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Tangentially degenerate fronts and their singularities: A survey.

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1 Introduction.

Tangentially degenerate submanifolds in projective spaces are studied from various aspects; differetial geometry, algebraic geometry, singularity theory and so on. In particular, P. Griffith and J. Harris [18] and A. Akivis and V.V. Goldberg [2][3] gave the description of tangentially degenerate submanifolds in detail.

Looking at unit normal vectors or tangent planes to space surfaces is the most fundamental method in differential geometry initiated by C.F. Gauss [16]. He, in particular, considered the class of tangentially degenerate surfaces by means of his (Gauss) mappings.

Naturally we are led to consider tangentially submanifolds in Euclidean spaces, or more naturally in projective spaces by means of Gauss mappings. One of important classes of tangentially degenerate submanifolds, then, consists of submanifolds with degenerate Gauss mappings. Another important class consists of hypersurfaces with degenerate projective dual. The tangential degeneracy of a hypersurface can be described by the deneneracy of its projective dual; the variety, in the dual projective space, consisting of tangent hyperplanes to the hypersurface. Moreover we notice that, also for submanifolds of codimension greater than two, the tangential degeneracy can be described by means of projective duality. This means that the Gauss

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mapping is degenerate, then the projective dual is necessarily degenerate [18]. Thus among tangentially degenerate submanifolds, we study, in this paper, submanifolds with degenerate projective duals, possibly with singularities.

The notions of projective duality and of incidence relation play the central role in projective geometry. In this survey article, on particular, we re-formulate the study on submanifolds with degenerate Gauss mappings using the incidence relation in projective geometry via contact geometry. Also we introduce the notion of "frontal mappings"and discuss the relations with "poly-symplectic geometry". Moreover the projective duality is generalized to "Grassmann duality"in very natural way.

2 Degenerate and bi-degenerate Legendre submanifolds.

We denote by $\mathbf{R}P^{n+1} = P(\mathbf{R}^{n+2})$ the n-dimensional projective space and by $\mathbf{R}P^{n+1*} = P((\mathbf{R}^{n+2})^*)$ the n-dimensional dual projective space. Here $(\mathbf{R}^{n+2})^*$ means the dual vector space to $\mathbf{R}^{n+2}$.

Any submanifold $M^m \subseteq \mathbf{R}P^{n+1}$ lifts to a Legendre submanifold $\widetilde{M}$ of the manifold $P(T^*\mathbf{R}P^{n+1})$ of contact elements (tangent hyperplanes) of $\mathbf{R}P^{n+1}$. Actually $\widetilde{M}$ is defined to be the projective conormal bundle $P(T^*_M\mathbf{R}P^{n+1})$ of M. Here $T^*_M\mathbf{R}P^{n+1} \subseteq T^*\mathbf{R}P^{n+1}$ is the conormal bundle of M in $\mathbf{R}P^{n+1}$. Note that, independently of m, the dimension of the Legendre lifting $\widetilde{M}$ is equal to n. In general the image of a Legendre submanifold by the projection $\pi : P(T^*\mathbf{R}P^{n+1}) \to \mathbf{R}P^{n+1}$ is called a *wave front* or simply a *front*. Therefore any submanifold of $\mathbf{R}P^{n+1}$ can be regarded as a front. This is not the case just only for $\mathbf{R}P^{n+1}$: any submanifold M of any manifold X lifts to a Legendre submanifold $P(T^*_MX)$ of $P(T^*X)$.

The special feature of $\mathbf{R}P^{n+1}$ is $P(T^*\mathbf{R}P^{n+1})$ has natural double Legendre fibration:

$$\mathbf{R}P^{n+1} \longleftarrow P(T^*\mathbf{R}P^{n+1}) \longrightarrow \mathbf{R}P^{n+1*},$$

to $\mathbf{R}P^{n+1}$ and to the dual projective space $\mathbf{R}P^{n+1*}$.

Inversing the process, first we can consider Legendre submanifolds in the manifold of cotact elements $P(T^*\mathbf{R}P^{n+1})$, the projective cotangent bundle, then second study their projections by $\pi : P(T^*\mathbf{R}P^{n+1}) \to \mathbf{R}P^{n+1}$ and by $\pi^* : P(T^*\mathbf{R}P^{n+1}) \to \mathbf{R}P^{n+1*}$.

The above constructions can be described in term of projective duality. Set

$$\widetilde{Q} := \{(x, y) \in \mathbf{R}^{n+2} \times (\mathbf{R}^{n+2})^* \mid x \cdot y = 0\},$$

where $x \cdot y$ denotes the canonical pairing of elements $x \in \mathbf{R}^{n+2}$ and $y \in (\mathbf{R}^{n+2})*$.

On $\widetilde{Q}$, we have $0 = d(x \cdot y) = dx \cdot y + x \cdot dy$. Moreover we set

$$Q := \{([x],[y]) \in \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*} \mid x \cdot y = 0\},$$

the manifold of incident pairs or *the incidence relation.* Then Q is of dimension $2n+1$ and Q has the contact structure

$$D := \{dx \cdot y = 0\} = \{x \cdot dy = 0\} \subset TQ.$$

Namely, a tangent vector $(u, v) \in T_{([x],[y])}Q$ belongs to the contact distribution D if and only if $u \cdot y = 0$ and, if and only if $x \cdot v = 0$.

The projection $\pi : Q \to \mathbf{R}P^{n+1}$ (resp. $\pi^* : Q \to \mathbf{R}P^{n+1*}$) indentify Q, as contact manifolds, with the fiberwise projectivization $P(T^*\mathbf{R}P^{n+1})$ of $T^*\mathbf{R}P^{n+1}$ (resp. $P(T^*\mathbf{R}P^{n+1*})$ of $T^*\mathbf{R}P^{n+1*}$).

