Jørgensen groups of parabolic type II
(Countable infinite case)

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ABSTRACT. This paper is the second part of the study on Jørgensen groups of parabolic type. We will show that there are countable infinite many Jørgensen groups of parabolic type on a certain cylinder in this case.

1. Introduction.

1.1. It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup \( G \) of the Möbius transformation group is discrete. For this problem there are two important and useful theorems: One is Poincaré’s polyhedron theorem, which is a sufficient condition for \( G \) to be discrete. The other is Jørgensen’s inequality, which is a necessary condition for a two-generator Möbius transformation group \( \langle A, B \rangle \) to be discrete. Here we will consider extreme discrete groups (Jørgensen groups) for Jørgensen’s inequality. This paper is the second part

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of a series of studies on Jørgensen groups (cf. Li - Oichi - Sato [4]).

1.2. Let Möb denote the set of all linear fractional transformations (Möbius transformations)

\[ A(z) = \frac{az + b}{cz + d} \]

of the extended complex plane \( \hat{C} = \mathbb{C} \cup \{\infty\} \), where \( a, b, c, d \) are complex numbers and the determinant \( ad - bc = 1 \). There is an isomorphism between Möb and \( PSL(2, \mathbb{C}) \). We always write elements of Möb as matrices with determinant 1 in this paper. We recall that Möb (= PSL(2, \( \mathbb{C} \))) acts on the upper half space \( H^3 \) of \( \mathbb{R}^3 \) as the group of conformal isometries of hyperbolic 3-space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of Möb. A Kleinian group \( G \) is of the first kind if the limit set \( \Lambda(G) \) of \( G \) is all of the extended complex plane \( \hat{C} \) and it is of the second kind otherwise. A subgroup \( G \) of Möb is said to be elementary if there exists a finite \( G \)-orbit in \( \mathbb{R}^3 \).

1.3. The trace \( \text{tr}(A) \) of the matrix

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1) \]

in \( SL(2, \mathbb{C}) \) is defined by \( \text{tr}(A) = a + d \). We remark that the trace of an element of Möb (= PSL(2, \( \mathbb{C} \))) is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

1.4. Let \( A^* \) and \( B^* \) be matrices in \( SL(2, \mathbb{C}) \) representing the Möbius transformations \( A \) and \( B \), respectively. As \( A^* \) and \( B^* \) are determined by \( A \) and \( B \) to within a factor of \(-1\), we see that the commutator \( A^*B^*(A^*)^{-1}(B^*)^{-1} \) (resp. \( (A^*)^2 \)) are uniquely determined by \( A \) and \( B \) (resp. \( A \)). Thus we may write \( \text{tr}(ABA^{-1}B^{-1}) = \text{tr}(A^*B^*(A^*)^{-1}(B^*)^{-1}) \) and \( \text{tr}^2(A) = \text{tr}^2(A^*) \).
In 1976 Jørgensen obtained the following important theorem, which gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B \rangle$ to be discrete.

**Theorem A (Jørgensen [1]).** Suppose that the Möbius transformations $A$ and $B$ generate a non-elementary discrete group. Then

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$  

The lower bound 1 is best possible.

1.5. **Definition 1.** Let $A$ and $B$ be Möbius transformations. The Jørgensen number $J(A, B)$ for the ordered pair $(A, B)$ is defined by

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2|.$$  

**Definition 2.** A subgroup $G$ of Möb is called a Jørgensen group if $G$ satisfies the following four conditions:

1. $G$ is a two-generator group.
2. $G$ is a discrete group.
3. $G$ is a non-elementary group.
4. There exist generators $A$ and $B$ of $G$ such that $J(A, B) = 1$.

1.6 Jørgensen and Kiikka showed the following.

**Theorem B (Jørgensen-Kiikka [2]).** Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$. Then $A$ is elliptic of order at least seven or $A$ is parabolic.

If $\langle A, B \rangle$ is a Jørgensen group such that $A$ is parabolic and $J(A, B) = 1$, then we call it a Jørgensen group of parabolic type. There are infinite many Jørgensen groups of parabolic type (Jørgensen-Lascurain-Pignataro [3], Sato [6]).

Now it gives rise to the following problem.
Problem 1. Find all Jørgensen groups of parabolic type.

1.7. Let $\langle A, B \rangle$ be a marked two-generator group such that $A$ is parabolic. Then we can normalize $A$ and $B$ as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\sigma, \mu} = \begin{pmatrix} \mu \sigma & \mu^2 \sigma - 1/\sigma \\ \sigma & \mu \sigma \end{pmatrix}$$

where $\sigma \in C \setminus \{0\}$ and $\mu \in C$. See [4] for this normalization.

We denote by $G_{\sigma, \mu}$ the marked group generated by $A$ and $B_{\sigma, \mu}$: $G_{\sigma, \mu} = \langle A, B_{\sigma, \mu} \rangle$. We say that $(\sigma, \mu) \in C \setminus \{0\} \times C$ is the point representing a marked group $G_{\sigma, \mu}$ and that $G_{\sigma, \mu}$ is the marked group corresponding to a point $(\sigma, \mu)$.

