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ON TWO DISTANCES ON TEICHMÜLLER SPACE

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We consider a distance d_L on the Teichmüller space $T(S_0)$ of a hyperbolic Riemann surface S_0. The distance is defined by the length spectrum of Riemann surfaces in $T(S_0)$ and we call it the length spectrum metric on $T(S_0)$. It is known that the distance d_L determines the same topology as that of the Teichmüller metric if S_0 is a topologically finite Riemann surface.

We shall announce that there exists a Riemann surface S_0 of infinite type such that the length spectrum distance d_L on $T(S_0)$ does not define the same topology as that of the Teichmüller distance. Also, we shall give a sufficient condition for these distance to have the same topology on $T(S_0)$. The proofs are given in [6].

1. Preliminaries

Let S_0 be a hyperbolic Riemann surface. We consider a pair (S, f) of a Riemann surface S and a quasiconformal homeomorphism f of S_0 onto S. Two such pairs (S_j, f_j) $(j = 1, 2)$ are called *equivalent* if there exists a conformal mapping $h : S_1 \to S_2$ which is homotopic to $f_2 \circ f_1^{-1}$, and the equivalence class of (S, f) is denoted by $[S, f]$. The set of all equivalence classes $[S, f]$ is called *the Teichmüller space* of S_0, which is denoted by $T(S_0)$.

The Teichmüller space $T(S_0)$ has a complete distance d_T called *the Teichmüller distance* which is defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \inf_f \log K[f],$$

where the infimum is taken over all quasiconformal mapping $f : S_1 \to S_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K[f]$ is the maximal dilatation of f.

We define another distance on $T(S_0)$ by length spectrum of Riemann surfaces. Let $\Sigma(S)$ be the set of closed geodesics on a hyperbolic Riemann surface S. For any two points $[S_j, f_j]$ $(j = 1, 2)$ in $T(S_0)$, we set

$$\rho([S_1, f_1], [S_2, f_2]) = \sup_{c \in \Sigma(S_1)} \max \left\{ \frac{\ell_{S_1}(c)}{\ell_{S_2}(f_2 \circ f_1^{-1}(c))}, \frac{\ell_{S_2}(f_2 \circ f_1^{-1}(c))}{\ell_{S_1}(c)} \right\},$$

where $\ell_S(\alpha)$ is the hyperbolic length of a closed geodesic on S freely homotopic to a closed curve α. For two points $[S_j, f_j] \in T(S_0)$ $(j = 1, 2)$, we define a distance d_L called *the length spectrum distance* by

$$d_L([S_1, f_1], [S_2, f_2]) = \log \rho([S_1, f_1], [S_2, f_2]).$$

Wolpert([8]) shows that $\ell_{S_2}(f(c)) \leq K[f]\ell_{S_1}(c)$ holds for every quasiconformal mapping $f : S_1 \to S_2$ and for every $c \in \Sigma(S_1)$. Thus, immediately we have:

Lemma 1.1. *An inequality*

$$d_L(p, q) \leq d_T(p, q)$$

holds for every $p, q \in T(S_0)$.

2. Results

On the Teichmüller space $T(S_0)$ of a hyperbolic Riemann surface S_0, we have the Teichmüller distance $d_T(\cdot, \cdot)$, which is a complete distance on $T(S_0)$. In this paper, we study another distance $d_L(\cdot, \cdot)$ which is defined by the length spectrum on Riemann surfaces in $T(S_0)$. Li [4] discussed the distance $d_L(\cdot, \cdot)$ on the Teichmüller space of a compact Riemann surface of genus $g \geq 2$ and showed that the distance d_L defines the same topology as that of the Teichmüller distance. Recently, Liu [5] showed that the same statement is true even if S_0 is a Riemann surface of topologically finite type, and he asked whether the statement holds for Riemann surface of infinite type. The following first result of us gives a negative answer to this question.

Theorem 2.1. *There exist a Riemann surface* S_0 *of infinite type and a sequence* $\{p_n\}_{n=0}^{\infty}$ *in* $T(S_0)$ *such that*

$$d_L(p_n, p_0) \to 0 \quad (n \to \infty)$$

while

$$d_T(p_n, p_0) \to \infty \quad (n \to \infty).$$

From the proof of this theorem, we show the incompleteness of the length spectrum distance.

Corollary 2.1. *There exists a Riemann surface of infinite type such that the length spectrum distance* d_L *is incomplete in the Teichmüller space.*

Next, we give a sufficient condition for the length distance to define the same topology as that of the Teichmüller distance as follows.

Theorem 2.2. *Let S_0 be a Riemann surface. Assume that there exists a pants decomposition $S_0 = \cup_{k=1}^{\infty} P_k$ of S_0 satisfying the following conditions.*

(1) *Each connected component of ∂P_k is either a puncture or a simple closed geodesic of S_0 (k=1, 2, ...).*

(2) *There exists a constant $M > 0$ such that if α is a boundary curve of some P_k then*

$$0 < M^{-1} < \ell_{S_0}(\alpha) < M$$

holds.

Then d_L defines the same topology as that of d_T.

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