数理解析研究所講究録 1328 巻 2003 年 165-173

THE BRIANÇON-SPEDER AND OKA FAMILIES ARE NOT BILIPSCHITS TRIVIAL

SATOSHI KOIKE

小池敏司 (兵庫教育大学)

At the symposium "Several Topics on Singularity Theory," Tadeusz Mostowski gave excellent lectures on Lipschitz equisingularity ([18]). Related to this topic, the author described some facts in the short communications. In this note we explain it in detail.

Let $f,\ g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic function germs. We say that they are *blow-analytically equivalent* if there are real modifications $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^n, 0)$, $\mu' : (M', \mu'^{-1}(0)) \to (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \to (M', \mu'^{-1}(0))$ which induces a homeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \phi$. A *blow-analytic homeomorphism* is such a homeomorphism induced by an analytic isomorphism via real modifications. Blow-analytic equivalence is a notion introduced by Tzee-Char Kuo ([14, 15]) as a natural equivalence relation for real analytic function germs. Concerning the blow-analyticity, he posed the following question:

Question 0.1. Is a blow-analytic homeomorphism a biLipschitz homeomorphism ?

A blow-analytic homeomorphism is an *arc-analytic* homeomorphism in the sense of K. Kurdyka [16]. Namely, any analytic arc is mapped to an analytic arc by a blow-analytic homeomorphism. The relation between blow-analyticity and arc-analyticity was discussed by E. Bierstone and P. Milman [1]. Originally, Kuo presented the following weak conjecture ([14]):

Conjecture 0.2. *A blow-analytic homeomorphism preserves the contact order of analytic arcs.*

For more properties of blow-analyticity, see [5].

We next consider the Briançon-Speder family ([2]) $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $t \in J = (-1 - \epsilon, \infty)$, defined by

$$f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}$$

1991 *Mathematics Subject Classification.* 14B05, 32S15, 57R45.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 13640070) of Ministry of Education, Science and Culture of Japan.

where ϵ is a sufficiently small positive number. Then, $\{f_t\}_{t\in J}$ is a family of functions with isolated singularities. It was shown by T. Fukui [4] that $\{f_t\}_{t\in J}$ admits a blow-analytic trivialisation along I for any closed subinterval I of J. At about the same time as Fukui, Koike observed in [10] that $(\mathbb{R}^3, f_0^{-1}(0))$ and $(\mathbb{R}^3, f_{-1}^{-1}(0))$ cannot be equivalent via a homeomorphism which preserves the tangency of analytic arcs contained in the zero-sets. We call such a homeomorphism a *strong homeomorphism.* Combining the results of Fukui and Koike, we get the fact that

(0.1) a blow-analytic equivalence is not always a "blow-analytic and strong C^0" equivalence.

This implies the following:

(0.2) A blow-analytic equivalence is not always a "blow-analytic and biLipschitz" equivalence.

These facts show that there are a negative example to Question 0.1 and a counterexample to Conjecture 0.2. In fact, after Fukui and Koike, L. Paunescu constructed a blow-analytic homeomorphism which is not a biLipschitz one ([24]).

For a family of analytic function germs with isolated singularities, T.C. Kuo ([15]) established a locally finite classification theorem on blow-analytic equivalence, but recently J.-P. Henry and A. Parusiński ([8, 9]) showed the appearance of Lipschitz moduli. Therefore, Question 0.1 is not so interesting any more. On the other hand, a local finiteness theorem was established on biLipschitz equivalence for a family of zero-sets of analytic functions (T. Mostowski [17], A. Parusiński [21, 22, 23]).

Example 0.3. Let $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $t \in I = [-2, 2]$, be a family of weighted homogeneous polynomials defined by

$$f_t(x, y) = x^3 + 3xy^4 + ty^6.$$

Then $\{f_t\}_{t\in I}$ admits a blow-analytic trivialisation along I ([4], T. Fukui and L. Paunescu [6]), and $\{(\mathbb{R}^2, f_t^{-1}(0))\}$ is biLipschitz trivial over I. But Lipschitz moduli as function germs appear in the family $\{f_t\}_{t\in I}$ ([8, 9]).

