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The order of conformal automorphisms of Riemann surfaces of infinite type — supplement

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1 Introduction

On a compact Riemann surface $R$ of genus $g \geq 2$, the order of a conformal automorphism of $R$ is not greater than $2(2g + 1)$ (see [7]). However for a Riemann surface with the infinitely generated fundamental group, the order of a conformal automorphism is not finite, in general. In [3], we showed a necessary and sufficient condition for a conformal automorphism of a Riemann surface to have finite order.

**Proposition 1** ([3]) Let $R = \mathbb{H}/\Gamma$, where $\Gamma$ is a Fuchsian group which is not necessarily torsion-free. Suppose that $R$ has the non-abelian fundamental group. Then a conformal automorphism $f$ of $R$ has finite order if and only if $f$ fixes either a simple closed geodesic, a puncture, a point or a cone point on $R$.

On the basis of Proposition 1, for a Riemann surface $R$ such that the injectivity radius at any point in $R$ is uniformly bounded from above, we estimated the order of conformal automorphisms of $R$ in terms of the injectivity radius. One of the results is the following.

**Proposition 2** ([3]) Let $R$ be a hyperbolic Riemann surface. Suppose that there exists a constant $M > 0$ such that the injectivity radius at any point in $R$ is less than $M/2$. Let $f$ be a conformal automorphism of $R$ such that $f(c) = c$ for a simple closed geodesic $c$ on $R$ whose length is $\ell > 0$. Then the order $n$ of $f$ satisfies

$$n < (e^M - 1) \cosh(\ell/2).$$
In this note, for a Riemann surface $R$ such that the injectivity radius at any point in $R$ is not necessarily uniformly bounded from above, we prove the same statements.

2 Statements of Theorems

Let $\mathbf{H}$ be the upper-half plane equipped with the hyperbolic metric $|dz|/\text{Im}z$. We say that a Riemann surface $R$ is hyperbolic if it is represented by $\mathbf{H}/\Gamma$ for a torsion-free Fuchsian group $\Gamma$ acting on $\mathbf{H}$. The hyperbolic distance on $\mathbf{H}$ or on $R$ is denoted by $d(\cdot, \cdot)$, and the hyperbolic length of a curve $c$ on $R$ is denoted by $\ell(c)$.

Definition For a constant $M > 0$, we define $R_M$ to be the subset of points $p \in R$ such that there exists a non-trivial simple closed curve $c_p$ passing through $p$ with $\ell(c_p) < M$.

Remark The injectivity radius at a point $p \in R$ is the supremum of radii of embedded hyperbolic discs centered at $p$. The $R_M$ is nothing but the set of points in $R$ where the injectivity radius is less than $M/2$.

We consider the following condition in terms of hyperbolic geometry on Riemann surfaces $R$.

Definition We say that $R$ satisfies the upper bound condition if there exist a constant $M > 0$ and a connected component $R_M^*$ of $R_M$ such that a homomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ that is induced by the inclusion map of $R_M^*$ into $R$ is surjective.

Remark (i) If the injectivity radius at any point in $R$ is uniformly bounded from above, then $R$ clearly satisfies the upper bound condition. (ii) If $R$ ($\neq \mathbf{H}$) is a normal covering surface of an analytically finite Riemann surface, then $R$ satisfies the upper bound condition (see [4]).

We state our theorems.

Theorem 1 (hyperbolic case) Let $R$ be a hyperbolic Riemann surface with the non-abelian fundamental group. Suppose that $R$ satisfies the upper bound condition for a constant $M > 0$ and a connected component $R_M^*$ of $R_M$. Let $f$ be a conformal automorphism of $R$ such that $f(c) = c$ for a simple closed geodesic $c$ on $R$ with $c \subset R_M^*$ and $\ell(c) = \ell > 0$. Then the order $n$ of $f$ satisfies

