Moduli of punctured Riemann surfaces
and the Takhtajan-Zograf metric

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ABSTRACT. We show a convergence theorem of Eisenstein series for degenerating
Riemann surfaces, which is an improved version of the former one of the author. We
will apply it to investigate $L_2$-cohomology of the Takhtajan-Zograf metric.

§1. PRELIMINARIES

1.1 Eisenstein series.

Let $S$ be a punctured hyperbolic surface of type $(g,n)$ ($n > 0$). It can be
represented as a quotient $H/\Gamma$ of the upper half plane $H = \{z \in \mathbb{C} | \text{Im} z > 0\}$
by the action of a torsion free finitely generated Fuchsian group $\Gamma \in \text{PSL}_2(\mathbb{R})$.
The group is generated by $2g$ hyperbolic transformations $A_1, B_1, \ldots , A_g, B_g$ and
parabolic transformations $P_1, \ldots , P_n$ satisfying the relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A^{-1}_g B^{-1}_g P_1 \ldots P_n = 1.$$ 

The fixed points of the parabolic elements $P_1, \ldots , P_n$ will be denoted by $z_1, z_2,$
$\ldots , z_n \in \mathbb{R} \cup \{\infty\}$ respectively and called inequivalent cusps. The projection of the
cusps $z_1, z_2, \ldots , z_n$ are the punctures $p_1, p_2, \ldots , p_n$ of $S$. For each $i = 1, \ldots , n$, 
denote by $\Gamma_i$ the stabilizer of $z_i$ in $\Gamma$ that is the cyclic subgroup of $\Gamma$ generated by
$P_i$. Pick $\sigma_i \in \text{PSL}_2(\mathbb{R})$ such that $\sigma_i \infty = z_i$ and $(\sigma_i^{-1} P_i \sigma_i) = (z \mapsto z + 1)$. Then,
for $a > 1$, the $a$--cusp region $C_i(a)$ associated to $p_i$ is represented as a quotient

$$\langle \sigma_i^{-1} P_i \sigma_i \rangle \backslash \{z \in H | \text{Im} z > a\} \simeq \Gamma \backslash \{z \in H | \text{Im} z > a\},$$

$C_i(a) \simeq [a, \infty) \times S^1$, equipped with the metric $ds^2 = (dy^2 + dx^2)/y^2$.

Let $\Delta : C^\infty(S) \rightarrow C^\infty(S)$ be the negative hyperbolic Laplacian of $S$. Regarded
as an operator in $L^2(S)$ with domain $C^\infty_0(S)$, $\Delta$ is essentially self-adjoint. Denote by
$\Delta$ the unique self-adjoint extension (that is, Friedrichs extension). Then the continuous spectrum of $\Delta$
can be described in terms of Eisenstein series ([He]Chap.Seven, [K]Chap.V, [V]§3.2).

The Eisenstein series attached to $z_i$ is defined by

$$E_i(z,s) = \sum_{\gamma \in \langle P_i \rangle \backslash \Gamma} \text{Im}(\sigma_i^{-1} \gamma z)^s, \quad \text{Re } s > 1.$$
The series is absolutely convergent in the half-plane Re $s > 1$ and in the upper half-plane, it satisfies

\[ \Delta E_i(z, s) = s(s - 1)E_i(z, s). \]

A. Selberg originally showed that the series admits meromorphic continuation to the whole complex $s$-plane, holomorphic on \{Re $s = \frac{1}{2}$\} and satisfies a system of functional equations ([SI]§7). Several mathematicians also verified it by the various methods ([dV], [He] Th.11.6, [K] pp.23–46, [Mu]). $E_i(z, s)$ has Fourier expansions at punctures $p_j$, ([He] Prop.8.6, [K] §2.2, [L-P] §8, [V] §3.1)

\[ E_i(\sigma z, s) = \delta_{ij}y^s + \sum_{m \neq 0} c_m(s) y^{\frac{s}{2}} K_{s-\frac{1}{2}}(2\pi|m|y) e^{2\pi \sqrt{-1}mx}, \]

$K_{s-\frac{1}{2}}$ the MacDonald-Bessel function ([Wa], p.78), that has the following asymptotics ([Wa], p.202)

\[ y^{\frac{1}{2}} K_{s-\frac{1}{2}}(y) \sim \sqrt{\frac{\pi}{2}} e^{-y}, \text{ as } y \nearrow \infty, \quad \text{for any complex } s. \]

In the proof of Theorem 1, we need a more precise information about the ratio of both terms in (1.3). We use 3.70(6)(p.78), 7.2(p.197) in [Wa] and can easily see

\[ \left| \frac{y^{\frac{1}{2}} K_{s-\frac{1}{2}}(y)}{\sqrt{\frac{\pi}{2}} e^{-y}} - 1 \right| < \frac{B_s}{y}, \quad \text{as } (R \ni) \quad y \nearrow \infty, \]

where $B_s$ can be chosen to be a positive number depending only on $s$.

