ T \end{matrix}$, X_t germs at 0, then $T \approx T \times 0 \subset X$. We take a loc. Lipschitz stratification of X compatible with T, and for $T' = T_{sing} \cup (T \cap \text{skeleton of dim } \tau - 1)$ where $\tau = \dim T$.

So the problem is to show that every X has a loc. Lipschitz trivial stratification.

A vector field v, defined on $A \subset k^n$, is tangent to a stratification $\mathfrak{X} = \{X^j\}$ if $\forall\, a \in \mathring{X}^i \cap A \quad v(a) \in T_a \mathring{X}^i$

Preliminary definitions.

I $\mathcal{X} = \{X^j\}$ is Lipschitz if it has the following extension property for Lipschitz vector fields:

any Lipschitz vector field v, tangent to $\mathcal{X}$, defined on X^j, has a Lipschitz extension, defined on k^n, tangent to $\mathcal{X}$.

II $\mathcal{X}$ is weakly Lipschitz if $\forall\, x_0 \in \mathring{X}^j$, $v_0 \in T_{x_0}\mathring{X}^j$

$\exists$ Lipschitz vector field v, defined on k^n, tangent to $\mathcal{X}$, such that $v(x_0) = v_0$.

Almost obvious: each of these properties implies loc. Lipschitz triviality of $\mathcal{X}$.

<u>Natural way</u> of extending vector fields, tangent to $\mathfrak{X}$:

suppose v is defined on X^j, tangent to $\mathfrak{X}$.

First we may extend it to a Lipschitz vector field V on $\mathbb{R}^n$ using a theorem of Kirszbraun:

If $\mathbb{R}^n \supset A \xrightarrow{\varphi} \mathbb{R}^m$

with the metric given by a scalar product

is Lipschitz with a constant L, then it extends to a L. function $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ without changing the L. constant

This is tricky if $m \geq 2$ (Federer, Geometric Measure theory).

If $m=1$, then an extension (without changing L) is for instance

$$\Phi(x) = \sup_{a\in A} \left(\varphi(a) - L\,|x-a| \right)$$

If we apply this formula to every component of

$$\varphi = (\varphi_1, \ldots, \varphi_m)$$

we get a Lipschitz extension of φ, with a Lipschitz constant $L\sqrt{m}$.

Now we define a vector field $\tilde{v}$ on X^{j+1}:

$$\tilde{v}(q) = \begin{cases} v(q) & q \in X^j \\ P_q V(q) & q \in \mathring{X}^{j+1} \end{cases}$$

where $P_q : T_q k^n \longrightarrow T_q \mathring{X}^{j+1}$ is the orthogonal projection.

<u>Definition</u> $\mathfrak{X}$ is Lipschitz $\Leftrightarrow$ $\forall v$ on X^j and every Lipschitz extension V, this formula produces a Lipschitz v.f. $\tilde{v}$.

<u>Proposition</u>: $\mathfrak{X}$ is Lipschitz $\Longleftrightarrow$

$$\exists\, C = C(\mathfrak{X})$$

$$\forall W, \quad X^j \subset \underset{\text{compact}}{W} \subset X^{j+1}$$

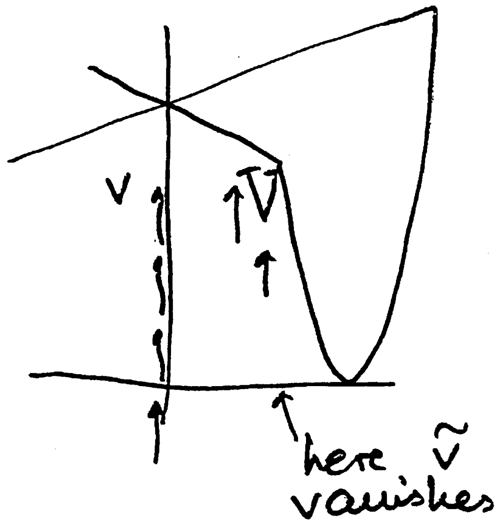
$\forall v$ — v is a Lipschitz vector field on W, tangent to $\mathfrak{X}$, with Lipschitz constant L, and $|v(x)| \leq K$ $\forall x \in W$

$\exists$ Lipschitz extension of v, tangent to $\mathfrak{X}$, defined on k^n, with a Lipschitz constant

$$C\,(K+L).$$

All this needs an obvious compactness assumption: everything lives in some compact in k^n.

Stupid and trivial example



$\overset{\circ}{X}{}^1$ = vertical axis

V extension of v

then $\tilde{v}$ is

discontinuous at 0

Example is stupid because Whitney's (A) fails.

It turns out that the Lipschitz condition is equivalent to a big system of estimates on angles between tangent planes to strata.

Definition Let $q \in \overset{\circ}{X}{}^j$. Put

$j = j_1$, $\quad q = q_{j_1}$.