In the next section we introduce the notion of a sea-tangle neighbourhood of a Lipschitz arc. Using some properties of the neighbourhood, we shall show in §2 that the Briançon-Speder family is not biLipschitz trivial as a family of zero-sets, and in §3 that the Oka family [20] is not so, either. But both families of functions are blow-analytically trivial. Therefore (0.2) is improved as follows:

(0.3) Blow-analytic equivalence for functions does not always imply biLipschitz equivalence for the zero-sets.

The author would like to thank Tzee-Char Kuo and Tadeusz Mostowski for useful communications.

1. Preliminaries

By a C^ω arc at $0 \in \mathbb{R}^n$, we mean the germ of an analytic map $\lambda : [0, \epsilon) \to \mathbb{R}^n$ with $\lambda(0) = 0$, $\lambda(s) \neq 0$, $s > 0$. The set of all such arcs is denoted by $A(\mathbb{R}^n, 0)$.

Let S^{n-1} denote the $(n-1)$-dimensional unit sphere. For $a = (a_1, \cdots, a_n) \in S^{n-1}$, let $L(a) : [0, \delta) \to \mathbb{R}^n$, $\delta > 0$, be a mapping defined by

$$L(a)(t) = (a_1 t, \cdots, a_n t).$$

Then $L(a) \in A(\mathbb{R}^n, 0)$. For any $\lambda \in A(\mathbb{R}^n, 0)$, there exists unique $a \in S^{n-1}$ such that λ is tangent to $L(a)$ at $0 \in \mathbb{R}^n$. Then we write $L(a) = T(\lambda)$.

For an analytic function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, let $C_0(f)$ denote the set of connected components of $f^{-1}(0) - \{0\}$ as germs at $0 \in \mathbb{R}^n$. We put

$$C_0(f) = \{C_1, \cdots, C_m\}, \ m \in \{0\} \cup \mathbb{N}.$$

For each i, let $D_i(f)$ be the set of $a \in S^{n-1}$ satisfying the following condition:

There exists an arc $\lambda \in A(\mathbb{R}^n, 0)$ such that $\lambda \subset \overline{C_i}$ and λ is tangent to $L(a)$ at $0 \in \mathbb{R}^n$.

For $1 \leq i, j \leq m$, $i \neq j$, we define *the cardinal number of common directions of C_i and C_j* as follows:

$$D_{ij}(f) = \sharp(D_i(f) \cap D_j(f)).$$

Let $\theta : [0, \epsilon) \to \mathbb{R}^n$ be a Lipschitz map not identically zero such that $\theta(0) = 0$. For d, $C > 0$, define a *sea-tangle neighbourhood of θ of degree d with width C* in a small neighbourhood of $0 \in \mathbb{R}^n$:

$$ST_d(\theta; C) = \{x \in \mathbb{R}^n : dist(x, \theta) \leq C|x|^d\}.$$

This notion originated in the *hornneighbourhood* $H_d(f; C)$ introduced by T.C. Kuo [11, 13].

Let $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a biLipschitz homeomorphism germ, namely, there are positive numbers K_1, $K_2 > 0$ with $K_1 \leq K_2$ such that

$$K_1|x_1 - x_2| \leq |\phi(x_1) - \phi(x_2)| \leq K_2|x_1 - x_2| \tag{1.1}$$

in a small neighbourhood of $0 \in \mathbb{R}^n$. Conversely, we have

$$\frac{1}{K_2}|y_1 - y_2| \leq |\phi^{-1}(y_1) - \phi^{-1}(y_2)| \leq \frac{1}{K_1}|y_1 - y_2| \tag{1.2}$$

in a small neighbourhood of $0 \in \mathbb{R}^n$. Then we have the following:

Lemma 1.1. *For $K > 0$, $ST_d(\phi(\theta); \frac{KK_1}{K_2^d}) \subset \phi(ST_d(\theta; K)) \subset ST_d(\phi(\theta); \frac{KK_2}{K_1^d})$ in a small neighbourhood of $0 \in \mathbb{R}^n$.*