A submanifold $L \subset Q$ is called *a Legendre submanifold* if L is an integral submanifold of the contact distribution D of dimension n. The integrality condition means that $TL \subset D|_L$.

Now, to any submanifold M of $\mathbf{R}P^{n+1}$ of any codimension m, there corresponds the Legendre submanifold in Q:

$$\widetilde{M} := \{([x],[y]) \in Q[x] \in M, \mid (T_x\widehat{M}) \cdot y = 0\},$$

which is called *the Legendre lifting* of M. Here $\widehat{M} \subseteq \mathbf{R}^{n+2} \setminus \{0\}$ is the corresponding $(m+1)$-dimensional submanifold to $M \subseteq \mathbf{R}P^{n+1}$.

Also to any submanifold N of $\mathbf{R}P^{n+1*}$ of any codimension m^*, there corresponds the Legendre submanifold in Q:

$$\widetilde{N} := \{([x],[y]) \in Q[y] \in N, \mid (x \cdot T_y\widehat{N}) = 0\},$$

which is also called *the Legendre lifting* of N. Here $\widehat{N} \subseteq \mathbf{R}^{n+2*} \setminus \{0\}$ is the corresponding (m^*+1)-dimensional submanifold to $N \subseteq \mathbf{R}P^{n+1*}$.

A *front* of L in $\mathbf{R}P^{n+1}$ (resp. in $\mathbf{R}P^{n+1*}$) is, by definition, the image of L by π (resp. π^*).

Thus any submanifold of $\mathbf{R}P^{n+1}$ (resp. $\mathbf{R}P^{n+1*}$) can be regarded as a front in $\mathbf{R}P^{n+1}$ (resp. in $\mathbf{R}P^{n+1*}$) of a Legendre submanifold of Q. However a front may have singularities, which also we are interested in.

Let $L \subset Q$ be a Legendre submanifold. Set

$$m := \sup\{\mathrm{rank}_q(d(\pi|_L) : T_qL \to T_{\pi(q)}\mathbf{R}P^{n+1}) \mid q \in L\}.$$

Moreover set

$$m^* := \sup\{\mathrm{rank}_q(d(\pi^*|_L) : T_qL \to T_{\pi^*(q)}\mathbf{R}P^{n+1*}) \mid q \in L\}.$$

We call L *degenerate* if $m^* < n$. Moreover we call L *bi-degenerate* if $m < n$ and $m^* < n$.

Now we call a font in $\mathbf{R}P^{n+1}$ (resp. in $\mathbf{R}P^{n+1*}$) *tangentially degenerate* or briefly *degenerate* if $m^* < n$ (resp. $m < n$). Moreover we call a front in $\mathbf{R}P^{n+1}$ (resp. in $\mathbf{R}P^{n+1*}$) *tangentially bi-degenerate* or briefly *bi-degenerate* if both $m^* < n$ and $m < n$.

Example 2.1 Let n, m be integers with $0 \leq m \leq n$ Let $M = \mathbf{R}P^m \subset \mathbf{R}P^{n+1}$ be a projective subspace of dimension m. We denote by $M^\vee \subset \mathbf{R}P^{n+1*}$ the projective dual to M; $M^\vee$ consists of hyperplanes containing M, and $M^\vee$ is a projective subspace of $\mathbf{R}P^{n+1*}$ of dimension $n - m$. Set $L := M \times M^\vee \subset Q$. Then L is the Legendre lifting of M. Then L is degenerate if and only if $0 < m \leq n$. Moreover L is bi-degenerate if and only if $0 < m < n$.

Example 2.2 Let $M^m \subset \mathbf{R}P^{n+1}$, $0 \leq m \leq n$, be a submanifold with degenerate Gauss mapping. Recall that the Gauss mapping $\gamma : M \to \mathrm{Gr}(m+1, \mathbf{R}^{n+2})$ is defined by $\gamma([x]) := T_x\widehat{M}, ([x] \in M)$. Then the required condition is that rank$\gamma < m$. Thus we are assuming $0 < m \leq n$. Lots of examples have been found of submanifolds with degenerate Gauss mappings [27][29]. Let L be the Legendre lifting of M. We have $M = \pi(L)$ and $\pi^*(L) =: M^\vee \subset \mathbf{R}P^{n+1*}$ is the projective dual of M. Then L is degenerate. Moreover L is bi-degenerate if $m < n$. In other words, a submanifold with degenerate Gauss mapping is a degenerate front. Moreover if it is of codimension ≥ 2, then it is a bi-degenerate front.

Example 2.3 Let $W \subset \mathbf{C}P^n$ be a complex submanifold of complex dimension $\ell \leq n$. Consider the Hopf fibration $h : \mathbf{R}P^{2n+1} \to \mathbf{C}P^n$. Set $M := h^{-1}N \subset \mathbf{R}P^{2n+1}$. Then M is a real submanifold of real dimension $2\ell + 1$ with degenerate Gauss mapping. Let $L := \widetilde{M} \subset Q \subset \mathbf{R}P^{2n+1} \times \mathbf{R}P^{2n+1*}$ be the Legendre lifting of M. Then L is bi-degenerate. In fact $\pi^*(L) = \overline{h^*}^{-1} W^\vee$, for the complex projective dual $W^\vee \subset \mathbf{C}P^{n*}$ and the Hopf fibration $\mathbf{R}P^{2n+1*} \to \mathbf{C}P^{n*}$. Now suppose W is a non-singular complex quadric hypersurface in $\mathbf{C}P^n$. Then $W^\vee$ is a non-singular complex quadric hypersurface in $\mathbf{C}P^{n*}$. Then both $\pi|_L$ and $\pi^*|_L$ are of constant rank $2n-1$. In this example $m = 2n - 1 = m^*$ and $m + m^* - 2n = 2n - 2$. If $n = 2$, then $m = 3 = m^*$, $\dim L = 4$ and $m + m^* - 4 = 2$.