A chain in $\mathcal{X}$ for q is a sequence

$j_1 > j_2 > j_3 > \cdots$ – uniquely determine

q_{j_1}, $\quad q_{j_2} \in \overset{\circ}{X}{}^{j_2}$, $q_{j_3} \in \overset{\circ}{X}{}^{j_3}$ not unique

j_2 = smallest integer for which

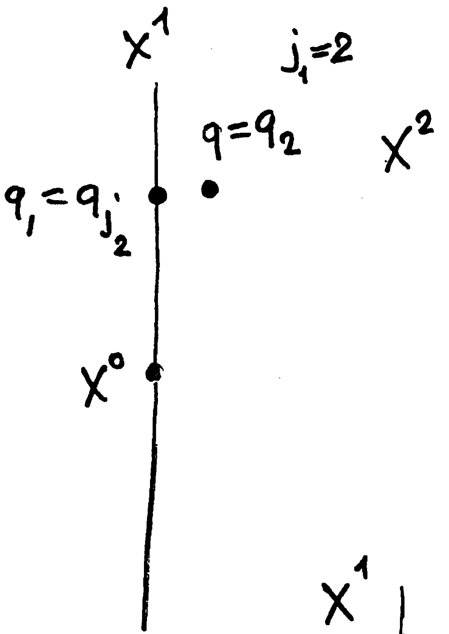
$$d(q, X^i) \geq 4\, d(q, X^{j_2})$$

$$\forall i, \quad j_2 < i < j_1$$

q_{j_2} = any point in $\mathring{X}^{j_2}$ satisfying

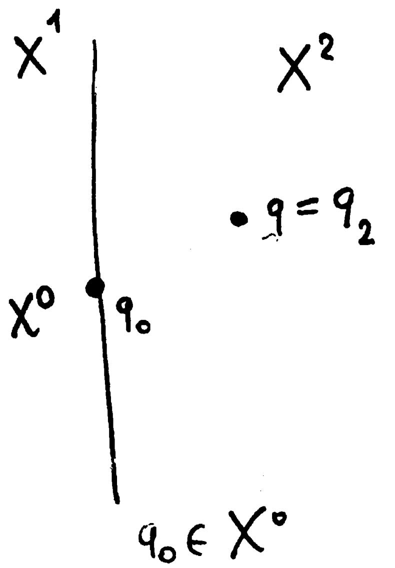
$$d(q, q_{j_2}) \leq 2\, d(q, X^{j_2})$$

(for instance the closest point to q)



distance of q to X^1 is much smaller than to X^0

so $j_2 = 1$, $q_1 = q_{j_2} \in X^1$



distance of q to X^1 is of the same order as the distance to X^0, so we put $j_2=0$

Now replace $q = q_{j_1}$ by q_{j_2} and repeat the construction; we get j_3, q_{j_3}, etc.

If $q \in \mathring{X}^j$, let – as before –

$$k^n = T_q(k^n) \xrightarrow{P_q} T_q \mathring{X}^j \subset k^n$$

orthogonal projection

We consider P_q as a matrix.

Workable definition of a Lipschitz stratification. Under the same obvious compactness assumption

$\mathcal{X} = \{X^j\}$ is Lipschitz $\iff$

$\exists C \quad \forall\ q = q_{j_1} \in \mathring{X}^{j_1}$,

every chain $q_{j_1}, q_{j_2}, \ldots$ for q

and every point $q' \in \mathring{X}^{j_1}$, such that

$$|q - q'| < \frac{1}{2}\, d(q, X^{j_1 - 1}) \quad :$$

a) $\left| (P_q - P_{q'}) P_{q_{j_2}} \cdots P_{q_{j_r}} \right| \leq$

$$\leq C \, \frac{|q - q'|}{d(q, X^{j_r - 1})}$$

b) $\left| P^{\perp}_{q_{j_1}} P_{q_{j_2}} \cdots P_{q_{j_r}} \right| \leq$

$$\leq C \, \frac{|q_{j_1} - q_{j_2}|}{d(q_{j_1}, X^{j_r - 1})} \, ,$$

for every r.

Existence Every set in any one of the mentioned classes has a Lipschitz stratification, with skeletons in the same class ; this stratification can be chosen to be compatible with any finite number of sets in the same class.

Unfortunately there is no uniqueness.

Examples & special cases.

1) in b) take $r = 2$; then

$$|P^{\perp}_{q_{j_1}} P_{q_{j_2}}| \le C \frac{|q_{j_1} - q_{j_2}|}{d(q_{j_1}, X^{j_2 - 1})} .$$

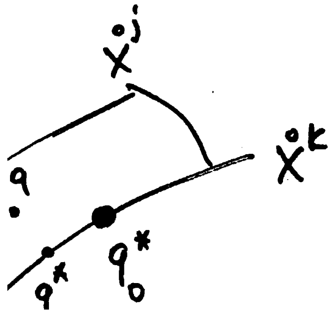
More generally, every Lipschitz stratification satisfies :

$$\forall q \in \mathring{X}^j , \quad q^* \in \mathring{X}^k , \quad k<j$$

$$\underbrace{|P^{\perp}_q P_{q^*}|}_{\uparrow \atop \approx \measuredangle (T_q \mathring{X}^j , T_{q^*} \mathring{X}^k)} \le C \frac{|q - q^*|}{d(\{q, q^*\}, X^{k-1})}$$

This can be compared to Kuo-Verdier's condition (w) :

$\forall q_0^* \in \mathring{X}^k$ there exists a nbd of q_0^* in k^n, $C_0 = C(q_0^*)$



$$\measuredangle (T_q \mathring{X}^j , T_{q^*} \mathring{X}^k) \le C_0 |q - q^*|$$

We may state it globally (get rid of q_0^*) for subanalytic sets:

$$C_0 = C(q_0^*) \leq \frac{C_1}{d(q_0^*, X^{k-1})^N}$$

for some $N \geq 0$

C_1 depends now only on $\mathcal{X}$

so Kuo–Verdier's condition can be stated:

$$\measuredangle \left(T_q \mathring{X}^j, T_{q^*} \mathring{X}^k \right) \leq C_1 \frac{|q - q^*|}{d(\{q, q^*\}, X^{k-1})^N}$$

Thus <u>every Lipschitz stratification satisfies K–V</u>, <u>with exponent $N=1$</u>

2) take $r = 1$ in a); then

$$|P_q - P_{q'}| \leq C \frac{|q - q'|}{d(q, X^{j-1})}$$

$q, q' \in \mathring{X}^j$, $|q - q'| < \cdots$

more generally, every Lipschitz stratification satisfies

$$|P_q - P_{q'}| \leq C \frac{|q - q'|}{d(\{q, q'\}, X^{j-1})}$$

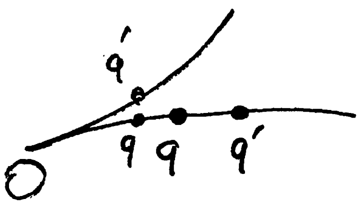
$\approx$

$T_q \mathring{X}^j, T_{q'} \mathring{X}^j)$ $\qquad q, q' \in \mathring{X}^j.$

This shows that X^{j-1} detects those places where $T_q \overset{\circ}{X}{}^j$ changes rapidly with q.

