TOPOLOGICAL TYPES OF GEOMETRIC LIMIT MANIFOLDS
OF QUASI-FUCHSIAN GROUPS

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This note is a survey of the author’s results given in [7]. Let $\Sigma$ be a closed orientable surface of genus greater than one. We fix a hyperbolic structure on $\Sigma$ for convenience, and set $\Pi = \pi_1(\Sigma)$. In [3], Jørgensen and Marden gave an example of faithful representations $\rho_n : \mathbb{Z} \rightarrow \text{PSL}_2(\mathbb{C})$ with $\rho_n(1)$ loxodromic such that the cyclic Kleinian groups $\rho_n(\mathbb{Z})$ converge geometrically to a rank two parabolic group. This is one of typical phenomena which appear in geometric limits. In fact, Kerckhoff and Thurston [4] considered the cyclic action on the Bers slice $B_{\sigma_+}$ at $\sigma_+ \in \text{Teich}(\Sigma)$ generated by the Dehn twist $\varphi$ on $\Sigma$ along a simple closed geodesic $l$. Then, they showed that any geometric accumulation point of the cyclic orbit $\{(\varphi_n^*(\sigma_-), \sigma_+)\} \subset B_{\sigma_+}$ is a Kleinian group $G$ such that $\mathbb{H}^3/G$ is homeomorphic to $\Sigma \times \text{Int}(0,1) - l \times \{1/2\}$. Then, a tubular neighborhood of $l \times \{1/2\}$ in $\Sigma \times (0,1)$ corresponds to a $\mathbb{Z} \times \mathbb{Z}$-cusp of $\mathbb{H}^3/G$ where Jørgensen-Marden phenomenon occurs, see Fig. 1(a). By using this method iteratively, it is also possible
to construct an example of a geometric limit $G'$ of quasi-Fuchsian groups such that $\mathbb{H}^3/G'$ has infinitely many $\mathbb{Z} \times \mathbb{Z}$-cusps. In particular, $G'$ is infinitely generated. Another important example of geometric limits of quasi-Fuchsian groups is given by Brock [2]. He considered the cyclic action on a Bers slice generated by a homomorphism $\psi : \Sigma \rightarrow \Sigma$ such that $\psi|\text{Int}H : \text{Int}H \rightarrow \text{Int}H$ is pseudo-Anosov for a proper subsurface $H$ of $\Sigma$ and $\psi|\Sigma - \text{Int}H$ is the identity. Then, any geometric accumulation point of the cyclic orbit $\{(\psi_n^*(\sigma_-), \sigma_+)\} \subset B_{\sigma_+}$ is a Kleinian group $G''$ such that $\mathbb{H}^3/G''$ is homeomorphic to $\Sigma \times (0,1) - H \times \{1/2\}$, see Fig. 1(b). However, all of these examples are very special ones. In this talk, we will present what kinds of topological types appear generally in geometric limits of quasi-Fuchsian groups.

Let $p : \Sigma \times I \rightarrow \Sigma$ and $q : \Sigma \times I \rightarrow I$ be the projections onto the first and second factors, where $I$ is the closed interval $[0,1]$. For any $y \in I$, the preimage

![Figure 1](image_url)

**Figure 1.** The dot in (a) represents $l \times \{1/2\}$ in the model of $\mathbb{H}^3/G$. The vertical bold segment in (b) represents $H \times \{1/2\}$ in the model of $\mathbb{H}^3/G''$. 
$\Sigma_y = q^{-1}(y)$ is supposed to have the hyperbolic structure so that $p|\Sigma_y : \Sigma_y \rightarrow \Sigma$ is isometric. A compact connected subsurface $F$ of $\Sigma_y$ with geodesic boundary is called a non-pant geodesic subsurface if $F$ is not homeomorphic to a genus-zero surface with three boundary components, a pair of pants. Note that the interior of a non-pant geodesic subsurface contains a simple closed geodesic.

For a closed subset $A$ of $\Sigma_y$, if $\Delta(A)$ is a minimal disjoint union of a geodesic subsurface $F$ and simple closed geodesics $l_1, \ldots, l_k$ in $\Sigma_y$ with $\Delta(A) \supset A$, then the frontier $\text{Fr}(\Delta(A))$ of $\Delta(A)$ in $\Sigma_y$ is the union $\partial F \cup l_1 \cup \cdots \cup l_k$ of mutually disjoint simple geodesic loops in $\Sigma_y$. Let $\lambda(A)$ be the union of all simple closed geodesics $l$ in $\Delta(A)$ such that, for the $\delta$-neighborhood $\mathcal{N}_\delta(l, \Sigma_y)$ with a small $\delta > 0$, at least one component of $\mathcal{N}_\delta(l, \Sigma_y) - l$ is disjoint from $A$. In particular, $\lambda(A)$ contains $\text{Fr}(\Delta(A))$, see Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The case of $A = l_1 \cup A_1 \cup A_2$. Then, $\Delta(A) = F \cup l_1$, where $F$ is the union of the shaded regions. $\text{Fr}(\Delta(A)) = l_1 \cup m_1 \cup m_3$, and $\lambda(A) = \text{Fr}(\Delta(A)) \cup m_2 = l_1 \cup m_1 \cup m_2 \cup m_3$.}
\end{figure}

Let $\mathcal{X}$ be the closed subset of $\Sigma \times I$ given below. Then, we set $\mathcal{Y} = q(\mathcal{X})$, $X_y = \Sigma_y \cap \mathcal{X}$ for $y \in \mathcal{Y}$, $\Lambda_y^- = \Sigma_y \cap \Sigma \times [y, 1] \cap \mathcal{X}$ for $y < 1$, $\Lambda_y^+ = \Sigma_y \cap \Sigma \times [0, y) \cap \mathcal{X}$ for $y > 0$.