Proof. We first show the right inclusion. Pick $x \in ST_d(\theta; K)$. Then there is $x_0 \in \theta$ such that $|x - x_0| \leq K|x|^d$. On the other hand, $K_1|x| \leq |\phi(x)| \leq K_2|x|$. Therefore

$$dist(\phi(x), \phi(\theta)) \leq |\phi(x) - \phi(x_0)| \leq K_2|x - x_0| \leq KK_2|x|^d \leq \frac{KK_2}{K_1^d}|\phi(x)|^d.$$

Thus $\phi(x) \in ST_d(\phi(\theta); \frac{KK_2}{K_1^d})$.

We can show the left inclusion similarly. By the above argument, we have

$$\phi^{-1}(ST_d(\phi(\theta); C)) \subset ST_d(\theta; \frac{C(\frac{1}{K_1})}{(\frac{1}{K_2})^d}).$$

Set $\frac{C(\frac{1}{K_1})}{(\frac{1}{K_2})^d} = K$, then $C = \frac{KK_1}{K_2^d}$. □

It is easy to see the following fact.

Lemma 1.2. *Let $0 < C < 1$ and $a \in S^{n-1}$. Let $\theta : [0, \epsilon) \to \mathbb{R}^n$ be a Lipschitz map not identically zero such that $\theta(0) = 0$. Suppose that $\theta \subset ST_1(L(a); C)$ as germs at $0 \in \mathbb{R}^n$. Then $L(a) \subset ST_1(\theta; C)$ as germs at $0 \in \mathbb{R}^n$.*

2. The Briançon-Speder family is not biLipschitz trivial

Let $D = \{x \in \mathbb{R} : |x| < 1 + \epsilon\}$ where ϵ is a sufficiently small positive number. Concerning the Briançon-Speder family [2], we have

Theorem 2.1. *Let $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $t \in D$, be a family of weighted homogeneous polynomial functions with isolated singularities defined by*

$$f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}.$$

Then $(\mathbb{R}^3, f_0^{-1}(0))$ is not biLipschitz equivalent to $(\mathbb{R}^3, f_{-1}^{-1}(0))$.

Proof. Put $f = f_0$ and $g = f_{-1}$. Then $f^{-1}(0)$ is the graph of a function differentiable at $0 \in \mathbb{R}^2$:

$$(x, y) \to z = -(y^7x + x^{15})^{\frac{1}{5}}.$$

Remark that there is a positive number $C > 0$ such that

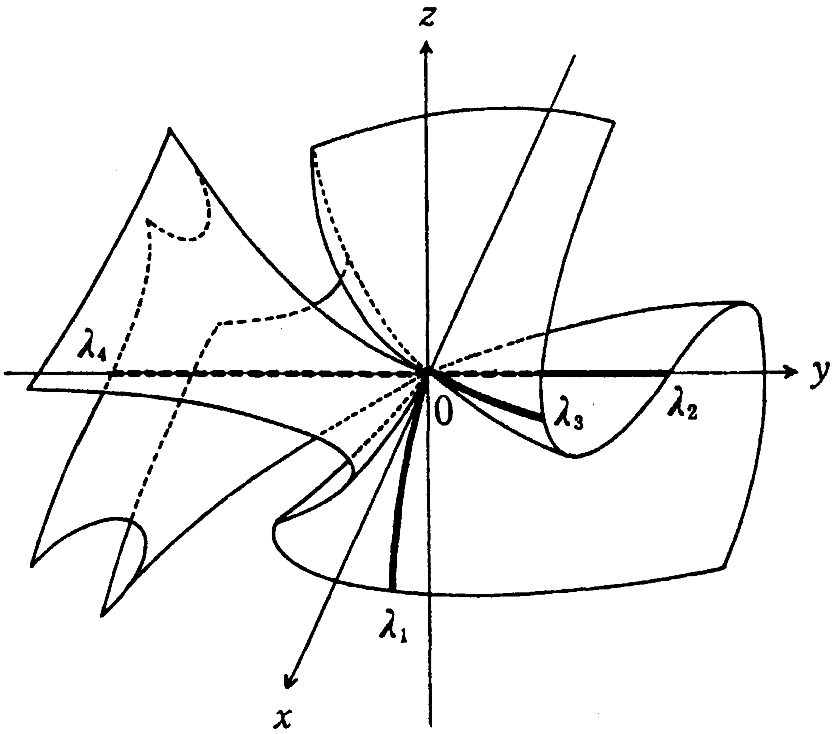
$$|z| \leq C|(x, y)|^{\frac{8}{5}} \tag{2.1}$$