This inequality has no analogue in Whitney's conditions.

3) X = germ of a curve at O, singular at O



$X^0 = \{0\} \subset X \subset k^n$ is a Lipschitz stratification.

In fact, the inequalities reduce to

$$\gamma(T_q \overset{\circ}{X}{}^1, T_{q'} \overset{\circ}{X}{}^1) \leq C \frac{|q-q'|}{\min(|q|, |q'|)}$$

which follows easily from Puiseux

<u>Remark</u> If q, q' are on different branches (as in the green) case, then the above estimate is optimal. If q, q' are on the same branch (as in the blue case) then, for some $\alpha < 1$

$$\gamma(T_q \overset{\circ}{X}{}^1, T_{q'} \overset{\circ}{X}{}^1) \leq C \frac{|q-q'|}{\min(|q|, |q'|)^\alpha}$$

This can be generalised to all dimensions there exist Lipschitz stratifications which, in some regions, allow to replace distances in the denominators by their powers with an exponent < 1.

4) X^2 = germ of a surface in k^3.

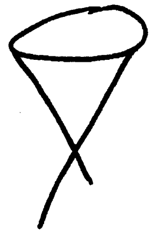
Suppose we admit to take $X^0 = \{0\}$

Then, by the curve selection lemma, the Lipschitz conditions reduce to

$$\sphericalangle(T_q \mathring{X}^2, T_{q'} \mathring{X}^2) \le C \frac{|q-q'|}{d(\{q,q'\}, X^1)}$$

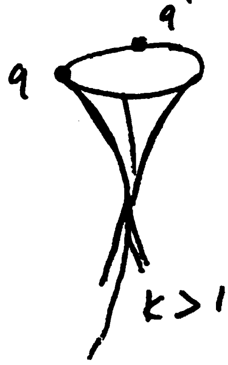
a) $k = \mathbb{R}$

$$X \quad : \quad x^2 + y^2 = z^{2k}$$



$k=1$

here we can take $X^1 = X^0$



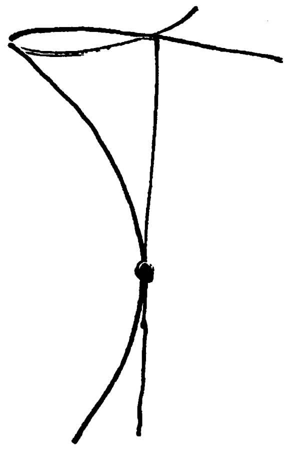
X^1

X^1 must contain something more since $\sphericalangle(T_q \mathring{X}^2, T_{q'} \mathring{X}^2) = 90°$

$|q-q'| = |q|^k = |q'|^k$

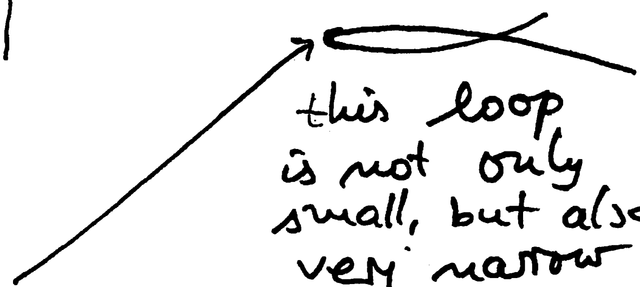
So there is no uniqueness of X'

b)



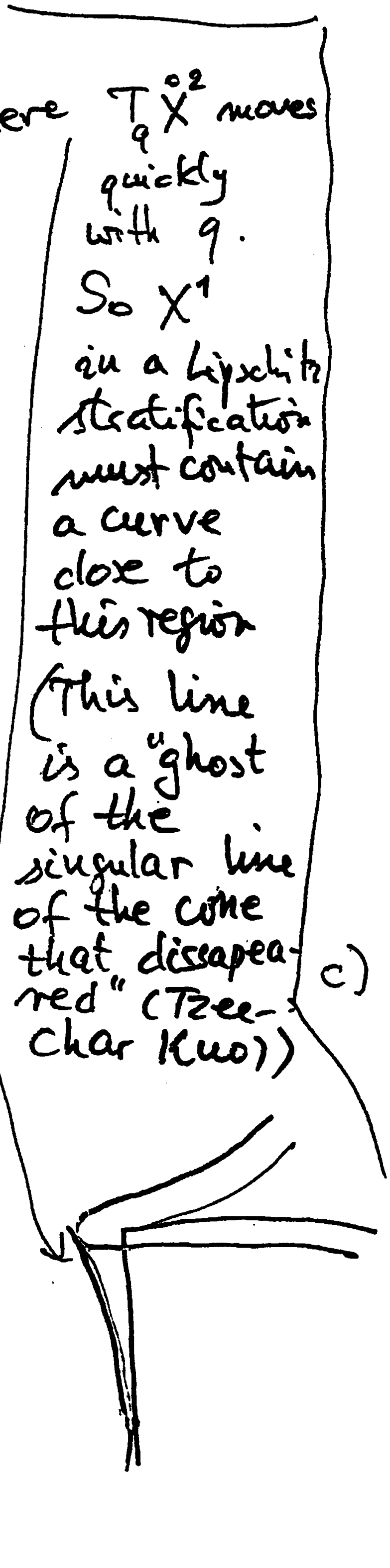
$X: y^2 = x^3 + z^2x^2$

$X \cap \{z = \text{const} \overset{\shortparallel}{z_0}\}$:



this loop is not only small, but also very narrow

here $T_q \overset{\circ}{X}{}^2$ changes very rapidly with q. So X' must contain a curve lying very close to this region.



