JØRGENSEN'S INEQUALITY FOR COMPLEX HYPERBOLIC 2- SPACE

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1 Introduction

Jørgensen's inequality gives a necessary condition for non-elementary two generator group of isometries of hyperbolic space to be discrete. We give analogues of Jørgensen's inequality for non-elementary groups of isometries of complex hyperbolic 2-space generated by two elements, one of which is either loxodromic or boundary elliptic.

This is a joint work with Jiang Yueping (Hunan University) and John. R. Parker (University of Durham).

2 The classical Jørgensen’s inequality

We discuss the original inequality of Jørgensen and reformulate in a way that we can generalize. Jørgensen takes two elements $A$ and $B$ in $\text{SL}(2,\mathbb{C})$ and says that if

$$|tr^2(A) - 4| + |tr(ABA^{-1}B^{-1}) - 2| < 1,$$

then the group $< A, B >$ generated by $A$ and $B$ is either elementary or not discrete. In this paper we will only be concerned with the cases where $A$ is loxodromic or elliptic. We may reformulate Jørgensen's inequality in terms of cross ratios of fixed points. Jørgensen's inequality is equivalent to

Theorem 1. Let $A$ and $B$ be elements of $\text{SL}(2,\mathbb{C})$ so that $A$ is either loxodromic or elliptic with fixed points $\mu$ and $\nu$ in $\hat{\mathbb{C}}$. Let $M = |tr^2(A) - 4|^\frac{1}{2}$.

If either

$$M^2(||B(\mu), \nu, B(\nu)|| + 1) < 1 \quad \text{or} \quad M^2(||B(\mu), \mu, \nu, B(\nu)|| + 1) < 1,$$

then the group $< A, B >$ is either elementary or not discrete.

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3 Preliminaries

Let $\mathbb{C}^{2,1}$ be a complex vector space of dimension 3 with the Hermitian form of signature (2,1). We choose the Hermitian form on $\mathbb{C}^{2,1}$ to be given by the matrix

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Thus $<z, w> = w^* J z = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1$.

We define the Siegel domain model of complex 2-space, $\mathbb{H}_C^2$ as follows: We identify points of $\mathbb{H}_C^2$ with their horospherical coordinates, $z = (\zeta, v, u) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ = \mathbb{H}_C^2$. Similarly, points in $\partial \mathbb{H}_C^2 = \mathbb{C} \times \mathbb{R} \cup \{\infty\}$ are either $z = (\zeta, v, 0) \in \mathbb{C} \times \mathbb{R} \times \{0\}$ or a point at infinity, which is denoted by $\infty$. Define the map $\phi : \overline{\mathbb{H}_C^2} \to \mathbb{P}\mathbb{C}^{2,1}$ by

$$\phi : (\zeta, v, u) \mapsto \left([(-|\zeta|^2 - u + iv)/2, \zeta, 1]^t, \phi : \infty \mapsto [1, 0, 0]^t. $$

The map $\phi$ is a homeomorphism from $\mathbb{H}_C^2$ to the set of points $z$ in $\mathbb{P}\mathbb{C}^{2,1}$ with $<z, z>$ < 0. Also $\phi$ is a homeomorphism from $\partial \mathbb{H}_C^2$ to the set of points $z$ in $\mathbb{P}\mathbb{C}^{2,1}$ with $<z, z> = 0$. Let $L$ be a complex line intersecting $\mathbb{H}_C^2$. Then $\phi(L)$ is a 2-dimensional subspace of $\mathbb{C}^{2,1}$. The orthogonal complement of this space is a one (complex) dimensional subspace of $\mathbb{C}^{2,1}$ spanned by a vector $p$ with $<p, p> > 0$. Without loss of generality, we take $<p, p> = 1$ and call $p$ the polar vector corresponding to the complex line $L$.

The Bergman metric on $\mathbb{H}_C^2$ is defined by the following formula for the distance $\rho$ between points $z$ and $w$ of $\mathbb{H}_C^2$:

$$\cosh\left(\frac{\rho(z, w)}{2}\right) = \frac{<\phi(z), \phi(w)> <\phi(w), \phi(z)>}{<\phi(z), \phi(z)> <\phi(w), \phi(w)>}. $$

The holomorphic isometry group of $\mathbb{H}_C^2$ with respect to the Bergman metric is the projective unitary group $PU(2, 1)$ and acts on $\mathbb{P}\mathbb{C}^{2,1}$ by matrix multiplication. A matrix $g \in \text{GL}(3, \mathbb{C})$ is in $PU(2, 1)$ if and only if it preserves the Hermitian form given by $J$. For four distinct points $z_1, z_2, w_1, w_2$ of $\overline{\mathbb{H}_C^2}$ the cross-ratio is defined as

$$\frac{<\phi(w_1), \phi(z_1)> <\phi(w_2), \phi(z_2)>}{<\phi(w_2), \phi(z_1)> <\phi(w_1), \phi(z_2)>}. $$

In order to represent the holomorphic isometries of $\mathbb{H}_C^2$, we work with the special unitary group $SU(2, 1)$ throughout this paper.
4 Subgroups with loxodromic generators

We give our results about the subgroups with loxodromic elements. Let \( A \) be a loxodromic element with fixed points \( \mu \) and \( \nu \) in \( \partial \mathbb{H}_2^3 \). Suppose that \( A \) has a complex dilation factor \( \lambda(A) \). We define a conjugation invariant factor \( M \) by

\[
M = |\lambda(A) - 1| + |\lambda(A)^{-1} - 1|.
\]

Theorem 2. Let \( A \) be a loxodromic element of \( SU(2,1) \) fixing \( \mu \) and \( \nu \), and let \( B \) be any element of \( SU(2,1) \). If either

\[
M(||B(\mu), \nu, \mu, B(\nu)||^{1/2} + 1) < 1 \quad \text{or} \quad M(||B(\mu), \mu, \nu, B(\nu)||^{1/2} + 1) < 1,
\]

then the group \( \langle A, B \rangle \) is either elementary or not discrete.