In the last example, we have observed the Legendre submanifold has the constant rank projections $\pi|_L$ and $\pi^*|_L$ so that $\pi(L)$ and $\pi^*(L)$ are both non-singular degenerate

3 Symmetric Ferus inequalities for degenerate Legendre submanifolds.

In this section, we give a formulation of Ferus inequality [14][15] in projective and symmetric form.

First we recall the Ferus inequality for submanifolds in a sphere or in a projective space with degenerate Gauss mappings [14][15]. See also [7][27].

Let $M^m \subset \mathbf{R}P^{n+1}$ be a submanifold with degenerate Gauss mapping. See Example 2.2. Set $r = \text{rank}\gamma$, the rank of Gauss mapping γ of M.

First recall the *Adams number* $A(k)$ for $k \in \mathbf{N}$ from algebraic topology. The number $A(k)$ is by definition the maximal number of independent vector fields over the sphere S^{k-1}. For example, since Euler number of S^2 is not equal to zero, there does not exist nowhere vanishing vector field over S^2, so we have $A(3) = 0$. Since S^1 and S^3 are parallelizable, namely, TS^1 and TS^3 are trivial, we have $A(2) = 1$ and $A(4) = 3$. One of great results in algebraic topology (or homotopy theory), is the following surprisingly simple formula due to Adams:

$$A((2b+1)2^{c+4d}) = 2^c + 8d - 1, (b, c, d \in \mathbf{N} \cup \{0\}, 0 \leq c \leq 3).$$

In particular $A(k)$ depends only on the exponent to 2 in the primary decomposition of k.

Then define the *Ferus number* for $m \in \mathbf{N}$ by

$$F(m) := \min\{k \in \mathbf{N} \mid A(k) + k \geq m\}.$$

Then Ferus showed, in the framework of Riemannian geometry, the following crucial result:

Theorem 3.1 *Let M^m be a closed and immersed submanifold of $\mathbf{R}P^{n+1}$ with $r = \text{rank}(\gamma) < m$. Then $r < F(m)$ implies $r = 0$. In particular, if M is a closed and connected submanifold of $\mathbf{R}P^{n+1}$ and M is not a projective subspace, then $F(m) \leq r$.*

We write down $F(m)$, for smaller m:
$F(1) = 1, F(2) = 2, F(3) = 2, F(4) = 4, F(5) = 4, F(6) = 4, F(7) = 4, F(8) = 8,$
$F(9) = 8, F(10) = 8, F(11) = 8, F(12) = 8, F(13) = 8, F(14) = 8, F(15) = 8,$
$F(m) = 16, (16 \leq m \leq 24), F(m) = 24, (25 \leq m \leq 31), F(m) = 32, (32 \leq m \leq 41),$
$F(m) = 40, (42 \leq m \leq 47), F(m) = 48, (48 \leq m \leq 56), F(m) = 56, (57 \leq m \leq 63),$
$F(m) = 64, (64 \leq m \leq 75), F(m) = 72, (76 \leq m \leq 79), F(m) = 80, (80 \leq m \leq 88),$
$F(m) = 88, (89 \leq m \leq 95), F(m) = 96, (96 \leq m \leq 105)$ and so on. Moreover we have $F(m) = m$ if m is a power of 2.

In this paper we call the inequality $F(m) \leq r$ *Ferus inequality*. Many examples satisfying in fact *Ferus equality* $F(m) = r$ have been found related to isoparametric submanifold, homogeneous submanifolds, austere subamnifolds and so on ([27][29]).

However we may feel something missing, by the fact that, in Ferus inequality or Ferus equality, there appear just m and r, but, there does not appear the number n, or the dimension of the ambient space $\mathbf{R}P^{n+1}$.

Now we are going to formulate Ferus type inequality in term of Legendre sugmanifolds and in more symmetric form.

Theorem 3.2 *Let L be a closed (compact without boundary) immersed Legendre submanifold of the incidence relation $Q \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$. Suppose $\pi|_L$ and $\pi^*|_L$ are constant rank m and m^* respectively, and L is not the Legendre lifting of a projective subspace. Then we have*

$$F(m) \leq m + m^* - n, \quad F(m^*) \leq m^* + m - n.$$

Note that $n \leq m+m^*$. Moreover we see, if $m+m^* = n$ in the situation of Theorem 3.2, then L is the Legendre lifting of a projective subspace (Example 2.1).

Proof of Theorem 3.2: Set $M = \pi(L)$. Then M is a closed and immersed submanifold in $\mathbf{R}P^{n+1}$. It is easy to see that

$$\mathrm{rank}(\gamma) \leq m + m^* - n.$$

Thus we have $F(m) \leq m+m^*-n$ if M is not a projective subspace. By the symmetry, we also have $F(m^*) \leq m^* + m - n$. Thus we have Theorem 3.2. $\square$

Now we are led to the following fundamental question:

Question: For any positive integers n, m, m^* satisfying

$$F(m) = m + m^* - n, \quad F(m^*) = m^* + m - n,$$

the symmetric Ferus equalities, find examples of closed Legendre submanifolds $L^n \subset Q^{2n+1} \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$ such that $\pi|_L$ is of constant rank m and $\pi^*|_L$ is of constant rank m^*.