here $T_q \overset{\circ}{X}{}^2$ moves quickly with q. So X^1 in a Lipschitz stratification must contain a curve close to this region (This line is a "ghost of the singular line of the cone that dissapeared" (Tzee-Char Kuo))

c) Take a homogenous eq. in k^3 which determines a cone with a singular line, e.g

$$zy^2 - x^3 = 0$$

Perturb it to get an isolated singularity, e.g.

$X: zy^2 - x^3 + z^4 = 0$

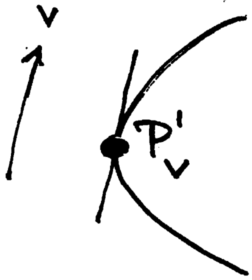
d) in the complex case, in any dimensions, there is a characterisation of all Y's that satisfy, for a given X,

$$\measuredangle\,(T_q X,\ T_{q'} X) \leq C \frac{|q-q'|}{d(\{q,q'\},Y)}$$

In particular, if $X^2 \subset \mathbb{C}^3$ then for $Y = X^1$ one can take a <u>generic polar curve</u> $\cup\ X_{sing}$:

Let $F=0$ be a reduced equation of X. If $v \in \mathbb{C}^3$ is a vector, then the polar curve $P_v = P_v(X)$ is defined as

$$\overline{X_{reg} \cap \{\partial_v F = 0\}} =$$
$$= \text{closure of } \{q \in X^2_{reg} : v \in T_q X^2\}$$



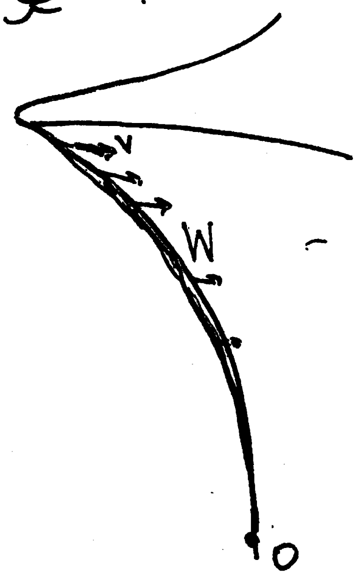
Ex.:

generic polar curves live in this region

e) Suppose $X^2 \subset k^3$ has an isolated singularity at 0. Then $\mathfrak{X}: \{0\} \subset X^2$ is Whitney and weakly Lipschitz. But not every Lipschitz vector field v, defined on a W, tangent to $\mathfrak{X}$,

$$\{0\} \subset W \subset X^2,$$

has a Lipschitz extension, tangent to $\mathfrak{X}$:



– this curve may be smooth

$$|v(q)| = |q|$$

D. Juniati, D. Trotman, G. Valette:

$$L \Leftrightarrow L^*$$

Lipschitz estimate

L^*: Take any $q \in \overset{\circ}{X}{}^i$ and let

W^m be a smooth mfd,

$$W \supset \overset{\circ}{X}{}^i$$

such that

$T_q W$ is generic in the space of all m-planes containing $T_q \overset{\circ}{X}{}^i$.

Then $\{W \cap X^j\}$ is a Lipschitz stratification in a nbd of q.

(such a condition – stability under generic intersections – was required by B. Teissier for a good equisingularity condition).

Construction of a Lipschitz stratification / $\mathbb{C}$

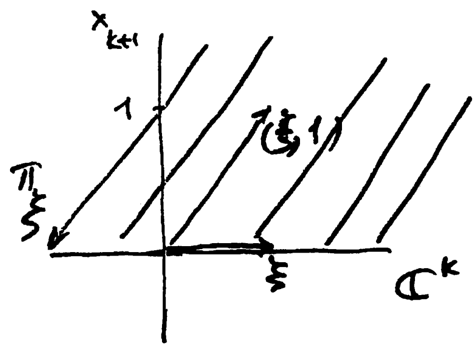
$X^d \subset \mathbb{C}^n$ analytic.

Notation: Consider the flag.

$$\underset{\overset{\shortparallel}{\mathbb{C}^1}}{\mathbb{C}^1_{x_1}} \subset \underset{\overset{\shortparallel}{\mathbb{C}^2}}{\mathbb{C}^2_{x_1 x_2}} \subset \cdots \subset \mathbb{C}^n .$$

If $\xi \in \mathbb{C}^k$, then ξ determines a projection

$$\pi_\xi : \mathbb{C}^{k+1} \longrightarrow \mathbb{C}^k$$



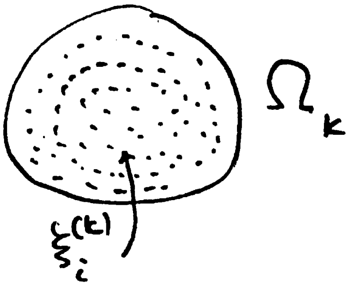
$(\xi, 1) \in \ker \pi_\xi$

In every $\mathbb{C}^k$ we choose any open $\Omega_k \subset \mathbb{C}^k$ and, for $\varepsilon << 1$, a <u>finite ε-net</u> in Ω_k

$$\{ \xi_1^{(k)}, \ldots, \xi_N^{(k)} \} \subset \Omega_k$$

(i.e. every $w \in \Omega_k$ lies in a distance $< \varepsilon$ from some $\xi_i^{(k)}$).