1.2 Modified infinite-energy harmonic maps.

In this part, we will introduce the modified infinite-energy harmonic functions that are defined by S. Wolpert ([W2]), while the infinite-energy harmonic maps are originally constructed by M. Wolf ([Wf]), for parametrizing degeneration of hyperbolic surfaces. Denote by $(S_l(l > 0), \rho_l(w)|dw|^2)$ a degenerating family of hyperbolic surfaces of type $(g, n)$. We assume that several disjoint simple closed geodesics $l_1, l_2, \ldots, l_k$ on $S_l$ will be pinched (We denote their hyperbolic lengths by the same notations). Let $\Delta_l$ be the negative Laplacian of $S_l$. To compare functions on the limit surface $(S_0, \rho(x)|dz|^2)$ and $(S_l, \rho_l(w)|dw|^2)$, M. Wolf has constructed infinite-energy harmonic maps $w^l : S_0 \to S_l \setminus \{l_1, l_2, \ldots, l_k\}$ ([J], [Wf], [W2]). A node on $S_0$ is a pair of cusps and distinct nodes involve distinct cusps. We call the cusps of $S_0$ that arise from the cusps (resp. arise from the pinching geodesics) of $S_l$ the old cusps (resp. the new cusps). But $w^l$ is not adequate for us to compare the Eisenstein series for $S_l$ and for $S_0$ on cusp regions around old cusps, because $w^l$ is not the identity map on the cusp regions and the Eisenstein series has a singularity at the associated cusp for Re $s > 1$. Thus we will use Wolpert’s infinite-energy harmonic map, denoted by $f^l$, that is modified from $w^l$ so that the meridians and longitudes of a cusp will be mapped to the meridians and longitudes of the collar or cusp in the image ([W2]).

Now we can arrange that given $b > 1$, for $b$-cusp regions $C^0_i(b)$ on $S_0$ and $b$-cusp regions $C'_i(b)$ on $S_l (i = 1, 2, \ldots, n)$,

\[ f^l|_{C^0_i(b)} = id : C^0_i(b) \to C'_i(b). \]
1.3 The Weil-Petersson and the Takhtajan-Zograf metrics.

Denote by $T_{g,n}$ Teichmüller space of hyperbolic surfaces of type $(g,n)$. Now we consider the tangent and cotangent spaces at $S$ of $T_{g,n}$. The cotangent space is $Q(S)$, the integrable holomorphic quadratic differentials on $S$. Let $B(S)$ be the $L^\infty$-closure of $\Gamma$-invariant, bounded, $(-1,1)$-forms, i.e. the Beltrami differentials for $S$. For $\mu \in B(S), \varphi \in Q(S)$, the integral $(\mu, \varphi) = \int_S \mu \varphi$ defines a paring, let $Q(S)^{\perp}$ be the annihilator of $Q(S)$. The tangent space at $S$ to $T_{g,n}$ is $B(S)/Q(S)^{\perp} \simeq HB(S)$, the Serre dual space of $Q(S)$, i.e. the harmonic Beltrami differentials on $S$. Then for $\mu, \nu \in HB(S)$, the Weil-Petersson and the Takhtajan-Zograf metrics are defined as follows ([T-Z]),

\begin{align}
\langle \mu, \nu \rangle_{WP} &= \int_{S} \mu(z)\overline{\nu(z)}y^{-2}dxdy \\
\langle \mu, \nu \rangle_{(i)} &= \int_{S} E_i(z, 2)\mu(z)\overline{\nu(z)}y^{-2}dxdy \\
\langle \mu, \nu \rangle_{T-Z} &= \sum_{i=1}^{\infty} \langle \mu, \nu \rangle_{(i)}.
\end{align}

In the theory of automorphic functions, those two inner products are called, respectively the Petersson product and the Rankin product, while they are defined for general automorphic forms in the setting (refer to [Hi] §5.4). Both Weil-Petersson and Takhtajan-Zograf metric are Kählerian and incomplete ([O1], [T-Z]).