If the symmetric Ferus equalities are satisfied, then we have

$$F(m) = F(m^*) \;\; \textit{and} \;\; n = m + m^* - F(m) (= m^* + m - F(m^*)).$$

Since $m \geq F(m)$ and $m^* \geq F(m^*)$, the inequalities $m \leq n, m^* \leq n$ are necessarily fulfilled. Thus the question can be re-written as follows:

Question': For any positive integers m, m^* satisfying $F(m) = F(m^*)$, find examples of closed Legendre submanifolds $L^n \subset Q^{2n+1} \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$, $n = n = m + m^* - F(m)(= m^* + m - F(m^*))$, such that $\pi|_L$ is of constant rank m and $\pi^*|_L$ is of constant rank m^*.

We give here some of known examples:

Example 3.3 By Example 2.3, we have examples for $(m, m^*) = (3,3), (5,5), (9,9), (17,17), (25,25), (33,33), (49,49), (57,57), (65,65), (81,81), (89,89), (97,97)$, and so on. Moreover, we have examples for the sequence : $(2^\ell + 1, 2^\ell + 1)$, $\ell = 1, 2, 3, \ldots$.

Example 3.4 (Cartan hypersurfaces.)

(1) $(m, m^*) = (3,2)$. Let $M^3 \in \mathbf{R}P^4$ be the Cartan hypersurface. Then $n = m = 3, m^* = 2$. Note that $F(3) = 2 = F(2)$. Thus we see the symmetric Ferus equalities hold.

(2) $(m, m^*) = (6,4)$. Let $M^6 \in \mathbf{R}P^7$ be the Cartan hypersurface. Then $n = m = 6, m^* = 4$. Note that $F(6) = 4 = F(4)$. Thus we see the symmetric Ferus equalities hold.

(3) $(m, m^*) = (12,8)$. Let $M^{12} \in \mathbf{R}P^{13}$ be the Cartan hypersurface. Then $n = m = 12, m^* = 8$. Note that $F(12) = 8 = F(8)$. Thus we see the symmetric Ferus equalities hold.

(4) $(m, m^*) = (24,16)$. Let $M^{24} \in \mathbf{R}P^{25}$ be the Cartan hypersurface. Then $n = m = 24, m^* = 16$. Note that $F(24) = 16 = F(16)$. Thus we see the symmetric Ferus equalities hold.

Moreover by Kimura's constructions([27]), we have examples, for instance, for $(m, m^*) = (6,5), (11,10), (21,20)$.

4 Bi-degenerate fronts in four dimensional spaces.

Now we turn to singularities. We study Legendre submanifolds in the incidence relation $Q \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$ with $\mathrm{rank}(\pi|_L) = m$ and $\mathrm{rank}(\pi^*|_L) = m^*$, not assuming $\pi|_L$ and $\pi^*|_L$ are of constant rank. Then $\pi(L)$ and $\pi^*(L)$ may have singularities.

In the case $(m, m^*) = (n, 1)$, the diffeomorphism classification of the singularities of degenerate fronts are studied in detail in [20][22][23]. Note that, if $n \geq 2$, $F(n) >$

$1 = n + 1 - n$, so $\pi|_L$ is never of constant rank. For example, in the case $n = 2$, the typical singularities of degenerate fronts of dimension 2 in $\mathbf{R}P^3$ are a *cuspidal edge*, a *folded umbrella* and a *swallowtail*. These are singularities of *tangent developables* of space curves of types $(1,2,3)$, $(1,2,4)$ and $(2,3,4)$, respectively.

For the classification by a weaker equivalence relation, namely by the homeomorphism classification is given in [25].

In this section, we give the classification of singularities of bi-degenerate Legendre submanifold in case $n = 3, m = 2, m^* = 2$. Note that, in this case, $F(2) = 2 > 1 = 2 + 2 - 3$, so that $\pi|_L$ and $\pi^*|_L$ are never of constant rank.

Consider the flag manifold

$$\mathcal{F} := \{V : \{0\} \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset \mathbf{R}^5\}.$$

Then we see $\dim \mathcal{F} = 10$. On $\mathcal{F}$, we define the *canonical distribution* $D \subset T\mathcal{F}$ by teh following: a curve

$$V(t) : \{0\} \subset V_1(t) \subset V_2(t) \subset V_3(t) \subset V_4(t) \subset \mathbf{R}^5$$

on $\mathcal{F}$ is tangent to D at $t = t_0$ if the infinitesimal defomation of $V_1(t)$ at t_0 belongs to $V_2(t_0)$, the infinitesimal defomation of $V_2(t)$ at t_0 belongs to $V_3(t_0)$, and the infinitesimal defomation of $V_3(t)$ at t_0 belongs to $V_4(t_0)$. Then we see $\text{rank}D = 4$.

We define the projection $\pi_1 : \mathcal{F} \to \mathbf{R}P^4$ (resp. $\pi_4 : \mathcal{F} \to \mathbf{R}P^{4*}$) by $\pi_1(V) = V_1$ ($\pi_4(V) = V_4$). Also we define the projection $\pi_{1,4} : \mathcal{F} \to Q \subset \mathbf{R}P^4 \times \mathbf{R}P^{4*}$ by $\pi_{1,4}(V) = (V_1, V_4)$. Then we have $\pi_1 = \pi \circ \pi_{1,4}$ and $\pi_4 = \pi^* \circ \pi_{1,4}$.