near $0 \in \mathbb{R}^2$. Let $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection defined by $\Pi(x, y, z) = (x, y)$. Then Π gives a homeomorphism from $f^{-1}(0)$ to $\mathbb{R}^2$.

Pick a point $P_0 = (1, y_1, z_1)$ on $g^{-1}(0)$ with $y_1 > 0$, $z_1 > 0$. Define the C^ω arcs $\lambda_j \in A(\mathbb{R}^3, 0)$, $1 \leq j \leq 4$, as follows:

$$\lambda_1(s) = (s, 0, -s^3), \quad \lambda_2(s) = (0, s, 0),$$
$$\lambda_3(s) = (s, y_1s^2, z_1s^3), \quad \lambda_4(s) = (0, -s, 0) \quad (s \geq 0).$$

Then λ_1 and λ_3 are tangent at $0 \in \mathbb{R}^3$, and they are perpendicular to λ_2, λ_4 at $0 \in \mathbb{R}^3$.



(Figure of $g^{-1}(0)$)

Assume that there is a biLipschitz homeomorphism $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\phi(g^{-1}(0)) = f^{-1}(0)$. It follows from the relations of λ_j's that there are positive numbers C_1, $C_2 > 0$ such that

$$\lambda_3 \subset ST_{\frac{3}{2}}(\lambda_1; C_1),\ \lambda_j \subset \mathbb{R}^3 - ST_1(\lambda_1; C_2),\ j = 2, 4.$$

By Lemma 1.1, there are positive numbers C_3, $C_4 > 0$ such that

$$\phi(\lambda_3) \subset ST_{\frac{3}{2}}(\phi(\lambda_1); C_3),\ \phi(\lambda_j) \subset \mathbb{R}^3 - ST_1(\phi(\lambda_1); C_4),\ j = 2, 4.$$

in a neighbourhood of $0 \in \mathbb{R}^3$. Then, by (2.1), there are positive numbers C_5, $C_6 > 0$ such that

$$(*)\ \Pi(\phi(\lambda_3)) \subset ST_{\frac{3}{2}}(\Pi(\phi(\lambda_1)); C_5),\ \Pi(\phi(\lambda_j)) \subset \mathbb{R}^2 - ST_1(\Pi(\phi(\lambda_1)); C_6),\ j = 2, 4,$$

in a neighbourhood of $0 \in \mathbb{R}^2$.

On the other hand, $\Pi(\phi(\lambda_2)) - \{0\}$ and $\Pi(\phi(\lambda_4)) - \{0\}$ are contained in different components of $\mathbb{R}^2 - \Pi(\phi(\lambda_1)) \cup \Pi(\phi(\lambda_3))$. This contradicts $(*)$. Thus $(\mathbb{R}^3, g^{-1}(0))$ is not biLipschitz equivalent to $(\mathbb{R}^3, f^{-1}(0))$.

3. The Oka family is not biLipschitz trivial

Let $D = (-1-\epsilon, 1+\epsilon)$ be an open interval as in §2. Concerning the Oka family [20], we have

Theorem 3.1. *Let $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $t \in D$, be a family of polynomial functions with isolated singularities defined by*

$$f_t(x,y,z) = x^8 + y^{16} + z^{16} + tx^5z^2 + x^3yz^3.$$

Then $(\mathbb{R}^3, f_0^{-1}(0))$ is not biLipschitz equivalent to $(\mathbb{R}^3, f_1^{-1}(0))$.