$$n < (e^M - 1) \cosh(\ell/2).$$
Theorem 2 (parabolic case) Let $R$ be a hyperbolic Riemann surface with the non-abelian fundamental group. Suppose that $R$ satisfies the upper bound condition for a constant $M > 0$. Let $f$ be a conformal automorphism of $R$ such that $f(p) = p$ for a puncture $p$ of $R$. Then the order $n$ of $f$ satisfies

$$n < e^M - 1.$$ 

Theorem 3 (elliptic case) (i) Let $R$ be a hyperbolic Riemann surface with the non-abelian fundamental group, and $f$ a conformal automorphism of $R$ such that $f(p) = p$ for a point $p$ in $R$ at which the injectivity radius is $M > 0$. Then the order $n$ of $f$ satisfies

$$n < 2\pi \cosh M.$$ 

(ii) Let $R = \mathbb{H}/\Gamma$, where $\Gamma$ is a Fuchsian group which is not torsion-free. Suppose that $R$ has the non-abelian fundamental group and satisfies the upper bound condition for a constant $M > 0$. Let $f$ be a conformal automorphism of $R$ such that $f(p) = p$ for a cone point $p$ in $R$ which is a projection of a fixed point $\tilde{p}$ of an elliptic element of $\Gamma$ with order $m > 1$. Then the order $n$ of $f$ satisfies

$$n < (e^M - 1) \frac{\pi}{m} \left( \frac{1}{\sin^2 \frac{\pi}{m}} + \frac{1}{\sinh^2 \frac{M}{2}} \right)^{\frac{1}{2}}.$$ 

Remark The upper bound of the order of $f$ obtained in Theorem 2 is the limiting case of that in Theorem 1 as $\ell \to 0$. It is also the limiting case of that in Theorem 3 (ii) as $m \to \infty$.

Remark In [5], we obtained a better estimate than that in Theorem 1 in the case where $\ell \geq M$.

3 Proofs of Theorems

We prove Theorem 1 only, for we can prove the other theorems by using the same argument in the proof of Theorem 1 and the proofs of Theorems 2 and 3 in [3].

Definition A subset $S \subset \mathbb{H}$ is said to be precisely invariant under a subgroup $\Gamma_S$ of a Fuchsian group $\Gamma$ if $\gamma(S) = S$ for all $\gamma \in \Gamma_S$ and $\gamma(S) \cap S = \emptyset$ for all $\gamma \in \Gamma - \Gamma_S$. 
Collar Lemma ([6], [8]) Let $\Gamma$ be a Fuchsian group (which is not necessarily torsion-free) acting on $\mathbb{H}$, and $L$ an axis of a hyperbolic element $\gamma \in \Gamma$ whose translation length is less than $\ell$. Assume that there exists no fixed points of elements in $\Gamma$ on $L$ and that $L$ is precisely invariant under the cyclic subgroup $\langle \gamma \rangle$ generated by $\gamma$. Then a collar

$$C(L) = \{ z \in \mathbb{H} \mid d(z, L) \leq \omega(\ell) \}$$

is precisely invariant under $\langle \gamma \rangle$, where $\sinh \omega(\ell) = (2\sinh(\ell/2))^{-1}$. Equivalently, the boundaries $\partial C(L)$ of $C(L)$ and the real axis make an angle $\theta$, where $\tan \theta = 2\sinh(\ell/2)$.

The proof of Theorem 1 follows from the fact that there exists a wider collar of the simple closed geodesic $c$, as the order of a conformal automorphism $f$ fixing $c$ increases.

Proof of Theorem 1: Let $\Gamma$ be a Fuchsian model of $R$, and $\tilde{f}$ a lift of $f$ which is a hyperbolic element in $\text{PSL}_2(\mathbb{R})$. Note that $\tilde{f}^n$ is a hyperbolic element in $\Gamma$ which is corresponding to $c$. We consider the quotient $\hat{R} = R/\langle f \rangle$ by the cyclic group $\langle f \rangle$ and its Fuchsian model $\hat{\Gamma} = \langle \Gamma, \tilde{f} \rangle$. Then $\hat{c} = c/\langle f \rangle$ is a non-trivial simple closed geodesic on $\hat{R}$ whose length is $\ell/n$. Since $\tilde{f}$ is corresponding to $\hat{c}$, we may assume that $\tilde{f}(z) = \exp(\ell/n)z$ with the axis $L = \{ iy \mid y > 0 \}$. Applying Collar Lemma for $\hat{\Gamma}$ and $\tilde{f}$, we can take a collar