1.8. In [6], Sato considered the case of $\mu = ik \ (k \in \mathbb{R})$. Namely, he considered marked two-generator group $G_{\sigma, ik} = \langle A, B_{\sigma, ik} \rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\sigma, ik} = \begin{pmatrix} ik \sigma & -k^2 \sigma - 1/\sigma \\ \sigma & ik \sigma \end{pmatrix}$$

where $\sigma \in C \setminus \{0\}$ and $k \in \mathbb{R}$.

Now we have the following conjecture.

CONJECTURE. For any Jørgensen group $G$ of parabolic type there exists a marked group $G_{\sigma, ik} (\sigma \in C \setminus \{0\}, k \in \mathbb{R})$ such that $G_{\sigma, ik}$ is conjugate to $G$.

If this conjecture is true, then it is sufficient to consider the case of $\mu = ik$ in order to find all Jørgensen groups of parabolic type. In this paper we only consider the case of $\mu = ik$.

1.9. Let $C$ be the following cylinder:

$$C = \{ (\sigma, ik) \mid |\sigma| = 1, k \in \mathbb{R} \}.$$

Theorem C (Sato [6]). If a marked two-generator group $G_{\sigma, ik} (\sigma \in C \setminus \{0\}, k \in \mathbb{R})$ is conjugate to $G_{\sigma, ik}$, then $G_{\sigma, ik}$ is conjugate to $G_{\sigma, ik}$.
R) is a Jørgensen group, then the point $(\sigma, ik)$ representing $G_{\sigma,ik}$ lies on the cylinder $C$.

By Theorem C, if $(\theta, k)$ is a point on the cylinder $C$, then we can set $\sigma = -ie^{i \theta} (0 \leq \theta \leq 2\pi)$. If a point $(-ie^{i \theta}, ik)$ on the cylinder $C$ represents a Jørgensen group, then we say that the group is a Jørgensen group of parabolic type $(\theta, k)$.

Now it gives rise to the following problem.

**Problem 2.** Find all Jørgensen groups of parabolic type $(\theta, k)$.

We divide Jørgensen groups of this type into three parts as follows:

- **Part 1.** $|k| \leq \sqrt{3}/2$, $0 \leq \theta \leq 2\pi$ (finite case).
- **Part 2.** $\sqrt{3}/2 < |k| \leq 1$, $0 \leq \theta \leq 2\pi$ (countable infinite case).
- **Part 3.** $1 < |k|$, $0 \leq \theta \leq 2\pi$ (uncountable infinite case).

By some lemmas in [6], it suffices to consider the case of $0 \leq \theta \leq \pi/2$ and $k \geq 0$ in order to find Jørgensen groups of parabolic type $(\theta, k)$.

In the previous paper [4] we find all Jørgensen groups in the case where $0 \leq \theta \leq \pi/2$ and $0 \leq k \leq \sqrt{3}/2$, that is, we obtain the following theorem.

**Theorem D** (finite case) (Li - Oichi - Sato [4]).

(i) There are sixteen Jørgensen groups in $D = \{ (\theta, k) \in \mathbb{R} | 0 \leq \theta \leq \pi/2, 0 \leq k \leq \sqrt{3}/2 \}$.

(ii) Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.

1.10. For a sufficient condition for a subgroup of the Möbius transformation group to be discrete, the following theorem is well-known.

**Theorem E** (Poincaré's Polyhedron Theorem (Maskit [5, p.73])).

Let $P$ be a polyhedron with side pairing transformations satisfying the following conditions (1) through (6). Then, $G$, the group generated by the side pairing trans-
formations, is discrete and $P$ is a fundamental polyhedron for $G$, and the reflection relations and cycle relations form a complete set of relations for $G$:

(1) For each side $s$ of $P$, there is a side $s'$ and there is an element $g_s \in G$ satisfying $g_s(s) = s'$ and $g_s' = g_s^{-1}$.

(2) $g_s(P) \cap P = \emptyset$.

(3) For every point $z \in P^*, p^{-1}(z)$ is a finite set. Here $P^*$ is the space of equivalence classes so that the projection $p : \bar{P} \rightarrow P^*$ is continuous and open.

(4) Let $e$ be an edge and let $h$ be the cycle transformation at $e$. Then for each edge $e$, there is a positive integer $t$ such that $h^t = 1$.

(5) Let $\{e_1, e_2, \ldots, e_m\}$ be any cycle of edges of $P$ and let $\alpha(e_k) \ (k = 1, 2, \ldots, m)$ be the angle measured from inside $P$ at the edge $e_k$. Let $q$ be the smallest positive integer such that $h^q = 1$, where $h$ is the cycle transformation at $e_1$. Then the equality

$$\sum_{k=1}^{m} \alpha(e_k) = 2\pi/q$$

holds.

(6) $P^*$ is complete.

2. Theorems.

In this section we will state that we find all Jørgensen groups in Part 2, that is, we obtain the following theorems. The proofs will appear elsewhere.

**Main Theorem.** There are countable infinite many Jørgensen groups on the region $\{ (\theta, k) \mid 0 \leq \theta \leq \pi/2, \sqrt{3}/2 < k \leq 1 \}$.