Theorem 3. Let \( A \) be a loxodromic element of \( SU(2,1) \) fixing \( \mu \) and \( \nu \), and let \( B \) be any element of \( SU(2,1) \). If \( M \leq \sqrt{2} - 1 \) and

\[
||B(\mu), \nu, \mu, B(\nu)|| + ||B(\mu), \mu, \nu, B(\nu)|| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},
\]

then the group \( \langle A, B \rangle \) is either elementary or not discrete.

We can show that neither theorem is a consequence of the other one.

5 Subgroups with boundary elliptic elements

Let \( A \) be a boundary elliptic element of \( SU(2,1) \). Then \( A \) fixes a complex line in \( \mathbb{H}_2^3 \). We denote this complex line by \( L_A \) and its polar vector by \( p_A \).

The fixed complex line of \( BAB^{-1} \) is \( B(L_A) \), which has the polar vector \( B(p_A) \). Normalizing \( p_A \) and \( B(p_A) \) so that \( \langle p_A, B(p_A) \rangle = \langle B(p_A), B(p_A) \rangle = 1 \), we have three cases:

1. If \( \langle p_A, B(p_A) \rangle \) < 1, then \( L_A \) and \( B(L_A) \) intersect at a point in \( \mathbb{H}_2^3 \). Moreover, \( \langle p_A, B(p_A) \rangle = \cos \psi \), where \( \psi \) is the angle of intersection between \( L_A \) and \( B(L_A) \). In particular, if \( \langle p_A, B(p_A) \rangle = 0 \), then \( L_A \) and \( B(L_A) \) intersect orthogonally.

2. If \( \langle p_A, B(p_A) \rangle = 1 \), then either \( B(L_A) = L_A \) or \( L_A \) and \( B(L_A) \) are asymptotic at a point in \( \partial \mathbb{H}_2^3 \).

3. If \( \langle p_A, B(p_A) \rangle > 1 \), then \( L_A \) and \( B(L_A) \) are ultraparallel, that is, they are disjoint and have a common orthogonal complex geodesic. Moreover, \( \langle p_A, B(p_A) \rangle = \cosh \rho \), where \( \rho \) is the distance between \( L_A \) and \( B(L_A) \).

For a boundary elliptic element \( A \in SU(2,1) \) we define the order of \( A \) as

\[
ord(A) = \inf\{m > 0 : A^m = I\}.
\]
Theorem 4. Let $A$ be a boundary elliptic element of $SU(2,1)$ which rotates through an angle $\theta = 2\pi/n$ about a complex line $L_A$. Let $B$ be any element of $SU(2,1)$ so that $B(L_A) \neq L_A$. If one of the following three conditions (1), (2) and (3) is satisfied, then the group $<A,B>$ is not discrete.

1. $L_A$ and $B(L_A)$ intersect at an angle $\psi \neq \pi/2$ and $\text{ord}(A) = n \geq 6$.
2. $L_A$ and $B(L_A)$ are asymptotic and $\text{ord}(A) = n \geq 7$.
3. $L_A$ and $B(L_A)$ are ultraparallel and

$$|\cosh \frac{\rho}{2} \sin \frac{\theta}{2}| < \frac{1}{2},$$

where $\rho$ is the distance between $L_A$ and $B(L_A)$.

If $L_A$ and $B(L_A)$ intersect orthogonally and

$$|\text{tr}(B) \sin \frac{\theta}{2}| < \frac{1}{2},$$

then the group $<A,B>$ is either elementary or not discrete.

Theorem 5. Let $A$ be a boundary elliptic element fixing the complex line $L_A$ spanned by $\mu$ and $\nu$ in $\partial \mathbb{H}_2^C$. Suppose that $B$ is any element of $SU(2,1)$ for which $L_A$ and $B(L_A)$ do not intersect orthogonally. If either

$$M(|[B(\mu), \nu, B(\nu)]|^{1/2} + 1) < 1 \quad \text{or} \quad M(|[B(\mu), \mu, \nu, B(\nu)]|^{1/2} + 1) < 1,$$

then the group $<A,B>$ is either elementary or not discrete.

Theorem 6. Let $A$ be a boundary elliptic element fixing the complex line $L_A$ spanned by $\mu$ and $\nu$ in $\partial \mathbb{H}_2^C$. Suppose that $B$ is any element of $SU(2,1)$ for which $L_A$ and $B(L_A)$ do not intersect orthogonally. If $M \leq \sqrt{2} - 1$ and

$$|[B(\mu), \nu, B(\nu)]| + |[B(\mu), \mu, \nu, B(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then the group $<A,B>$ is either elementary or not discrete.
References


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