Proof. We first recall some facts which we have observed in the proof of Theorem B in [10]. Put

$$f(x,y,z) = f_0(x,y,z) = x^8 + y^{16} + z^{16} + x^3yz^3.$$

In each coordinate plane, $f^{-1}(0) - \{0\} = \emptyset$. Here we put

$$A_1 = \{x > 0,\ y > 0,\ z < 0\},\quad A_2 = \{x > 0,\ y < 0,\ z > 0\},$$
$$A_3 = \{x < 0,\ y > 0,\ z > 0\},\quad A_4 = \{x < 0,\ y < 0,\ z < 0\}.$$

Set $S_i = f^{-1}(0) \cap A_i$, $1 \le i \le 4$. Then $f^{-1}(0) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{0\}$ and each $\overline{S_i} = S_i \cup \{0\}$ is homeomorphic to S^2. As seen in [10], $D_{ij}(f) = 1$, $i \neq j$.

Next put

$$g(x,y,z) = f_1(x,y,z) = x^8 + y^{16} + z^{16} + x^5z^2 + x^3yz^3.$$

In (x,y)-plane or (y,z)-plane, $g^{-1}(0) - \{0\} = \emptyset$. Here we put

$$B_1 = \{x > 0,\ y > 0,\ z < 0\},\quad B_2 = \{x > 0,\ y < 0,\ z > 0\},$$
$$B_3 = \{x < 0,\ z > 0\},\quad B_4 = \{x < 0,\ z < 0\}.$$

Set $P_i = g^{-1}(0) \cap B_i$, $1 \le i \le 4$. Then $g^{-1}(0) = P_1 \cup P_2 \cup P_3 \cup P_4 \cup \{0\}$ and each $\overline{P_i} = P_i \cup \{0\}$ is homeomorphic to S^2. By the observations seen in [10], there are λ_1, $\lambda_2 \in A(\mathbb{R}^3, 0)$ with $\lambda_1 \subset \overline{P_3} \cap \{y = 0\}$ and $\lambda_2 \subset \overline{P_4} \cap \{y = 0\}$ such that $T(\lambda_1) = T(\lambda_2) = L((-1,0,0))$, and there are μ_1, $\mu_2 \in A(\mathbb{R}^3, 0)$ with $\mu_1 \subset \overline{P_3} \cap \{y = x\}$ and $\mu_2 \subset \overline{P_4} \cap \{y = x\}$ such that $T(\mu_1) = T(\mu_2) = L((-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0))$. Therefore there are C, $D > 0$ and $d > 1$ such that $\lambda_1 \subset ST_d(\lambda_2; D) \subset ST_1(\lambda_1; C)$, $\mu_1 \subset ST_d(\mu_2; D) \subset ST_1(\mu_1; C)$ and $ST_1(\lambda_1; C) \cap ST_1(\mu_1; C) = \{0\}$ in a small neighbourhood of 0 in $\mathbb{R}^3$.

Suppose that there is a biLipschitz homeomorphism germ $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ with Lipschitz constants K_1, $K_2 > 0$ in (1.1) such that $\phi(g^{-1}(0)) = f^{-1}(0)$. Set $\gamma_i = \phi(\lambda_i)$ and $\nu_i = \phi(\mu_i)$, $i = 1, 2$. Here we put $E = \frac{CK_1}{K_2} > 0$. Then it follows from Lemma 1.1 that $ST_1(\gamma_1; E) \subset \phi(ST_1(\lambda_1; C))$ and $ST_1(\nu_1; E) \subset \phi(ST_1(\mu_1; C))$ in a small neighbourhood of 0 of $\mathbb{R}^3$. Remark that

(3.1) $\quad ST_1(\gamma_1; E) \cap ST_1(\nu_1; E) = \{0\}$ in a neighbourhood of 0 of $\mathbb{R}^3$.

