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ON TWO DISTANCES ON TEICHM"ULLER SPACE

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We consider a distance $d_L$ on the Teichmüller space $T(S_0)$ of a hyperbolic Riemann surface $S_0$. The distance is defined by the length spectrum of Riemann surfaces in $T(S_0)$ and we call it the length spectrum metric on $T(S_0)$. It is known that the distance $d_L$ determines the same topology as that of the Teichmüller metric if $S_0$ is a topologically finite Riemann surface.

We shall announce that there exists a Riemann surface $S_0$ of infinite type such that the length spectrum distance $d_L$ on $T(S_0)$ does not define the same topology as that of the Teichmüller distance. Also, we shall give a sufficient condition for these distance to have the same topology on $T(S_0)$. The proofs are given in [6].

1. Preliminaries

Let $S_0$ be a hyperbolic Riemann surface. We consider a pair $(S, f)$ of a Riemann surface $S$ and a quasiconformal homeomorphism $f$ of $S_0$ onto $S$. Two such pairs $(S_j, f_j)$ ($j = 1, 2$) are called equivalent if there exists a conformal mapping $h : S_1 \rightarrow S_2$ which is homotopic to $f_2 \circ f_1^{-1}$, and the equivalence class of $(S, f)$ is denoted by $[S, f]$. The set of all equivalence classes $[S, f]$ is called the Teichmüller space of $S_0$, which is denoted by $T(S_0)$.

The Teichmüller space $T(S_0)$ has a complete distance $d_T$ called the Teichmüller distance which is defined by

\[ d_T([S_1, f_1], [S_2, f_2]) = \inf_{f} \log K[f], \]

where the infimum is taken over all quasiconformal mapping $f : S_1 \rightarrow S_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K[f]$ is the maximal dilatation of $f$.

We define another distance on $T(S_0)$ by length spectrum of Riemann surfaces. Let $\Sigma(S)$ be the set of closed geodesics on a hyperbolic Riemann surface $S$. For any two points $[S_j, f_j]$ ($j = 1, 2$) in $T(S_0)$, we set

\[ \rho([S_1, f_1], [S_2, f_2]) = \sup_{c \in \Sigma(S_1)} \max \left\{ \frac{\ell_{S_1}(c)}{\ell_{S_2}(f_2 \circ f_1^{-1}(c))}, \frac{\ell_{S_2}(f_2 \circ f_1^{-1}(c))}{\ell_{S_1}(c)} \right\}, \]
where $\ell_S(\alpha)$ is the hyperbolic length of a closed geodesic on $S$ freely homotopic to a closed curve $\alpha$. For two points $[S_j, f_j] \in T(S_0)$ ($j = 1, 2$), we define a distance $d_L$ called the length spectrum distance by

$$d_L([S_1, f_1], [S_2, f_2]) = \log \rho([S_1, f_1], [S_2, f_2]).$$

Wolpert([8]) shows that $\ell_{S_2}(f(c)) \leq K[f]\ell_{S_1}(c)$ holds for every quasiconformal mapping $f : S_1 \rightarrow S_2$ and for every $c \in \Sigma(S_1)$. Thus, immediately we have:

**Lemma 1.1.** An inequality

$$d_L(p, q) \leq d_T(p, q)$$

holds for every $p, q \in T(S_0)$.

2. Results

On the Teichmüller space $T(S_0)$ of a hyperbolic Riemann surface $S_0$, we have the Teichmüller distance $d_T(\cdot, \cdot)$, which is a complete distance on $T(S_0)$. In this paper, we study another distance $d_L(\cdot, \cdot)$ which is defined by the length spectrum on Riemann surfaces in $T(S_0)$. Li [4] discussed the distance $d_L(\cdot, \cdot)$ on the Teichmüller space of a compact Riemann surface of genus $g \geq 2$ and showed that the distance $d_L$ defines the same topology as that of the Teichmüller distance. Recently, Liu [5] showed that the same statement is true even if $S_0$ is a Riemann surface of topologically finite type, and he asked whether the statement holds for Riemann surface of infinite type. The following first result of us gives a negative answer to this question.

**Theorem 2.1.** There exist a Riemann surface $S_0$ of infinite type and a sequence $\{p_n\}_{n=0}^{\infty}$ in $T(S_0)$ such that

$$d_L(p_n, p_0) \rightarrow 0 \quad (n \rightarrow \infty)$$

while

$$d_T(p_n, p_0) \rightarrow \infty \quad (n \rightarrow \infty).$$

From the proof of this theorem, we show the incompleteness of the length spectrum distance.

**Corollary 2.1.** There exists a Riemann surface of infinite type such that the length spectrum distance $d_L$ is incomplete in the Teichmüller space.

Next, we give a sufficient condition for the length distance to define the same topology as that of the Teichmüller distance as follows.
Theorem 2.2. Let $S_0$ be a Riemann surface. Assume that there exists a pants decomposition $S_0 = \bigcup_{k=1}^{\infty} P_k$ of $S_0$ satisfying the following conditions.

1. Each connected component of $\partial P_k$ is either a puncture or a simple closed geodesic of $S_0$ ($k=1, 2, \ldots$).
2. There exists a constant $M > 0$ such that if $\alpha$ is a boundary curve of some $P_k$ then
   \[ 0 < M^{-1} < \ell_{S_0}(\alpha) < M \]
   holds.

Then $d_L$ defines the same topology as that of $d_T$.

REFERENCES


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