$$\tilde{C}(L) = \{ re^{i\theta} \in \mathbb{H} \mid 0 < r, \theta_0 < \theta < \pi - \theta_0 \}$$

so that it is precisely invariant under $\langle \tilde{f} \rangle \subset \hat{\Gamma}$, where

$$\tan \theta_0 = 2\sinh(\ell/2n).$$

In particular, $\gamma(\tilde{C}(L)) \cap \tilde{C}(L) = \emptyset$ for any $\gamma \in \Gamma - \langle \tilde{f}^n \rangle$. Then we can take a tubular neighborhood $C(c) = \tilde{C}(L)/\langle \tilde{f}^n \rangle$ of $c$ on $R$ whose fundamental region is

$$A = \{ re^{i\theta} \in \mathbb{H} \mid 1 < r < e^\ell, \theta_0 < \theta < \pi - \theta_0 \}.$$ 

We may assume that $d(c, \partial C(c)) = \omega(\ell/n) > M/2$. Indeed, suppose that

$$\omega(\ell/n) = \arcsinh \left( \frac{1}{2\sinh(\ell/2n)} \right) \leq \frac{M}{2}.$$ 

Using the fact that $x^{-1}\sinh x$ is a monotone increasing function for $x > 0$, we see that

$$\cosh \frac{\ell}{2} \exp \frac{M}{2n} \geq \cosh \frac{\ell}{2} > \sinh \frac{\ell}{2} = \frac{\ell}{2n} \sinh \frac{\ell}{2} \geq \frac{\ell}{2n} \sinh \frac{\ell}{2n} = \sinh \frac{\ell}{2n}.$$
for $n > 1$, $\ell > 0$ and $M > 0$. Then
\[
\frac{n}{2 \cosh \frac{\ell}{2} \exp \frac{M}{2}} < \frac{1}{2 \sinh \frac{\ell}{2n}} \leq \sinh \frac{M}{2}.
\]
This implies that
\[
n < 2 \exp(M/2) \sinh(M/2) \cosh(\ell/2) = (e^M - 1) \cosh(\ell/2),
\]
and we have nothing to prove.

We can take a point $p$ in $C(c)$ which satisfies $d(p, \partial C(c)) = M/2$ and $p \in R^*_M$. Here $\partial C(c)$ is the boundary of $C(c)$. Indeed, if there exist no such points, then any point in two simple closed curves $\{p \in C(c) \mid d(p, \partial C(c)) = M/2\}$ does not belong to $R^*_M$. This means that $R^*_M$ is a tubular neighborhood of $c$, and this contradicts the upper bound condition.

By the definition of $R_M^*$, the length $r_p$ of the shortest non-trivial simple closed curve $\alpha$ passing through $p$ is less than $M$. Since $d(p, \partial C(c)) = M/2$, the curve $\alpha$ is in $C(c)$. Let $\tilde{p} = re^{i\theta} \in A$ ($\theta_0 < \theta < \pi/2$) be a lift of $p$. Setting $z_1(t) = re^{it}$ for $t \geq 0$, we have
\[
\frac{M}{2} = d(\tilde{p}, \partial \tilde{C}(L)) = \int_{\theta_0}^{\theta} \frac{|z_1'(t)|}{\text{Im} z_1(t)} dt = \int_{\theta_0}^{\theta} \frac{1}{\sin t} dt = \log \frac{\theta}{\theta_0}.
\]
Hence $\theta \leq \exp(M/2)\theta_0$. We put $a = \exp(i\theta)$ and $b = \exp(\ell + i\theta)$. Then $r_p = d(a, b)$. From Theorem 7.2.1 in [1], we have
\[
\frac{1}{2} \sinh \frac{1}{2} d(a, b) = \frac{|a - b|}{2 \sqrt{\text{Im} a \text{Im} b}} = \frac{\cosh \frac{\ell}{2}}{2 \exp \frac{\ell}{2} \sin \theta} = \frac{\sinh \frac{\ell}{2}}{\sin \theta} \geq \frac{\sinh \frac{\ell}{2}}{\theta_0 \exp \frac{M}{2}} = \frac{n \sinh \frac{\ell}{2n}}{\ell \exp \frac{M}{2}}.
\]
For the last inequality, we used the fact that $x(\sinh x)^{-1}$ is a monotone decreasing function for $x > 0$. Since $r_p < M$, this implies that
\[
n < 2 \exp(M/2) \sinh(M/2) \cosh(\ell/2) = (e^M - 1) \cosh(\ell/2).
\]
\[
\blacksquare
\]
4 Application