Put $\pi_i^{(k)} = \pi_{\xi_i^{(k)}}$.



Thus for every k, we have a number of projections

$$\pi_i^{(k)} : \mathbb{C}^{k+1} \longrightarrow \mathbb{C}^k$$

If $\ell > k$, we may consider all compositions

$$\begin{tikzcd} \mathbb{C}^{\ell} \arrow[r, "\pi^{(\ell-1)}_{i_1}"] \arrow[rrr, bend right, "\pi_I"'] & \mathbb{C}^{\ell-1} \arrow[r, "\pi^{(\ell-2)}_{i_2}"] & \cdots \arrow[r] & \mathbb{C}^{k} \end{tikzcd}$$

<u>Construction of X^{d-1}</u> :

Take all $\mathbb{C}^n \xrightarrow{\pi_I} \mathbb{C}^d$. For every I, consider the variety

$$Y_I = \pi_I(X_{sing}) \cup$$

$$\text{critical values of } \pi_I \big|_{X_{reg}}$$

(after a proper choice of Ω_i's we may assume that $\pi_I \big|_X$ is proper, so Y_I is an analytic set.

Let

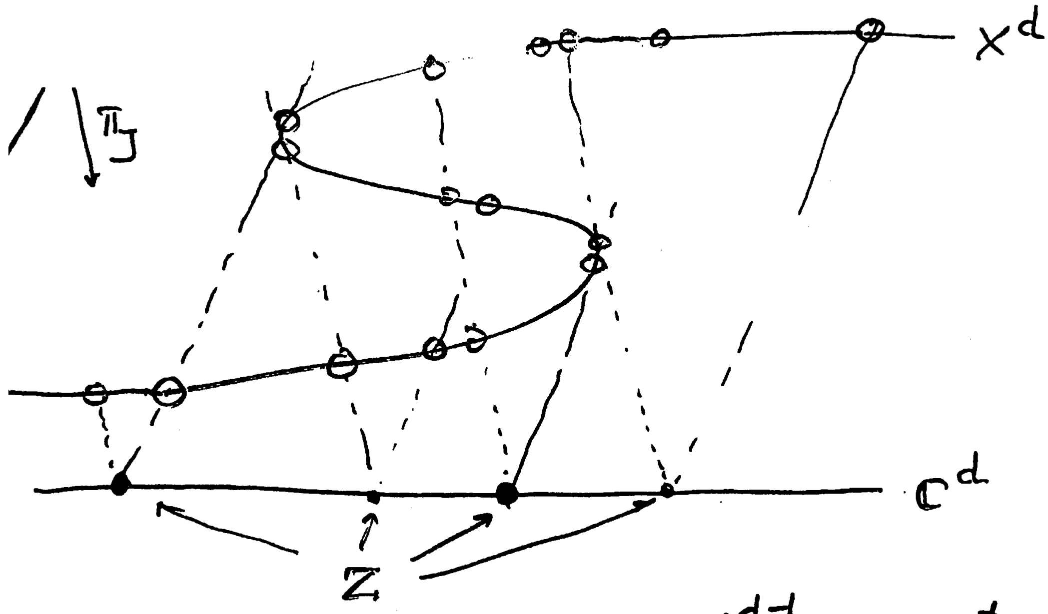
$$Z \subset \mathbb{C}^d$$

be any hypersurface containing all Y_I:

$$Z \supset \bigcup_I Y_I .$$

Put

$$X^{d-1} = \bigcup_I \pi_I^{-1}(Z) \cap X^d$$



X^{d-1} consists on this picture of all points o o

Inductive step – similar. Suppose we have already X^j. Replace X^d by X^j and repeat the construction.

Clearly this construction is so clumsy and depends on so many choices, that it should be regarded as an existence proof only.

$\mathbb{R}$: semianalytic.

<u>Corollary</u> (interesting mainly in $\mathbb{C}$):

$X \downarrow T$ family of germs of analytic sets.

Then the complement of

$\{t$: for all t' sufficiently close to t the fibers $X_{t'}$ have the same Lipsch type as $X_t\}$ is analytic.

Low-dimensional examples of Lipschitz-equisingular families of germs

1) $\dim X_t = 1$, $\mathbb{C}$

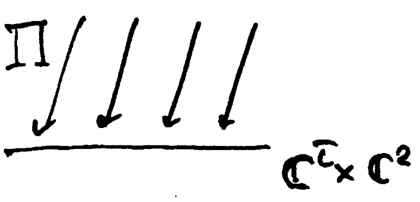
$$\begin{array}{ccc} X & \subset & \mathbb{C}^\tau \times \mathbb{C}^n \\ \downarrow & & \\ T & & \end{array}$$

T - smooth so $\approx \mathbb{C}^\tau$

a) $n=2$ (family of plane curves)

Lipschitz equisingularity $\Leftrightarrow$ topological equisingularity

b) $n \geq 3$. Take a generic projection



$$\mathbb{C}^\tau \times \mathbb{C}^n \xrightarrow{\Pi} \mathbb{C}^\tau \times \mathbb{C}^2$$

Then : $\begin{array}{c} X \\ \downarrow \\ T \end{array}$ is Lipschitz equisingular

$\Longleftrightarrow$

$\begin{array}{c} \Pi(X) \\ \downarrow \\ T \end{array}$ is Lipschitz equisingular

(see for example the case
$X^1_t = A^1_t \cup B^1_t \subset \mathbb{C}^3$)

<u>Remark</u> A. Fernandes proved:
consider a germ of a curve $X' \subset \mathbb{C}^2$;
on X' put the induced metric from $\mathbb{C}^2$.
Let Y' be another germ, also with this metric.