Typical singularites appearing in bi-degenerate fronts in this situation are *cones* and *1-developables.*

Let $c : \mathbf{R} \to \mathbf{R}P^4$,

$$c(t) = [x(t)] = [x_0(t), x_1(t), x_2(t), x_3(t), x_4(t))]$$

be a smooth curve. Consider the surface ruled by tangent (projective) lines to the curve. We call it *1-developable* of the curve. Then the tangent planes to regular points of the 1-developable are constant along each ruling. In fact the tangent plane to the 1-developable at a point on a tangent line coincides with the osculating 2-plane at the tangent point of the tangent line to the curve.

Let a_1, a_2, a_3, a_4 be integers with $1 \leq a_1 < a_2 < a_3 < a_4$. The curve c is called of type (a_1, a_2, a_3, a_4) at $t_0 \in \mathbf{R}$ if there exist a smooth coordinate t of $\mathbf{R}$ centered at t_0 and an affine coordinate x_1, x_2, x_3, x_4 such that $c(t)$ is represented near t_0 in the form

$$x_1(t) = t^{a_1} + o(t^{a_1}),\ x_2(t) = t^{a_2} + o(t^{a_2}),\ x_3(t) = t^{a_3} + o(t^{a_3}),\ x_4(t) = t^{a_4} + o(t^{a_4}).$$

The curve c is of finite type at t_0 if there exist such integers a_1, a_2, a_3, a_4 so that c is of type (a_1, a_2, a_3, a_4). The curve itself is called of finite type if it is of finite type at every point. Any curve $c : \mathbf{R} \to \mathbf{R}P^4$ of finite type lifts to unique D-integral curve $\widetilde{c} : \mathbf{R} \to \mathcal{F}$, by using osculating subspaces of dimension 1 (the tangent line), of dimension 2, of dimension 3 and of dimension 4. Moreover $c^* := \pi_4 \circ \widetilde{c} : \mathbf{R} \to \mathbf{R}P^{4*}$ is of finite type. If the original c is of type (a_1, a_2, a_3, a_4) at $t_0 \in \mathbf{R}$, then c^* is of type $(a_4 - a_3, a_4 - a_2, a_4 - a_1, a_4)$ at $t_0 \in \mathbf{R}$. We call c^* the *dual curve* to c ([40]).

Then we have the following fundamental result:

Theorem 4.1 *The* 1*-develpable of a curve c in $\mathbf{R}P^4$ of type (a_1, a_2, a_3, a_4) is a bi-degenerate front with $m = 2, m^* = 2$. Its projective dual is the* 1*-developable of the dual curve c^* of type $(a_4 - a_3, a_4 - a_2, a_4 - a_1, a_4)$.*

To classify singularities of subsets in $\mathbf{R}P^{n+1}$ we must define, at least, a local equivalence relation: a subset $A \subseteq N$ of a manifold N at a point $p_0 \in N$ and a subset $A' \subseteq N'$ of a manifold N' at a point $p'_0 \in N'$ are called *diffeomorphic* if there exists a diffeomorphism $\varphi : U \to U'$ of an open neighbourhood U of p_0 in N and an open neighbourhood U' of p'_0 in N' which maps $A \cap U$ to $A' \cap U'$.

Since an open dense part of $\pi(L)$ is a submanifold of dimension m, it is natural to consider a parametrization by an m dimensional manifold. Then smooth mappings $f : M \to N$ at a point $t_0 \in M$ and $f' : M' \to N'$ at a point $t'_0 \in M'$ are called *diffeomorphic* if there exist a diffeomorphism $\psi : V \to V'$ of of an open neighbourhood V of t_0 in M and an open neighbourhood V' of t'_0 in M' and a diffeomorphism $\varphi : U \to U'$ of of an open neighbourhood U of $p_0 = f(t_0)$ in M and an open neighbourhood V' of $p'_0 = f'(t')$ in M' such that $\varphi \circ f = f' \circ \psi$ on U.

Theorem 4.2 (cf. [22]) *Let $c : \mathbf{R} \to \mathbf{R}P^4$ be a smooth curve and $t_0 \in \mathbf{R}$. Suppose c at t_0 is of one of following types:*

$(\mathrm{I})_r : (1, 2, 3, 3 + r),\ r = 1, 2, \ldots,$

$(\mathrm{II})_0 : (2, 3, 4, 5),$

$(\mathrm{II})_1 : (1, 3, 4, 5),$

$(\mathrm{II})_2 : (1, 2, 4, 5),$

$(\mathrm{III}) : (3, 4, 5, 6).$

Then the diffeomorphism class in $\mathbf{R}P^4$ of the 1*-developable of the curve c at the point $c(t_0)$ is determined only by its type. In other words, if two curves have the same type, then their* 1*-developables are locally diffeomorphic.*

For a generic curve in $\mathbf{R}P^4$, only points of types $(\mathrm{I})_1 : (1, 2, 3, 4)$ and $(\mathrm{I})_2 : (1, 2, 3, 5)$ appear. Moreover, for the dual curve of a generic curve, only points of types $(\mathrm{I})_1 : (1, 2, 3, 4)$ and $(\mathrm{II})_0 : (2, 3, 4, 5)$ appear.

We call the 1-developable surface *cuspidal edge* in the case of type $(1, 2, 3, 4)$, and *open swallowtail* in the case of type $(2, 3, 4, 5)$.