Each P_i, $1 \leq i \leq 4$, is mapped to some S_j by ϕ as set-germs at $0 \in \mathbb{R}^3$. Let $\phi(P_3) = S_{j(3)}$ and $\phi(P_4) = S_{j(4)}$. Then $D_{j(3)}(f) \cap D_{j(4)}(f) = \{a\}$ where a is a member of $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. We consider the sea-tangle neighbourhood $V = ST_1(L(a); E)$ of $L(a)$. If there is a neighbourhood U of $0 \in \mathbb{R}^3$ such that γ_1, $\nu_1 \subset V$ in U, then it follows from Lemma 1.2 that $L(a) \subset ST_1(\gamma_1; E)$ and $L(a) \subset ST_1(\nu_1; E)$ in a neighbourhood of $0 \in \mathbb{R}^3$. This cotradicts (3.1). Thus

(3.2) $\quad \gamma_1$ or ν_1 is not contained in $V \cap U$ for any neighbourhood U of $0 \in \mathbb{R}^3$.

Put $H = \frac{DK_2}{K_1^d}$. By Lemma 1.1, $\gamma_1 \subset ST_d(\gamma_2; H)$ and $\nu_1 \subset ST_d(\nu_2; H)$ in a neighbourhood of $0 \in \mathbb{R}^3$. This contradicts (3.2) because γ_1, $\nu_1 \subset A_{j(3)} \cup \{0\}$ and γ_2, $\nu_2 \subset A_{j(4)} \cup \{0\}$ near $0 \in \mathbb{R}^3$. Therefore, $(\mathbb{R}^3, f^{-1}(0))$ and $(\mathbb{R}^3, g^{-1}(0))$ are not biLipschitz equivalent as germs at $0 \in \mathbb{R}^3$.

$\square$

Remark 3.2. We drew the pictures of the zero-sets of the Briançon-Speder family and the Oka family in [10] using the analysis at high school mathematics level. In [27] T. Fukui is explaining in detail how to draw and understand the pictures of their zero-sets using the Newton Polyhedra determined by the defining functions.

4. Problem

In the previous sections, we discussed the Lipschitz equisingularity of the Briançon-Speder family and the Oka family. In the complex case they are well-known as examples of families of functions which are μ-constant but not μ^*-constant. From our observations, it gives rise to the following problem:

Problem 4.1. Let $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $t \in J$, be a family of analytic functions, and let $f_t^{\mathbb{C}} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ denote the complexification of f_t. Assume that $\mu(f_t^{\mathbb{C}})$ $(< \infty)$ is constant but $\mu^*(f_t^{\mathbb{C}})$ is not constant. Then is the family of zero-sets $\{(\mathbb{R}^n, f_t^{-1}(0))\}_{t \in J}$ not biLipschitz trivial ?

Remark 4.2. In the case $n = 2$, there is no family which satisfies the assumption of the above problem.

The problem whether the biLipschitz triviality implies the Kuo-Verdier (w)-regularity in the complex case has been worked on by J.-P. Henry, T. Mostowski and M. Merle. The case $n = 3$ was shown affirmatively ([19]). See [12, 26] for the (w)-regularity which is equivalent to the Whitney (b)-regularity in the complex case (J.-P. Henry and M. Merle [7]). This condition is also equivalent to the μ^*-constancy (B. Teissier [25], J. Briançon and J.P. Speder [3]). Therefore, "not μ^*-constancy" implies "not biLipschitz triviality" in the complex 3 variables case. But this does not hold in the real case, namely, we cannot omit the assumption of μ-constancy from our problem.

Example 4.3. Let $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}$, be a family of polynomial functions defined by

$$f_t(x, y, z) = x^2 + y^2 + t^2 z^2 + z^4.$$

Then $\mu(f_t^{\mathbb{C}}) < \infty$, $t \in \mathbb{C}$, and $\{f_t^{\mathbb{C}}\}$ is not μ-constant. As a result, it is not μ^*-constant, either. On the other hand, $\{(\mathbb{R}^3, f_t^{-1}(0))\}_{t\in\mathbb{R}}$ is biLipschitz trivial.

References

[1] E. Bierstone, P. D. Milman, *Arc-analytic functions*, Invent. math. **101** (1990), 411–424.

[2] J. Briançon, J.P. Speder, *La trivialité topologique n'implique pas les conditions de Whitney*, C. R. Acad. Sci. Paris **280** (1975), 365–367.