Then the metric spaces X', Y' are bi-Lipschitz homeomorphic $\Leftrightarrow$ X', Y' have the same topological type (i.e.
$$\exists \ (\mathbb{C}^2, X', 0) \xrightarrow[\text{homeom.}]{} (\mathbb{C}^2, Y', 0)$$

2) For o-minimal structures moduli exist with respect to Lipschitz equisingularity:
the Lipschitz types of
$$X_t : \quad y=0 \ \text{ or } \ y = x^t, \ x>0$$

X_t

are all different $(t>1)$

It is conjectured that Lipschitz stratification exist in every <u>bounded</u> o-minimal structure.

3) germs of surfaces $X_t^2 \subset \mathbb{C}^3$, $T = \mathbb{C}^\tau$.

Zariski's equisingularity :

take generic projections.

$$\mathbb{C}^3 \times T \longrightarrow \mathbb{C}^2 \times T \longrightarrow \mathbb{C}^1 \times T$$

and take coordinates x_1, x_2, x_3 so that these projections are parallel to the axes x_3, x_2.

Let

$$X : F = 0$$

be a reduced equation of X.

We may assume that F is a distinguished polynomial in x_3.

Let $F_1(t; x_1, x_2)$ be its discriminant and

$G(t; x_1, x_2) = 0$ — the reduced equation of $F_1 = 0$

Again G may be supposed to be a distinguished polynomial in x_2.

Let $\delta(t; x_1)$ be its discriminant.

<u>Zariski's equisingularity</u> :

$$\delta(t; x_1) = x_1^p \cdot \text{unit} \quad \text{(for some } p\text{)}$$

In all dimensions:

Varchenko : Z-equisingularity $\Rightarrow$ top. equising.

Briançon : Z-equisingularity $\Rightarrow$ Whitney equisingularity

For surfaces in $\mathbb{C}^3$:

Z equisingularity $\Longleftrightarrow$ Lipschitz equisingularity

The "reason" for that:

$\Leftarrow$

I. we look for "non-trivial" invariants of Lipschitz equisingularity.

Take X_{t_0} and its generic polar curve

$$P_v(X_{t_0}) = \overline{\{q \in X_{t_0, reg} : v \in T_q X_{t_0}\}}$$

For simplicity suppose it is irreducible and consider its Puiseux expansion:

$$x' = \lambda(x_1^{1/m}) + x_1^{r} \underbrace{A(v)}_{\not\equiv \text{const}} + \cdots$$

$x' = (x_2, x_3)$

$\lambda = (\lambda_2, \lambda_3)$

$A = (A_2, A_3)$

after a coordinate change we may assume that there are no terms of degree < 1

$r = r(t_0) \in \mathbb{Q}$ = order of contact of two generic polar curves

<u>Fact</u> Lipschitz triviality $\Rightarrow$ $r(t_0)$ doesn't depend on t_0.

This follows from a very weak form of preservation of polar curves under the trivialising homeomorphism h_t:

if $q \in P_V(X_{t_0})$ and $\varepsilon \ll 1$,

then for t sufficiently close to t_0

$$h_t(q) \in P_{V_{t,q}}(X_t)$$

where $\measuredangle(V, V_{t,q}) < \varepsilon$.

Using this one shows that LE $\Rightarrow$ ZE

<u>Remark</u> One of the examples of Briançon – Speder provide a family of surfaces in $\mathbb{C}^3$, Whitney equisingular, such that $r(t) \neq$ const (by a direct easy calculation). This gives an example in codimension 1, over $\mathbb{C}$, of a family Whitney equisingular, but not Lipschitz eq.

$\Rightarrow$ Assuming the family Z-equisingular, one constructs, repeating the clumsy construction of a Lipschitz stratification (with some simplifications – dimension is small!) having $\mathbb{C}^\tau$ as the smallest stratum.

4) Lipschitz equivalence of <u>functions</u>

f, g germs at 0 of function on $\mathbb{C}^n$

$f \sim g \iff g = f \circ H$

germ of a bi-Lipschitz homeomorphism

Here <u>moduli exist</u>

Example: $f_t = x^3 - 3t^2 x y^4 + y^6$

If $t, t', 1 \pm 2t^3, 1 \pm 2t'^3$ are all $\neq 0$ and

$t' \neq \varepsilon t, \quad \varepsilon^3 = \pm 1$

then

$f_t \not\sim f_{t'}$

Idea of proof $f = f_t$, $g = f_{t'}$

I. suppose that t' is close to t and the homeomorphism H is of the form

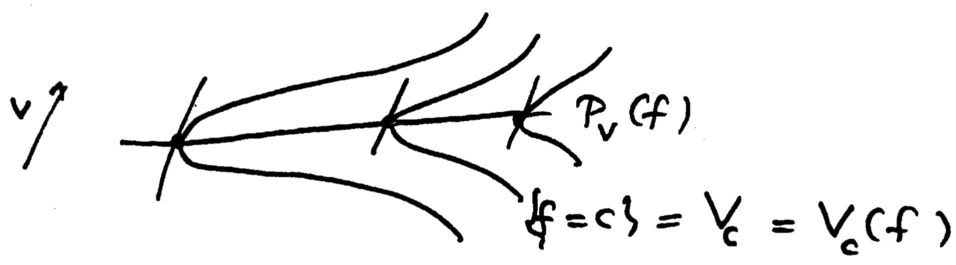
$$H = id + \varphi(x)$$

Lipschitz constant for φ is small.