For simplicity we write $B_{\theta, k}$ for $B_{-ie^{i\theta}, ik}$.

This theorem consists of the following Theorem 1 through Theorem 6.
Let $A$ and $B_{\theta,k}$ ($k \in \mathbb{R}, 0 \leq \theta \leq \pi/2$) be the following matrices:

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & i e^{-i\theta} (k^2 e^{2i\theta} - 1) \\ -i e^{i\theta} & ke^{i\theta} \end{pmatrix}
\]

We can prove these theorems by using Jørgensen's inequality and Poincaré's polyhedron theorem.

**Theorem 1** (Li - Oichi - Sato [4]). Let $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ be the group generated by $A$ and $B_{\theta,k}$. If $0 < \theta < \pi/6$, $\pi/6 < \theta < \pi/4$, $\pi/4 < \theta < \pi/3$, $\pi/3 < \theta < \pi/2$, then $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$ are not Kleinian groups and so not Jørgensen groups for $k \in \mathbb{R}$.

**Theorem 2** (the case of $\theta = 0$). Let

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{0,k} = \begin{pmatrix} k & i(k^2 - 1) \\ -i & k \end{pmatrix} (k \in \mathbb{R}),
\]

and let $G_k = \langle A, B_k \rangle$. Then the following hold.

(i) In the case where $\cos(\pi/2m) < k < \cos(\pi/(2m + 2))$ and $k \neq \cos(\pi/(2m + 1))$ ($m = 3, 4, \cdots$), $G_k$ are not Kleinian groups and not Jørgensen groups.

(ii) In the case of $k = 1$, $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signature $(0; 2, 3, \infty)$.

(iii) In the case of $k = \cos(\pi/n)$ ($n = 7, 8, \cdots$), $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a union of two Riemann surfaces with signatures $(0; 2, 3, n)$ and $(0; 2, 3, \infty)$.

**Theorem 3** (the case of $\theta = \pi/6$). Let

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{\pi/6,k} = \begin{pmatrix} ke^{\pi i/6} & i(k^2 e^{\pi i/6} - e^{-\pi i/6}) \\ -i e^{\pi i/6} & ke^{\pi i/6} \end{pmatrix} (k \in \mathbb{R}),
\]
and let $G_k = \langle A, B_k \rangle$. Then $G_k$ are not Kleinian groups and not Jørgensen groups for $k$ with $\sqrt{3}/2 < k \leq 1$.

**Theorem 4** (the case of $\theta = \pi/4$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{\pi/4, k} = \begin{pmatrix} ke^{\pi i/4} & i(k^2e^{\pi i/4} - e^{-\pi i/4}) \\ -ie^{\pi i/4} & ke^{\pi i/4} \end{pmatrix} \quad (k \in \mathbb{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then the following hold.

(i) In case of $\sqrt{3}/2 < k < 1$, $G_k$ are not Kleinian groups and not Jørgensen groups.

(ii) In the case of $k = 1$, $G_k$ is a Kleinian group of the first kind and a Jørgensen group. The volume $V(G_{\pi/4, 1})$ of the 3-orbifold for $G_{\pi/4, 1}$ is

$$V(G_{\pi/4, 1}) = 8[2L(\pi/4) - L(\pi/12) - L(5\pi/12)],$$

where $L(\theta)$ is the Lobachevskii function:

$$L(\theta) = -\int_0^\theta \log|2\sin u| \, du.$$

**Theorem 5** (the case of $\theta = \pi/3$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{\pi/3, k} = \begin{pmatrix} ke^{\pi i/3} & i(k^3e^{\pi i/3} - e^{-\pi i/3}) \\ -ie^{\pi i/3} & ke^{\pi i/3} \end{pmatrix} \quad (k \in \mathbb{R}),$$

and let $G_k = \langle A, B_k \rangle$. Then $G_k$ are not Kleinian groups and not Jørgensen groups for $k$ with $\sqrt{3}/2 < k \leq 1$.

**Theorem 6** (the case of $\theta = \pi/2$). Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{\pi/2, k} = \begin{pmatrix} ik & -(k^2 + 1) \\ 1 & ik \end{pmatrix} \quad (k \in \mathbb{R}),$$
and let $G_k = \langle A, B_k \rangle$. Then the following hold.

(i) In the case where $\cos(\pi/(2n-1)) < k < \cos(\pi/(2n + 1))$ and $k \neq \cos(\pi/2n)$ ($n = 3, 4, \ldots$), $G_k$ are not Kleinian groups and not Jørgensen groups.

(ii) In the case of $k = 1$, $G_k$ is a Kleinian group of the second kind and a Jørgensen group, and $\Omega(G_k)/G_k$ is a Riemann surface with signature $(0; 3, 3, \infty)$.

(iii) In the case of $k = \cos(\pi/n)$ ($n = 7, 8, \ldots$), $G_k$ are Kleinian groups of the second kind and Jørgensen groups, and $\Omega(G_k)/G_k$ is a Riemann surface with signature $(0; 3, 3, n)$.

References


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