Example 4.3 (Cuspidal edge.) The 1-developable surface of a curve of type $(1,2,3,4)$ has the normal form under the diffeomorphisms:

$$(x,t) \mapsto (x,\ 3t^2+2xt,\ 2t^3+xt^2,\ \frac{3}{4}t^4+\frac{1}{3}xt^3).$$

Moreover it is diffeomorphic to

$$(x,t) \mapsto (x,\ t^2,\ t^3, 0).$$

Example 4.4 The 1-developable surface of a curve of type $(1,2,3,5)$ has the normal form under the diffeomorphisms:

$$(x,t) \mapsto (x,\ 3t^2+2xt,\ 2t^3+xt^2,\ \frac{2}{5}t^5+\frac{1}{6}xt^4).$$

However it is actually deffieomorphic to

$$(x,t) \mapsto (x,\ t^2,\ t^3, 0),$$

namely, diffeomorphic to the cuspidal edge.

Actually we can prove the following:

Theorem 4.5 *The* 1 *developable of a curve of type* $(\mathrm{I})_r : (1,2,3,3+r), (r=1,2,3,\dots)$ *is diffeomorphic to the cuspidal edge.*

Also we observe that the dual of 1-developable of a curve of type $(1,2,3,4)$ and the dual of 1-developable of a curve of type $(1,2,3,5)$ are not diffeomorphic:

Example 4.6 (Open swallowtail.) The 1-developable surface of a curve of type $(2,3,4,5)$ has the normal form under the diffeomorphisms:

$$(x,t) \mapsto (x,\ 3t^3+2xt,\ \frac{9}{4}t^4+xt^2,\ \frac{9}{10}t^5+\frac{1}{3}xt^3).$$

This is not diffeomorphic to the cuspidal edge.

5 Frontal mappings.

In this section, we introduce the notion of frontal mappings and show an attempt to generalize Legendre singularity theory, clarifying their applications to the study of singularities appearing in the Grassmannian duality, or more generally in the Flag duality, and to poly-symplectic geometry.

Let $f : M^m \longrightarrow N^{n+1}$, $m < n+1$, be a C^∞ mapping. Assume f is immersive outside of a nowhere dense subset $\Sigma(f)$ of M. Then f is called a *frontal mapping* if, for any $x \in M$, there exists a unique limit

$$\lim_{x' \to x} f_*(T_{x'}M) =: T_x, \qquad (x' \in M - \Sigma(f)).$$

in the Grassmann bundle $Gr(m, TN)$, such that the correspondence $x \mapsto T_x$ is of class C^∞.

Examples of frontal mappings are (0) submanifolds, (1) singular curves with no infinitely flat point, (2) their arbitrarily intermediate developables, (3) wave front sets in the ordinary sense and (4) varieties of irregular orbits of finite reflection groups [17]. If we take a transverse intersection of wave front sets, then we get a "frontal variety", which does not necessarily admit a parametrization by a single non-singular manifold.

Let $f : M \longrightarrow N$ be a frontal mapping. Then f lifts naturally to a mapping $\widetilde{f} : M \longrightarrow Gr(m, TN)$, which is called the *Nash lifting* of f.

Let $D \subset TGr(m, TN)$ be the *tautological subbundle* (or the *canonical system* in the sense of [43]) of codimension $n + 1 - m =: r$. Notice that, if $r = 1$, then $Gr(m, TN) = Gr(r, T^*N) = PT^*N$, and D is the canonical *contact distribution* over PT^*N. Then the Nash lifting $\widetilde{f} : M \longrightarrow (Gr(m, TN), D)$ is a (not necessarily maximal dimensional) *integral mapping* of the distribution D on $Gr(m, TN)$. The Nash lifting $\widetilde{f}$ is characterized as the unique integral lifting of the frontal mapping f.

6 Relation to poly-symplectic singularity theory.

Let B be a manifold of dimension m. For a positive integer r, consider the Whitney sum

$$T^{*(r)}B = T^*B \oplus \cdots \oplus T^*B \xrightarrow{\pi} B$$

endowed with the system of closed 2-forms $\omega_i = d\theta_i, 1 \le i \le r$, where θ_i is the Liouville 1-form on the i-th factor [6].

A C^∞ mapping $\varphi : M^m \longrightarrow T^{*(r)}B$ from an m-dimensional manifold M is called *isotropic* if $\varphi^*\omega_i = 0, 1 \le i \le r$. If we take the universal covering $\rho : \widetilde{M} \longrightarrow M$ of M, then there exist functions $e_i : \widetilde{M} \longrightarrow \mathbf{R}$ such that $de_i = (\varphi \circ \rho)^*\theta_i, 1 \le i \le r$. We define the *graph* of φ by

$$f = (\pi \circ \varphi \circ \rho, e) : \widetilde{M} \longrightarrow B \times \mathbf{R}^r =: N.$$

If $\Sigma(f)$ is nowhere dense in $\widetilde{M}$, then f is frontal: The Nash lifting is

$$\widetilde{f} = (\varphi \circ \rho, e) : \widetilde{M} \longrightarrow T^{*(r)}B \times \mathbf{R}^r \hookrightarrow Gr(m, TN).$$

We compare equivalence relations for isotropic mappings, integral mappings and frontal mappings.

Two isotropic mappings φ and $\varphi' : M \longrightarrow T^{*(r)}B$ are called *Lagrange equivalent* if there exist diffeomorphisms $\sigma : M \longrightarrow M$ and $\tau : T^{*(r)}B \longrightarrow T^{*(r)}B$ such that $\tau^*\omega_i = \omega_i, 1 \leq i \leq r$, τ covers a diffeomorphism $\bar{\tau} : B \longrightarrow B$ with respect to $\pi : T^{*(r)}B \longrightarrow B$, and that $\tau \circ \varphi = \varphi' \circ \sigma$.