We apply Theorem 1 to investigating a certain property on hyperbolic geometry on Riemann surfaces. The following proposition is an extension of Proposition 3 in [3].

Definition We say that $R$ satisfies the lower bound condition if there exists a constant $\epsilon > 0$ (which is smaller than the Margulis constant) such that $R_{\epsilon}$ consists only of cusp neighborhoods and neighborhoods of geodesics which are homotopic to boundary components.

Proposition 3 Let $R$ be a hyperbolic Riemann surface, and $\tilde{R}$ a normal covering surface of $R$. If $\tilde{R}$ satisfies the lower and upper bound conditions, then $R$ also satisfies these conditions.

Proof. It is clear that $R$ satisfies the upper bound condition. Suppose that $R$ does not satisfy the lower bound condition. Then $R$ has a sequence $\{c_{n}\}$ of disjoint simple closed geodesics which are not homotopic to boundary components of $R$ with $\ell_{n} = \ell(c_{n}) \to 0 \ (n \to \infty)$. Let $\tilde{c}_{n} \subset \tilde{R}$ be a connected component of the preimage of $c_{n}$. Then $\tilde{c}_{n}$ is not homotopic to a boundary component of $\tilde{R}$. Since $\tilde{R}$ satisfies the lower bound conditions, there exists a constant $\epsilon > 0$ such that $\ell(\tilde{c}_{n}) > \epsilon$ for all $n$. We take a constant $M > 0$ so that $\tilde{R}$ satisfies the upper bound condition for $M$ and for a connected component $\tilde{R}_{M}^{*}$ of $\tilde{R}_{M}$. We may assume that $\tilde{c}_{n} \subset \tilde{R}_{M}^{*}$. Assume that $\ell(\tilde{c}_{n}) \leq M$ for infinitely many $n$. Then, by Theorem 1, the order of a conformal automorphism $\tilde{f}_{n}$ of $\tilde{R}$ fixing $\tilde{c}_{n}$ is less than $N = (e^{M} - 1) \cosh(M/2)$. Then we have $\ell(c_{n}) > \epsilon/N$. However, this contradicts $\ell(c_{n}) \to 0 \ (n \to \infty)$. Next, we assume that $\ell(\tilde{c}_{n}) > M$ (including the case that $\tilde{c}_{n}$ is not closed) for infinitely many $n$. By Collar Lemma, there exists a tubular neighborhood $C(c_{n})$ of $c_{n}$ with width $\omega(\ell_{n})$, where $\sinh \omega(\ell_{n}) = (2 \sinh(\ell_{n}/2))^{-1}$. From the proof of Theorem 1, there exists a (tubular) neighborhood of $\tilde{c}_{n}$ with width $\omega(\ell_{n})$. Since $\tilde{c}_{n} \subset \tilde{R}_{M}^{*}$, there exists a non-trivial simple closed curve passing through $\tilde{p}_{n} \in \tilde{c}_{n}$ whose length is less than $M$. However, since $\ell(\tilde{c}_{n}) > M$ and since $\omega(\ell_{n}) \to \infty$ as $n \to \infty$, we have a contradiction.

For applications of Proposition 3 to the action of Teichmüller modular groups, see [2].

References