§2. A REFINED VERSION OF CONVERGENCE THEOREM OF EISENSTEIN SERIES

In this section we will show a new convergence theorem of Eisenstein series, which is improved from the former version in [O2]. A little improvement is involved but, is essential for us to investigate the behavior of Takhtajan-Zograf metric near the boundary of moduli space more precisely than in [O2].

2.1 The Harish-Chandra transformation.

Here we prepare several fundamental notations from T. Kubota's book. ([K], Theorem 1.3.2). For $\epsilon > 0$, set a $\mathrm{PSL}_2(\mathbb{R})$-invariant kernel function on $H \times H$

$$k_{\epsilon}(z, z') = \begin{cases} 1, & \text{if } d(z, z') < \epsilon \\ 0, & \text{otherwise,} \end{cases}$$

where $d(z, z')$ denotes the hyperbolic distance between $z$ and $z'$ in $H$. Then there exists a constant $\Lambda_{\epsilon}(s)$ depending only on $\epsilon$ and the index $s$ such that for any $\sigma \in \mathrm{PSL}_2(\mathbb{R})$,

$$\Lambda_{\epsilon}(s) \Im(\sigma z)^{s} = \int_{H} k_{\epsilon}(z, z') \Im(\sigma z')^{s} \frac{dz'dy'}{y'^2}, \quad (z' = x' + y').$$

([K], Theorem 1.3.2). The correspondence $s(s - 1) \mapsto \Lambda_{\epsilon}(s)$ is sometimes called the Harish-Chandra transformation. We set $B(z, \epsilon) = \{ w \in H | d(w, z) < \epsilon \}$ for $z \in H, \epsilon > 0$. With the help of Mathematica ([Mt]), we find...
\[ \Lambda_{\epsilon}(s) = \int \int \frac{y^{s-2}}{B(\epsilon)} \, dx \, dy = \int \int \frac{y^{s-2}}{x^{2} + (y - \cosh \epsilon)^{2} \leq \sinh^{2} \epsilon} \, dx \, dy \]

\[ = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sinh \epsilon} (\cosh \epsilon + r \sin \theta)^{s-2} \, r \, dr \, d\theta \]

(Here we set \( x = r \cos \theta, y - \cosh \epsilon = r \sin \theta \))

\[ (2.1) \]

\[ = \pi \Gamma^{2} \left( \frac{3-s}{2} \right) (\cosh \epsilon)^{\epsilon} (\tanh \epsilon)^{2} {}_{2}F_{1} \left( 1-\frac{s}{2}, \frac{3-s}{2}; 2; (\tanh \epsilon)^{2} \right). \]

Here \( {}_{2}F_{1}(\alpha, \beta; \gamma; z) \) \((\gamma \neq 0, -1, -2, \ldots)\) is the hypergeometric function,

\[ {}_{2}F_{1}(\alpha, \beta; \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n-1) \cdot \sqrt{\sqrt{+1}} \cdots (\sqrt{n-1})}{\gamma(\gamma+1) \cdots (\gamma+n-1) \cdot 1 \cdot 2 \cdots n} \, z^{n} \]

and satisfies the differential equation,

\[ z(1-z) \frac{d^{2}u}{dz^{2}} + [\gamma - (\alpha + \beta + 1)z] \frac{du}{dz} - \alpha \beta u = 0 \]

([Wa]).

Then

\[ \Lambda_{\epsilon}(s) \sim \pi \Gamma^{2} \left( \frac{3-s}{2} \right) \epsilon^{2} \] as \( \epsilon \to 0 \)

holds. But we need the next more precise estimate of the ratio of both terms above in our proof of Theorem 1. It can be easily seen from the definitions that, for \( \Re s > 1 \),

\[ (2.2) \]

\[ \Lambda_{\epsilon}(\Res)^{-1} \leq \frac{c(\Res) \epsilon^{-2}}{\pi \Gamma^{2} \left( \frac{3-\Res}{2} \right)} \] as \( \epsilon \to 0 \)

holds, where \( c(\Res) \) is a positive constant depending only on \( \Re s \).