Two integral mappings F and $F' : M \longrightarrow T^{*(r)}B \times \mathbf{R}^r$ are called *s-Legendre equivalent* if there exist diffeomorphisms $\sigma : M \longrightarrow M$ and $\widetilde{\tau} : T^{*(r)}B \times \mathbf{R}^r \longrightarrow T^{*(r)}B \times \mathbf{R}^r$ such that $\widetilde{\tau}$ preserve the distribution and the fibration $\Pi : T^{*(r)}B \times \mathbf{R}^r \longrightarrow B \times \mathbf{R}^r$ and that $\widetilde{\tau} \circ F = F' \circ \sigma$.

Two frontal mappings f and $f' : M \longrightarrow B \times \mathbf{R}^r$ are called *s-equivalent* if there exist diffeomorphisms $\sigma : M \longrightarrow M$ and $\kappa : B \times \mathbf{R}^r \longrightarrow B \times \mathbf{R}^r$ of the form $\kappa(y, z) = (\bar{\tau}(y), z + \rho(y))$ and that $\kappa \circ f = f' \circ \sigma$.

Then we have

Proposition 6.1 *Let $\varphi : M \longrightarrow T^{*(r)}B$ be an isotropic mapping with nowhere dense singular set $\Sigma(\pi \circ \varphi)$. Then the following conditions are equivalent to each other:*

(1) *Isotropic mappings φ and $\varphi' : M \longrightarrow T^{*(r)}B$ are* Lagrange equivalent.

(2) *Nash liftings $\widetilde{f}$ and $\widetilde{f}' : \widetilde{M} \longrightarrow T^{*(r)}B \times \mathbf{R}^r$ are* s-Legendre equivalent.

(3) *Frontal mappings f and $f' : \widetilde{M} \longrightarrow B \times \mathbf{R}^r$ are* s-equivalent.

It holds also the local version of this result. The concrete classification of isotropic mappings to a poly-symplectic manifold under the Lagrange equivalence will be given in a forthcoming paper.

7 Projective duality and Grassmannian duality.

The projective duality plays an essential role, for instance, to formulate the famous Plücker-Klein's formula, to analyze generic projective hypersurface (Bruce, Platonova, Landis [4]), tangent developables (Scherbak [40], I [20][22]) and Monge-Ampère equations ([26]).

Let $f : M^n \longrightarrow \mathbf{R}P^{n+1}$ be a frontal mapping (e.g. a parametrization of a submanifold). Then we have the Nash lifting $\widetilde{f} : M \longrightarrow Gr(n, T\mathbf{R}P^{n+1}) = PT^*\mathbf{R}P^{n+1}$. Set $Q = \{(p,q) \in \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*} \mid p \subseteq q^\vee\}$, the manifold of incident pairs. Then Q is endowed with a contact structure and contact diffeomorphisms $PT^*\mathbf{R}P^{n+1} \cong Q \cong PT^*\mathbf{R}P^{n+1*}$. Then we get the *projective dual* $f^\vee : M \longrightarrow \mathbf{R}P^{n+1*}$ of f by the composition of $\widetilde{f}$ with the projection $PT^*\mathbf{R}P^{n+1*} \longrightarrow \mathbf{R}P^{n+1*}$. If f is sufficiently generic, then $f^\vee$ is also frontal, and we get the presumable equality $f^{\vee\vee} = f$.

With the notion of frontal mappings, we are naturally led to the following generalization of the projective duality.

Let $f : M^m \longrightarrow \mathbf{R}P^{n+1}$ be a frontal mapping of codimension $r = n+1-m$. Then, consider the Nash lifting of f :

$$\begin{aligned} \widetilde{f} : M \longrightarrow Gr(m, T\mathbf{R}P^{n+1}) &\hookrightarrow Gr(1, \mathbf{R}^{n+2}) \times Gr(m+1, \mathbf{R}^{n+2}) \\ &\cong Gr(1, \mathbf{R}^{n+2}) \times Gr(r, \mathbf{R}^{n+2*}). \end{aligned}$$

The image is again $Q = \{(p,q) \mid p \subseteq q^\vee\}$. Therefore we naturally define the *Grassmannian dual* $f^\vee : M \longrightarrow Gr(r, \mathbf{R}^{n+2*})$ of $f : M \longrightarrow \mathbf{R}P^{n+1}$. The equality "$f^{\vee\vee} = f$", however, does not have any meaning, even if $f^\vee$ is a front mapping in the meaning of previous definition. Therefore, for a mapping into a Grassmannian, it seems natural to specialize the definition of frontal mappings as follows:

Let $f : M^m \longrightarrow Gr(r, \mathbf{R}^{n+2})$ be a C^∞ mappings with $m + r \leq n + 1$. Set $s = n+2-m-r$. Then f is called *Grassmann-frontal* if there exists a *unique* integral lifting $\widetilde{f} : M \longrightarrow (Q, D)$ of f with respect to a fibration $\pi : Q \longrightarrow Gr(r, \mathbf{R}^{n+2})$ and a distribution D on Q defined as follows: Set first

$$Q = \{(p,q) \in Gr(r, \mathbf{R}^{n+2}) \times Gr(s, \mathbf{R}^{n+2*}) \mid p \subseteq q^\vee\},$$

and

$$P = \{(p,q,p') \in Gr(r, \mathbf{R}^{n+2}) \times Gr(s, \mathbf{R}^{n+2*}) \times Gr(r, \mathbf{R}^{n+2}) \mid p \subseteq q^\vee, p' \subseteq q^\vee\}.$$

Then we get the special divergent diagram $(\rho,\ \pi \circ \rho)$:

$$P \xrightarrow{\rho} Q \xrightarrow{\pi} Gr(r, \mathbf{R}^{n+2}),$$

where ρ (resp. π) is the projection to the first and second factors (resp. to the first factor). To define the tautological subbundle $D \subset TQ$ of codimension rs, for each $c = (p,q) \in Q$, we set $D_c \subset T_cQ$ by $D_c = \pi_*^{-1}(T_p(Gr(r, \mathbf{R}^{r+m})))$, where $Gr(r, \mathbf{R}^{r+m}) = \pi(\rho^{-1}(c))$ is embedded in $Gr(r, \mathbf{R}^{n+2})$ as $\{p' \in Gr(r, \mathbf{R}^{n+2}) \mid p' \subseteq q^\vee\}$. Notice that, if $r \neq 1$, or, $r \neq n+1$, then the "system of tangential linear subspaces" on the Grassmannian $Gr(r, \mathbf{R}^{n+2})$ defined by D does not represent general tangential linear subspaces of the Grassmannian.