Then the idea is based also on polar curves.

Polar curves of f:

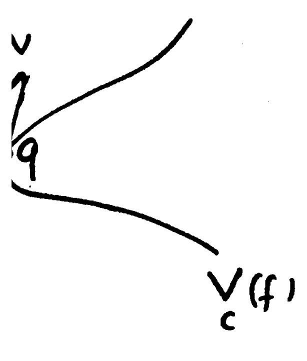
$$P_v(f) \quad : \quad \partial_v f = 0 \qquad v \in \mathbb{C}^2$$



thus $P_v(f) = \bigcup_c P_v(V_c)$.

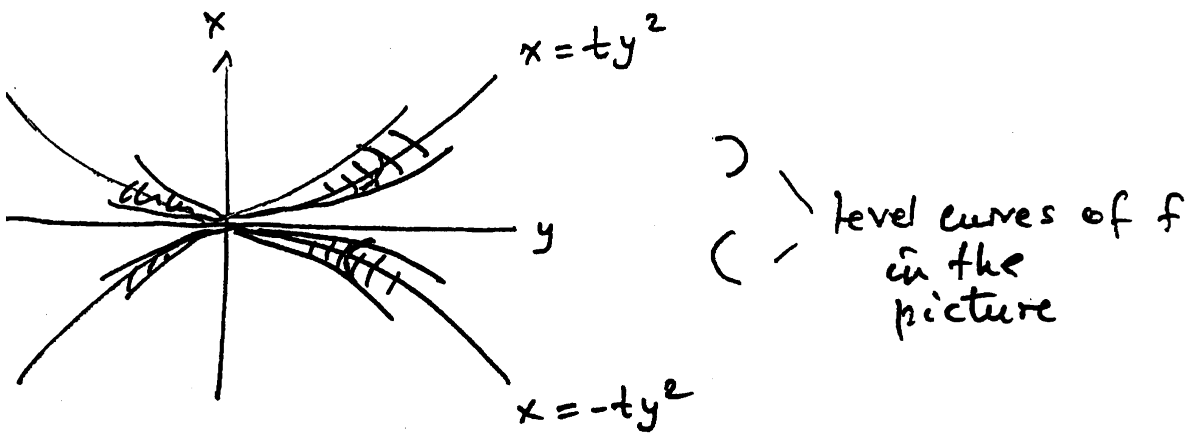
Easy to prove: if $g = f \circ H$, H - as above,

and $q \in V_c(f) \cap P_v(f)$



then $H(q) \in V_c(g) \cap P_{v'}(g)$

where $\measuredangle(v', v)$ small

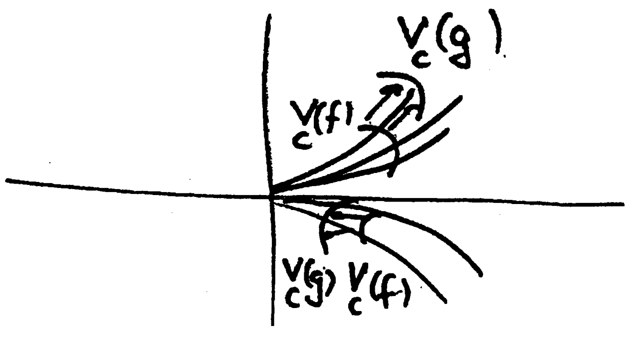


$P_v(f)$ (v close to $\uparrow$) lie in the shadowed region

Similarly for $P_v(g)$.

So H would have to move the shadowed region for f into the shadowed region for g.

Looking at level curves of f and g in these regions it is easy to prove that H couldn't be Lipschitz.



II. If we don't suppose H ~~to~~ be in a special form, we replace in the argument polar curves by "places where curvature of level curves is big"

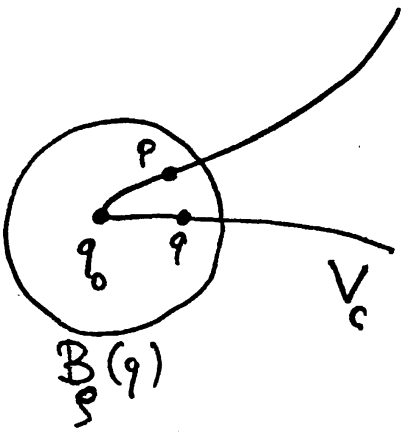
Let $q_0 \in V_c(f) = f^{-1}(c) = V_c$

Let $\rho > 0$.

For $K >> 0$ we define, for $p, q \in V_c \cap B_\rho(q_0)$

$d_{q_0, \rho, K}(p,q) =$

= internal in $V_c \cap B_{K\rho}(q_0)$ length distance between p, q



= inf lengths of curves lying in $V_c \cap B_{K\rho}(q_0)$ joining p and q

$$\varphi(q_0, \rho, K) = \sup_{p,q} \frac{d_{q_0,\rho,K}(p,q)}{|p-q|}$$

here internal distance between p, q is $>> |p-q|$,

so $\varphi(q_0, \rho, K)$ is big

To make φ monotone with respect to ρ, replace it by

$$\psi(q_0, \rho, K) = \sup_{\rho' \leq \rho} \varphi(q_0, \rho', K)$$

Places on level curves where curvature is big : sets defined by

$$Y(\rho, K, A) = \{ q_0 : \psi(q_0, \rho, K) > A \}$$

Let Y' be defined in the same way for g.

If $g = f \circ H$, H Lipschitz with a Lipschitz constant C,

then

$$Y'\left(\frac{\rho}{C}, K, AC^2\right) \subset H\left(Y(\rho, K, A)\right) \subset Y'\left(C\rho, K, \frac{A}{C^2}\right)$$

One replaces generic polar curves by sets $Y(\rho, K, A)$, $\rho = |q_0|^{1+\delta}$, $0 < \delta < 1$.