Now we quote the next lemma ([O2], Lemma 1.).

**Lemma 1.** We use the same notations as in § 1. Let the index \( \Res > 1 \). For any \( i = 1, 2, \ldots, n \) and any \( a > 1 \),

\[ (2.3) \]

\[ |E_{i}(z, s)| \leq E_{i}(z, \Res) < M_{1}(\Res, s, a), \quad \text{for } z \in \partial C_{i}(a). \]

Here \( M_{1}(\Res, s, a) \) is a constant depending only on \( \Res, a \), independent of complex structure and topological type of the surface, precisely represented as follows;

\[ M_{1}(\Res, s, a) = \frac{3 \cdot (2a)\Res^{1}}{\left( \Res - 1 \right) \Lambda_{e_{0}(a)}(\Res)} \] (we may set \( e_{0}(a) = \frac{1}{2a} \)).

Since \( E_{i}(z, \Res) \) is subharmonic on \( S \), we finally see

\[ (2.4) \]

\[ |E_{i}(z, s)| < M_{1}(\Res, s, a), \quad \text{on } S - C_{i}(a). \]

We use the setting as in §1, 1.1. Let \( \Gamma \) be the Fuchsian group uniformizing \( S \) of type \((g, n)\) \((n > 0)\) with \( z_{1} = \infty \) and \( P_{1}(z) = z + 1 \). The next proposition is a new version of Wolpert's result ([W2] p.260).
Proposition 1. Let the index of Eisenstein series $Re \ s > 1$. Then

$$|E_1(z, s)| < C(Re \ s) (Imz)^{-\text{Res}}$$

for $Imz < 1$.

Here $C(Re \ s)$ is a constant depending only on $Re \ s$, independent of complex structure and topological type of the surface.

Furthermore, the coefficients $\{c_m(s)\}_{m \neq 0}$ appearing in the Fourier expansion of $E_1(z, s)$ around $z_1 = \infty$ (1.2) satisfy

$$\sum_{m \neq 0} |c_m(s)|^2 |m|^{-2(\text{Res}+1)-1-\delta} < \infty, \quad \text{for any } \delta > 0.$$

Remark 1. The order $-(\text{Res} + 1)$ of $y$ in (1) are different from $-\text{Res}$ the one in p.260, [W2]. The reason is that our constant $C(\text{Res})$ is universal, while the constant $C$ in [W2] depends on complex structure of the surface $S$.

2.2 the convergence of Eisenstein series.

We will show a convergence theorem of Eisenstein series, which is refined from the old version stated in [O2] Theorem 1., concerning convergence on the cusp regions around the old cusps. We state

Theorem 1. We set the same notations as in §1. Let the index $Re \ s > 1$. Let $(f^1)^*E_1^i(z, s)$ be the pull-back of $E_1^i(z, s)$ on $S_i$ by the modified harmonic map $f^1: S_0 \rightarrow S_1$ introduced in §1, 1.2.

1) Assume that $\{l_1, \ldots, l_k\}$ do not separate $S_i$. Let $q_j$ ($j = 1, \ldots, k$) be the new cusp arising from $l_j$. Denote by $C_j(b)$ ($b > 1$) be the cusp region around $q_j$ in $S_0$, each composed of usual two $b$-cusp regions. Then for any $i = 1, \ldots , n$, as $\rightarrow \rightarrow 0$,

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$$\sum_{m \neq 0} |c_m(s)|^2 |m|^{-2(\text{Res}+1)-1-\delta} < \infty, \quad \text{for any } \delta > 0.$$
uniformly on \( S_{0,2}^i - \bigcup_{j=1}^k C_j(b) \).

Furthermore, for \( b > 1 \) fixed, \( |(f^i)^* E_1^i(z, s)| = O \left( \max_{j=1,\ldots, k} l_j^{(2-\delta)\text{Res} - 2} \right) \), for any small \( \delta > 0 \) on \( S_{0,2}^i - \bigcup_{j=1}^k C_j(b) \).

The new part of Theorem 1 is restated as the following proposition.

Proposition 2. We set the same notations as in Theorem 1. Assume that \( \text{Re} \, s > 1 \). We state our claim just on \( E_1^i(z, s) \) for notational simplicity. For any \( i = 1, 2, \ldots, n \), as \( \vec{t} \to \vec{0} \),

\[
E_1^i(z, s) - E_1^0(z, s) \to 0
\]

uniformly on \( C_i(B) \) (\( B > b \)), where \( b \) is the number taken in (1.5).