If we take local coordinates $(a_{ij})_{1\leq i\leq r, 1\leq j\leq m+s}$ of $Gr(r,\mathbf{R}^{n+2})$ and $(b_{k\ell})_{1\leq k\leq m+r, 1\leq \ell\leq s}$ of $Gr(s,\mathbf{R}^{n+2*})$, then Q is defined by the system of equations

$$b_{ij}+a_{i1}b_{r+1\,j}+\cdots+a_{im}b_{r+m\,j}+a_{i\,m+j}=0,\ 1\leq i\leq r, 1\leq j\leq s,$$

and D is defined by the system of 1-forms

$$b_{r+1\,j}da_{i1}+\cdots+b_{r+m\,j}da_{im}+da_{i\,m+j}=0,\ 1\leq i\leq r, 1\leq j\leq s.$$

The integral lifting $\widetilde{f}$ is called the *Grassmann-Nash lifting* of f. The relation to the original definition of frontal mappings is as follows:

Lemma 7.1 *Let $F:\mathbf{R}^m,0\longrightarrow Q,(p_0,q_0)$ be an integral map-germ. Then $f=\pi\circ F:\mathbf{R}^m,0\longrightarrow Gr(r,\mathbf{R}^{n+2}),p_0$ is Grassmann-frontal if and only if $\Sigma(\rho\circ f)\subset\mathbf{R}^m,0$ is nowhere dense, for some projection*

$$\rho: Gr(r,\mathbf{R}^{n+2}),p_0\hookrightarrow Hom(\mathbf{R}^r,\mathbf{R}^{m+s}),0\xrightarrow{i^*}Hom(\mathbf{R},\mathbf{R}^{m+s}),0\hookrightarrow\mathbf{R}P^{m+s-1},$$

induced from a linear inclusion $i:\mathbf{R}\hookrightarrow\mathbf{R}^r$.

Now, from the duality, we have another distribution $D'\subset TQ$ from the projection $\pi':Q\longrightarrow Gr(s,\mathbf{R}^{n+2*})$ to the second factor, setting

$$P'=\{(q',p,q)\in Gr(s,\mathbf{R}^{n+2*})\times Gr(r,\mathbf{R}^{n+2})\times Gr(s,\mathbf{R}^{n+2*})\mid q\subseteq p^\vee,\ q'\subseteq p^\vee\}.$$

Then the fundamental result is the following:

Proposition 7.2 *Two distributions D and D' on the incidental manifold Q coincide.*

Based on this fact and a version of the transversality theorem, we have the following *Grassmannian duality theorem*:

Theorem 7.3 *There exists an open dense subset $\mathcal{O}$ in the space of integral mappings $M^m\to Q\subset Gr(r,\mathbf{R}^{n+2})\times Gr(s,\mathbf{R}^{n+2*})$ with $m+r+s=n+2$ of kernel rank at most one, with the following property: For any $F:M\longrightarrow Q$ belonging to $\mathcal{O}$, F is the unique integral lifting of $\pi\circ F=:f$ and of $\pi'\circ F=:f'$ respectively, and the singular loci $\Sigma(f)$ and $\Sigma(f')$ are both nowhere dense in M. In particular, in this case, we have that f and f' are both Grassmann-frontal, $f'=f^\vee, f=f'^\vee$ and that $f^{\vee\vee}=f$.*

The proofs of these results will be given in forthcoming papers. We conclude this survey by giving just several illustrative examples.

Example 7.4 If $f : M^2 \longrightarrow \mathbf{R}P^4$ is the natural parametrization of the 1-developable of a curve in $\mathbf{R}P^4$. Then $f^\vee : M^2 \longrightarrow Gr(2, \mathbf{R}^{5*})$ collapses to a curve (Grassmannian dual curve).

Example 7.5 Let $f : \mathbf{R}P^2 \to \mathbf{R}P^5$ be the Veronese embedding. Then the dual $f^\vee : \mathbf{R}P^2 \to Gr(3, \mathbf{R}^{6*})$ is also an embedding. In fact, $f^\vee$ composed with the Plücker embedding $Gr(3, \mathbf{R}^6) \to \mathbf{R}P^{19}$ is decomposed into the Veronese embedding $\mathbf{R}P^2 \to \mathbf{R}P^9$ and a linear embedding $\mathbf{R}P^9 \to \mathbf{R}P^{19}$.

Example 7.6 Let $f : M^2 \longrightarrow Gr(2, \mathbf{R}^5)$ be an embedding. If $f(M) \subset Gr(2, \mathbf{R}^3) \subset Gr(2, \mathbf{R}^5)$, then f has infinitely many integral liftings $\widetilde{f} : M \longrightarrow Q$. The "dual" $f^\vee : M \longrightarrow \mathbf{R}P^{4*}$ collapses to a point on the projective line dual to $\mathbf{R}^3 \subset \mathbf{R}^5$.

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