[3] J. Briançon, J.P. Speder, *Les conditions de Whitney impliquent μ^* constant*, Ann. Inst. Fourier **26**.2 (1976), 153–163.

[4] T. Fukui, *The modified analytic trivialization via the weighted blowing up*, J. Math. Soc. Japan **44** (1992), 455–459.

[5] T. Fukui, S. Koike, T.-C. Kuo, *Blow-analytic equisingularities, properties, problems and progress*, Real Analytic and Algebraic Singularities, Pitman Research Notes in Mathematics Series **381**, Longman, 1998, pp. 8–29.

[6] T. Fukui, L. Paunescu, *Modified analytic trivialization for weighted homogeneous function-germs*, J. Math. Soc. Japan **52** (2000), 433–446.

[7] J.-P. Henry, M. Merle, *Limites de normales, conditions de Whitney et éclatement d'Hironaka*, Proceedings of Symposia in Pure Mathematics **40** Part I, Singularities, A.M.S., 1983, pp. 575–584.

[8] J.-P. Henry, A. Parusiński, *Existence of Moduli for bi-Lipschitz equivalence of analytic functions*, Compositio Math. (to appear).

[9] J.-P. Henry, A. Parusiński, *Invariants of bi-Lipschitz equivalence of real analytic functions*, preprint.

[10] S. Koike, *On strong C^0-equivalence of real analytic functions*, J. Math. Soc. Japan **45** (1993), 313–320.

[11] T.-C. Kuo, *A complete determination of C^0-sufficiency in $J^r(2,1)$*, Invent. math. **8** (1969), 226–235.

[12] T.-C. Kuo, *The ratio test for analytic Whitney stratifications*, Liverpool Singularities Sympos. I, Lecture Notes in Math. **192**, Springer-Verlag, Berlin and New York, 1971, pp. 141–149.

[13] T.-C. Kuo, *Characterizations of v-sufficiency of jets*, Topology **11** (1972), 115–131.

[14] T.-C. Kuo, *The modified analytic trivialization of singularities*, J. Math. Soc. Japan **32** (1980), 605–614.

[15] T.-C. Kuo, *On classification of real singularities*, Invent. math. **82** (1985), 257–262.

[16] K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann. **282** (1988), 445–462

[17] T. Mostowski, *Lipschitz equisingularity*, Dissertationes Math. **243** (1985).

[18] T. Mostowski, *Lipschitz equisingularity problems*, in this volume.

[19] T. Mostowski, Personal communication (April, 2003).

[20] M. Oka, *On the weak simultaneous resolution of a negligible truncation of the Newton boundary*, Contemporary Math. **90** (1989), 199–210.

[21] A. Parusiński, *Lipschitz stratification of real analytic sets*, Singularities, Banach Center Publications **20** (1988), 323–333.

[22] A. Parusiński, *Lipschitz properties of semi-analytic sets*, Ann. Inst. Fourier **38** (1988), 189–213.

[23] A. Parusiński, *Lipschitz stratification of subanalytic sets*, Ann. Sci. Ec. Norm. Sup. **27** (1994), 661–696.

[24] L. Paunescu, *An example of blow analytic homeomorphism*, Real Analytic and Algebraic Singularities, Pitman Research Notes in Mathematics Series **381**, Longman, 1998, pp. 62–63.

[25] B. Teissier, *Cycles évanescents, sections planes, et conditions de Whitney*, Singularités à Cargèse, Astérisque, **7** et **8** (1973), 285–362.

[26] J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. math. **36** (1976), 295–312.

[27] E. Yoshinaga, T. Fukui, 特異点とニュートン図形、「解析関数と特異点」第 I 部, 特異点の数理 **3**, pp. 1–163, 共立出版, 2002.

DEPARTMENT OF MATHEMATICS, HYOGO UNIVERSITY OF TEACHER EDUCATION, 942-1 SHIMOKUME, KATO, YASHIRO, HYOGO 673-1494, JAPAN

E-mail address: `koike@sci.hyogo-u.ac.jp`