§3. Comparisons of the W-P and T-Z Metric along General Degenerations

3.1 A review of H. Masur's work.

We review a construction of a basis of quadratic differentials spanning the cotangent spaces of a general degenerating family of punctured Riemann surfaces. In [Ms], he has constructed such a basis for any degenerating family of compact surfaces. But from his result, we can easily obtain the same kinds of quadratic differentials for a family of surfaces with cusps.

Assume that \( S_0 \) has singularities at points \( q_j \) (\( j = 1, 2, \ldots, k \)), these have neighbourhoods \( N_j = \{(z_j, w_j) \in C^2 \mid |z_j|, |w_j| < 1, z_j \cdot w_j = 0 \} \), respectively, \( N_j = N_j^1 \cup N_j^2 \) is a union of disks; \( N_j^1 = \{z_j \mid 0 \leq |z_j| \leq 1\} \), \( N_j^2 = \{w_j \mid 0 \leq |w_j| \leq 1\} \). The components \( S_\alpha \) of \( S_0 \) (\( \alpha = 1, \ldots, r \)) are called parts of \( S_0 \). We have to assume that the \( S_\alpha \) are hyperbolic, i.e. \( 2g_\alpha - 2 + n_\alpha + \tilde{n}_\alpha > 0 \) where \( g_\alpha \) is the genus of \( S_\alpha \) and \( n_\alpha + \tilde{n}_\alpha \) are the numbers of the old (new) cusps of \( S_\alpha \) respectively (we regard a node attached to just one component as a pair of two new cusps). Let \( g, n \) be the genus and the number of old cusps of \( S_0 \). Then we see the equations \( g = g_1 + \ldots + g_r + k - (r - 1), n = n_1 + \ldots + n_r, 2k = \tilde{n}_1 + \ldots + \tilde{n}_r \). These yield

\[
3g - 3 + n = \sum_{\alpha=1}^r (3g_\alpha - 3 + n_\alpha + \tilde{n}_\alpha) + k.
\]

Any part \( S_\alpha \) possesses a \( (3g_\alpha - 3 + n_\alpha + \tilde{n}_\alpha) \)-dimensional universal family, \( T_{g_\alpha, n_\alpha + \tilde{n}_\alpha} \), i.e. Teichmüller space of type \((g_\alpha, n_\alpha + \tilde{n}_\alpha)\). We set a basis of Beltrami differentials \( \nu^i \) on \( S_\alpha \) with compact supports in \( S_\alpha - \bigcup_{j=1}^k N_j - \{\text{cusp neighborhoods around old cusps of } S_\alpha\} \) (For example, we may take restrictions to compact support of the duals of a basis of integrable quadratic differentials): let

\[
\tau^\alpha = (\tau_1^\alpha, \tau_2^\alpha, \ldots, \tau_{3g_\alpha - 3 + n_\alpha + \tilde{n}_\alpha}^\alpha)
\]
be associated local coordinates for $T_{g_{\alpha}, n_{\alpha}+n_{\alpha}}$ around $S_{\alpha}$, where we set $\mu_{\alpha}(\tau) = \sum_{i=1}^{g_{\alpha}} \tau_{i}^{\alpha} r_{i}^{\alpha}$. If we vary the complex structure of parts $S_{\alpha}$, and set $\tau = (\tau^{1}, \tau^{2}, \ldots, \tau^{r})$, we obtain a family $\{S_{\tau}\}$ and quasiconformal homeomorphisms $f^{\mu_{\alpha}(\tau)}: S_{\alpha} \rightarrow S_{\alpha, \tau^{\alpha}}$ which comprise a quasiconformal homeomorphism $f^{\tau}: S_{0} = \bigcup_{\alpha=1}^{r} S_{\alpha} \rightarrow \bigcup_{\alpha=1}^{r} S_{\alpha, \tau^{\alpha}}$, the last set denoted by $S_{\tau}$. The map $f^{\tau}$ is conformal on $N_{j}^{1}, N_{j}^{2}$ ($j = 1, \ldots, k$) and thus $z_{i}, w_{i}$ serve as local coordinates for $f^{\tau}(N_{j}^{1}), f^{\tau}(N_{j}^{2})$ respectively. For each $t = (t_{1}, t_{2}, \ldots, t_{k}), |t_{j}| < 1$, the new Riemann surface $S_{t, \tau}$ is constructed from $S_{\tau}$ by removing the disks $\{z_{j}||z_{j}| < |t_{j}|\}$ and $\{w_{j}||w_{j}| < |t_{j}|\}$ and identifying $z_{j}$ with $w_{j} = t/z_{j}$, constructing annuli $N_{j}^{1} = \{z_{j}||t_{j}| < |z_{j}| < 1\} \simeq \{w_{j}||t_{j}| < |w_{j}| < 1\}$. If $K$ is a compact subset of $S_{0}\{q_{1}, \ldots, q_{k}\}$, for small $(t, \tau)$ we will consider $K$ as a compact subset of $S_{t, \tau}$ via $f^{\tau}$ and the natural inclusion of $S_{t, \tau}$ in $S_{\tau}$ (See [Ms], p.625).