<u>Degenerations on curves</u> (more natural $\mathbb{R}$)

$$\begin{array}{ccc} X & \subset & \mathbb{R}^m_t \times \mathbb{B}^n_x \\ \downarrow & & \\ T & \subset & \mathbb{R}^m_t \end{array}$$

$\mathbb{B}^n_x$ closed unit ball

Curves in T : germs of continuous maps $[0,\varepsilon] \xrightarrow{p} \mathbb{R}^m_t$ such that

$$p(\mu) \in T \quad \text{for } \mu > 0$$

We say that $p \sim_{\mathrm{rel}\,X} q$ (p, q – curves)

if $\exists$

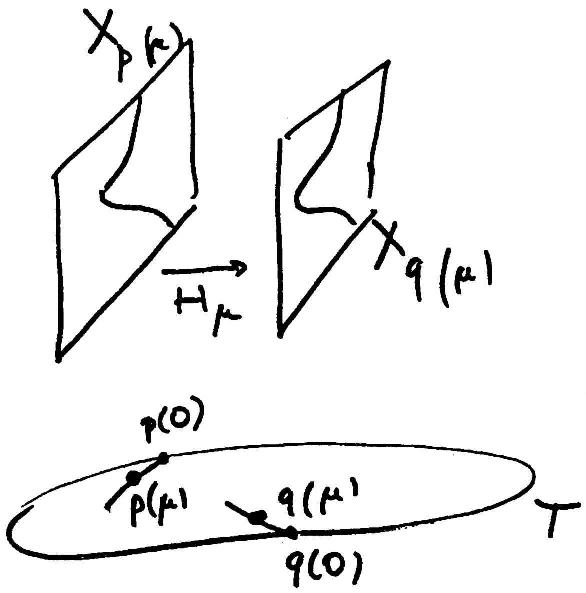
$$(0,\varepsilon] \times \mathbb{R}^n_x \xrightarrow{H} \mathbb{R}^n_x$$

$$H(\mu, x) = H_\mu(x)$$

continuous as function of (μ, x) such that $\forall \mu > 0$

$$(\mathbb{R}^n_x, X_{p(\mu)}) \xrightarrow{H_\mu} (\mathbb{R}^n_x, X_{q(\mu)})$$

is bi-Lipschitz with a Lipschitz constant independent of μ



So $X_{p(\mu)}$, $X_{q(\mu)}$ degenerate as $\mu \to 0$, but they degenerate in the same way (from the Lipschitz point of view)

Suppose now X, T are semialgebraic. Let us bound ourselves to semialgebraic curves of bounded complexity

(i.e. graphs of these curves $p(\mu)$ are semialgebraic, and can be described by at most N polynomial equations & inequalities of degree $\leq N$)

The number of equivalence classes $\sim_{\mathrm{rel}\ X}$ is finite.

In the subanalytic case: take a suitable stratification of $\mathbb{R}^m_t$ and restrict to curves $p(\mu)$ such that their orders of contact with strata (being rational numbers) are bounded, and also their denominators are bounded:

The main lemma is a contribution toward undestanding the question of lifting Lipschitz vector fields: Let $\begin{array}{c} X \\ \downarrow \\ T \end{array}$; does thee exist a stratification of T such that evey Lipschitz vector field on T, tangent to it, lifts to a Lipschitz vector field, whose flow preserves X? (Probably no).

But there is a slightly weaker result.
The flow of v we denote by χ_λ^v

$$\left.\begin{array}{l}\dfrac{dx}{d\lambda} = v(x(\lambda)) \\ x(0) = x_0\end{array}\right. \iff x(\lambda) = \chi_\lambda^v(x_0).$$

<u>Main lemma</u> There exists a stratification of $\mathbb{R}^m_t$, compatible with T, such that for any Lipschitz vector field v, tangent to it, and every stratum $\mathring{T}^j$, there exists

$$\mathring{T}^j \times \mathbb{R}^n_x \xrightarrow{H_\lambda} \mathring{T}^j \times \mathbb{R}^n_x$$

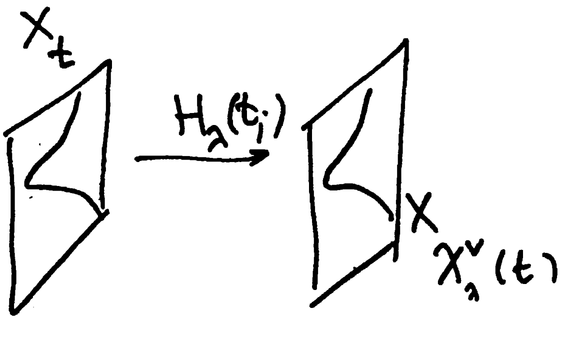
$\lambda \in [0,1]$

continuous

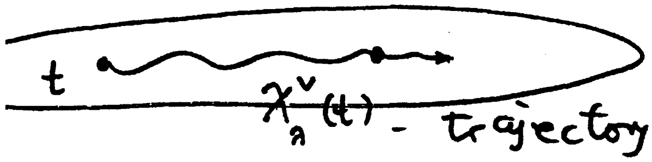
$\forall t \in \mathring{T}^j$

$$(\mathbb{R}^n, X_t) \xrightarrow{H_\lambda(t,\cdot)} (\mathbb{R}^n, X_{\chi_\lambda^v(t)})$$

bi-Lipschitz, Lipschitz constant independent of λ, t



$\mathring{T}^j$



$\chi_\lambda^v(t)$ - trajectory of v, time restricted to $[0,1]$