Here following Wolpert ([W1] Lemma1.1), we take a modification $F^{\tau}$ of $f^{\tau}$ so that each lift $F^{\tau}_{\alpha}: H \rightarrow H$ of $F^{\tau}|_{S_{\alpha}}: S_{\alpha} \rightarrow S_{\alpha, \tau^{\alpha}}$ will coincide with an element of $\mathrm{PSL}_{2}(\mathbb{R})$ on any cusp region corresponding to old cusps of $S_{\alpha}$ (See also [P-S] Lemma2.2). It should be remarked that $F^{\tau}$ and $f^{\tau}$ have equivalent initial tangents $\partial/\partial \tau_{\alpha}$, that is, the corresponding Beltrami differentials will have the same pairing with each integrable quadratic differential on $S_{\alpha}$, which we need in the proof of Proposition 4 (See [W1] Remark1.2).

Thus we have gotten a local parameter space $(t, \tau) = (t_{1}, t_{2}, \ldots, t_{k}, \tau^{1}; \tau^{2}; \ldots; \tau^{r}) \in D \subset C^{3g-3+n}$, a neighbourhood of the origin. By changing the indicies, we sometimes use a notation $\tau = (\tau_{k+1}, \tau_{k+2}, \ldots, \tau_{3g-3+n-k})$. We state the important proposition, essentially due to H. Masur [Ms] (see also [B], [Sc], [Tr]). We arrange his results so that they should fit to our setting, that is, a degenerating family of punctured Riemann surfaces.

Proposition 3. There is a basis of regular quadratic differentials

$$\{\phi_{j}(z, t, \tau)dz^{2}, \phi_{\nu}(z, t, \tau)dz^{2}\}_{j=1, \ldots, k; \nu \geq k+1},$$

dual to $\{\partial/\partial t_{j}, \partial/\partial \tau_{\nu}\}_{j=1, \ldots, k; \nu \geq k+1}$, satisfying the next properties:

i) $\phi_{\nu}(z, 0, 0)$ has support in the component of $S_{0}$ where the Beltrami differential corresponding to $\partial/\partial \tau_{\nu}$ has support.

ii) the followings hold, where $(\cdot, \cdot)$ means the Serre dual pairing:

(3.1) $$(\phi_{i}, \partial/\partial t_{j}) = \delta_{ij}, \quad \text{for } i, j \leq k$$

(3.2) $$(\phi_{i}, \partial/\partial \tau_{\nu}) = O(|t_{i}|), \quad \text{for } i \leq k, \nu \geq k+1$$

(3.3) $$(\phi_{\mu}, \partial/\partial t_{j}) = 0, \quad \text{for } \mu \geq k+1, j \leq k$$

(3.4) $$\lim_{(t, \tau) \rightarrow (0, 0)} (\phi_{\mu}, \partial/\partial \tau_{\nu}) = \delta_{\mu-k, \nu}, \quad \text{for } \mu, \nu \geq k+1.$$

iii) On $z_{j} \in N_{j}^{1}$, for $i \leq k$,

(3.5) $$\phi_{i}(z_{j}, t, \tau) = -\frac{t_{i}}{\pi} \left[ \frac{\delta_{ij}}{z_{j}^{2}} + a_{-1}(z_{j}, t, \tau) + \frac{1}{z_{j}^{2}} \sum_{r=1}^{\infty} \left( \frac{t_{j}^{r}}{z_{j}} \right)^{\tau} \cdot t_{j}^{m(r)} \cdot a_{r}(t, \tau) \right],$$
where \( m(r) \geq 0 \), \( a_{-1} \) has at most a simple pole at \( z_j = 0 \), \( a_r \) \((r \geq 1)\) is holomorphic. On \( z_j \in N_j^1 \), \( \nu \geq k + 1 \),

\[
(3.6) \quad \phi_{\nu}(z_j, t, \tau) = \phi_{\nu}(z_j, 0, 0) + \frac{1}{z_j^2} \sum_{r=1}^{\infty} \left( \frac{t_j}{z_j} \right)^r \cdot t^m(r) \cdot b_r(t, \tau) + \sum_{r=-1}^{\infty} z_j^r \cdot c_r(t, \tau),
\]
where \( \tilde{m}(r) \geq 0 \), \( \phi_{\nu}(z_j, 0, 0) \) has at most a simple pole and \( b_r, c_r \) is holomorphic.

On \( z_j \in N_j^1 \), \( \nu \geq k + 1 \),

\[
(3.7) \quad <\phi_{ij}, \phi_{\nu}> \approx -|t_i|^2 (\log |t_i|)^3, \quad \text{for } i \leq k,
\]

\[
(3.8) \quad <\phi_{ij}, \phi_{ji}> = O(|t_i|) O(|t_j|), \quad \text{for } i, j \leq k, i \neq j
\]

\[
(3.9) \quad <\phi_{ij}, \phi_{\mu}> = O(|t_i|), \quad \text{for } i \leq k, \mu \geq k + 1,
\]

\[
(3.10) \quad \lim_{(t, \tau) \to (0, 0)} <\phi_{\mu}(z, t, \tau), \phi_{\nu}(z, t, \tau)> = <\phi_{\mu}(z, 0, 0), \phi_{\nu}(z, 0, 0)>, \quad \text{for } \mu, \nu \geq k + 1.
\]

Remark 2. It seems difficult that we would apply the method of Masur’s original proof to our case because he used compactness of general fibers of the degenerating family in his proof.

### 3.2 Comparisons along general degenerations.

Before we state the main theorem, we give a preparatory tool for investigating the boundary behaviors of the Takhtajan-Zograf metric. We get the representation of \( \{\partial/\partial t_j, \partial/\partial \tau_\nu\}_{j=1, \ldots, k; \nu \geq k+1} \) in terms of harmonic Beltrami differentials approximately.

**Proposition 4.** Let \( \rho(z, t, \tau)|dz| \) be the Poincaré metric with curvature \(-1\). We define harmonic Beltrami differentials

\[
\eta_j(z, t, \tau) = \rho(z, t, \tau)^{-2} \phi_j(z, t, \tau), \quad \eta_{\nu}(z, t, \tau) = \rho(z, t, \tau)^{-2} \overline{\phi_{\nu}(z, t, \tau)}, \quad (j = 1, \ldots, k, \nu = k+1, \ldots, 3g-3+n).
\]

And for \( i \leq k, \mu \geq k + 1 \), we put

\[
(3.11) \quad \partial/\partial t_i = \sum_{j=1}^{k} u_{ij}(t, \tau) \eta_j(z, t, \tau) + \sum_{\nu=k+1}^{3g-3+n} u_{i\nu}(t, \tau) \eta_\nu(z, t, \tau)
\]

\[
(3.12) \quad \partial/\partial \tau_\mu = \sum_{j=1}^{k} u_{\mu j}(t, \tau) \eta_j(z, t, \tau) + \sum_{\nu=k+1}^{3g-3+n} u_{\mu\nu}(t, \tau) \eta_\nu(z, t, \tau).
\]

Then we obtain the followings:

\[
i) \quad u_{ii}(t, \tau) \approx -1/|t_i|^2 (\log |t_i|)^3, \quad \text{for } i \leq k
\]

\[
ii) \quad u_{ij}(t, \tau) = O(1/|t_i| |t_j| (\log |t_i|)^3 (\log |t_j|)^3), \quad \text{for } i, j \leq k, i \neq j
\]

\[
iii) \quad u_{i\nu}(t, \tau) = O(-1/|t_i| (\log |t_i|)^3), \quad \text{for } i \leq k, \nu \geq k + 1
\]

\[
iv) \quad u_{\mu j}(t, \tau) = O(-1/|t_j| (\log |t_j|)^3), \quad \text{for } \mu \geq k + 1, j \leq k
\]

\[
v) \quad u_{\mu\nu}(t, \tau) = \delta_{\mu\nu} + \sum_{i=1}^{k} O(-1/(\log |t_i|)^3), \quad \text{for } \mu, \nu \geq k + 1
\]

Finally we combine Theorem 1 and Proposition 1, 3 to get one of our main theorems.
Moduli of punctured surfaces, T-Z metric

\textbf{Theorem 2.} We obtain order estimates of the Riemannian tensors \( h_{ij}(t, \tau) \) of the Takhtajan-Zograf (the Weil-Petersson) metric near the boundary of Teichmüller space;

\begin{itemize}
  \item[i)] \( g_{ii}(t, \tau) = -1/|t_i|^2(\log|t_i|)^3 + O(-1/|t_i|^3(\log|t_i|)^6) \), for \( i \leq k \)
  \item[ii)] \( g_{ij}(t, \tau) = O(1/|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3) \), for \( i, j \leq k, i \neq j \)
  \item[iii)] \[ \lim_{(t, \tau) \to (0, 0)} g_{\mu\overline{\nu}}(t, \tau) = g_{\mu\overline{\nu}}(0, 0) \], for \( \mu \), \( \nu \geq k+1 \)
  \item[iv)] \( g_{\overline{i} \mu}(t, \tau) = O(-1/|t_i|(\log|t_i|)^3) \), for \( i \leq k \), \( \mu \geq k+1 \)
\end{itemize}

We state a conjecture that is inspired by M.Wolf's asymptotic formula of the hyperbolic metrics for degenerating Riemann surfaces ([Wf], Corollary 5.4). The second-term conjecture (Obitsu and Wolpert). Use the notations as in Theorem 2. The next asymptotic formula for the Weil-Petersson metric for \( T_g \) holds; for \( \mu, \nu \geq k+1 \),

\begin{equation}
  g_{\mu\overline{\nu}}(t, \tau) = g_{\mu\overline{\nu}}(0, \tau) + \frac{4\pi^4}{3} \sum_{i=1}^{k} (\log|t_i|)^{-2} \left\langle \eta_{\mu}, (E_{i,1}(z, 2) + E_{i,2}(z, 2)) \eta_{\nu} \right\rangle_{WP}(0, \tau) + O\left( \sum_{i=1}^{k} (\log|t_i|)^{-3} \right).
\end{equation}

Here, \( E_{i,1}(z, 2), E_{i,2}(z, 2) \) are the Eisenstein series associated with the \( i \)-th node and the components of the degenerate Riemann surface. i.e. in the second-term, the associated Takhtajan-Zograf metrics appear.

\section{AN APPLICATION TO \( L_2 \)-COHOMOLOGY OF MODULI SPACE}

First of all, we review the result of L.Saper.\textbf{Theorem 3 ([Sa])}. Denote by \( M_g \) the moduli space of compact Riemann surfaces of genus \( g > 1 \). Then, We have the isomorphisms

\[ H^*_2(M_g, \omega_{WP}) \cong H^*(\overline{M_g}, \mathbb{R}), \]

where the left-hand sides are the \( L_2 \)-cohomology groups with respect to the Weil-Petersson metric, and the right-hand sides are the usual cohomology groups of the Deligne-Mumford compactification of the moduli space with coefficients in \( \mathbb{R} \).

We can mimic the proof of Theorem 3 with using Theorem 2 to deduce the next generalization.
Theorem 4. Denote by $M_{g,n}$ the moduli space of punctured Riemann surfaces of genus $g$ with $n$ punctures, $3g - 3 + n > 0$. Then, we have the isomorphisms

$$H^*_\omega(M_{g,n}, \omega_{WP}) \simeq H^*_\omega(M_{g,n}, \omega_{TZ}) \simeq H^*(\overline{M}_{g,n}, \mathbb{R}),$$

where the middle are the $L_2$-cohomolgy groups with respect to the Takhtajan-Zograf metric, and the left-hand sides and the right-hand sides are respectively the obvious counterparts of them in Theorem 3.

REFERENCES


Moduli of punctured surfaces, T-Z metric


Addendum.

Very recently, I and S. Wolpert have proved the second-term conjecture! Precise proof and several applications will appear elsewhere.