**Theorem 1.** Let $\{\rho_n : \Pi \rightarrow \text{PSL}_2(\mathbb{C})\}_{n=1}^{\infty}$ be any algebraically convergent sequence of quasi-Fuchsian representations such that $\{\rho_n(\Pi)\}_{n=1}^{\infty}$ converges geometrically to a Kleinian group $G$. Then, the hyperbolic 3-manifold $\mathbb{H}^3/G$ is homeomorphic to $\Sigma \times I - \mathcal{X}$ such that $\mathcal{X}$ is a closed subset of $\Sigma \times I$ satisfying the following conditions (i)-(iii).

(i) $\Sigma \times I - \mathcal{X}$ is connected, containing $\Sigma_{1/2}$, and disjoint from $\Sigma_0 \cup \Sigma_1$.

(ii) For any $y \in \mathcal{Y}$, $X_y$ is a disjoint union of a geodesic subsurface and simple geodesic loops in $\Sigma_y$. For $\epsilon = \pm$, each non-peripheral component of $X_y - \Delta(\Lambda_y^\epsilon) \cup \text{Fr}(X_y)$ is an open non-pant geodesic subsurface of $\Sigma_y$.

(iii) For any $y, z \in \mathcal{Y}$ with $y < z$, if a component $l_y$ of $\text{Fr}(X_y) \cup \lambda(\Lambda_y^\pm)$ is parallel to a component $l_z$ of $\text{Fr}(X_z) \cup \lambda(\Lambda_z^\pm)$ in $\Sigma \times I - \mathcal{X}$, then $l_y$ and $l_z$ are horizontally parallel in $\mathcal{X}$.

The property $\Sigma_0 \cup \Sigma_1 \subset \mathcal{X}$ in the condition (i) is immediate from that $\mathbb{H}^3/G$ is an open manifold. A subsurface of $X_y$ is peripheral if it is horizontally parallel in $\mathcal{X}$ to a subsurface of either $\Sigma_0$ or $\Sigma_1$. Any component of $X_y - \Delta(\Lambda_y^\pm) \cup \text{Fr}(X_y)$ is called a subsurface of type $B^\epsilon$, see Fig. 3. Thus, the latter part of the condition (ii) is restated that any non-peripheral subsurface of type $B^\epsilon$ is not an open pair of pants. In fact, the ends of $\mathbb{H}^3/G$ corresponding to such subsurfaces are necessarily
Figure 3. $A^+_i$, $B^+_j$, $C_k$ represent respectively subsurfaces of $\Sigma_y$ of types $A^+$, $B^+$ and $C$.

geometrically infinite tame. The condition (iii) is derived from the fact that any two parabolic cusps in a hyperbolic 3-manifold $M$ is not parallel in $M$.

According to Myers [6], there exists a simple loop $l$ in $\Sigma \times (0, 1)$ which is not parallel to a loop in $\Sigma_0 \cup \Sigma_1$ and such that $N = \Sigma \times (0, 1) - l$ admits a geometrically finite hyperbolic metric $\sigma$. By Hyperbolic Dehn Surgery Theorem in [8], $N(\sigma)$ is a geometric limit of geometrically finite hyperbolic 3-manifolds without parabolic cusps. However, $N$ is not homeomorphic to $\mathbb{H}^3/G$ for any geometric limit $G$ of quasi-Fuchsian groups. This fact is proved by Theorem 1 or directly as an exercise without invoking the theorem.

In general, a closed subset $\mathcal{X}$ in $\Sigma \times I$ satisfying the conditions (i)-(iii) is very complicated. When $\mathcal{Y}$ is a totally disconnected subset of $I$, $\mathcal{Y}$ is not a perfect set and each connected component of $\mathcal{X}$ is either a geodesic subsurface or a geodesic loop. Even in this rather simple case, there may exist a doubly (or more multiply) accumulation point $y$ in $\mathcal{Y}$. This means that $y$ is an accumulation point of a subset $\{y_n\}$ of $\mathcal{Y}$ such that each $y_n$ is also an accumulation point of $\mathcal{Y}$, see Fig. 4.

Figure 4. '1/2' is a doubly accumulation point of $\mathcal{Y}$.

Remark 2. In particular, Theorem 1 implies that, for any geometric limit $G$ of an algebraically convergent sequence $\{\rho_n\}$ of quasi-Fuchsian representations, $\mathbb{H}^3/G$ is homeomorphic to an open subset of $\Sigma \times (0, 1)$. One may suppose that the assertion is obvious since each $\mathbb{H}^3/\rho_n(\Pi)$ is homeomorphic to $\Sigma \times (0, 1)$, and since moreover there
exists a $K_n$-quasi-isometry $g_n : \mathcal{N}_R(x_n, \mathbb{H}^3/\rho_n(\Pi)) \to \mathcal{N}_R(x_\infty, \mathbb{H}^3/G)$ between the $R_n$-neighborhoods centered at suitable base points $x_n$ and $x_\infty$ with $R_n \nrightarrow \infty$ and $K_n \not\subseteq 1$. Though the $g_n^{-1}$ and $g_{n+1}^{-1}$-images of $\mathcal{N}_R(x_\infty, \mathbb{H}^3/G)$ are mutually homeomorphic, their complements in $\mathbb{H}^3/\rho_n(\Pi)$ and $\mathbb{H}^3/\rho_{n+1}(\Pi)$ do not necessarily have the same topological type. Thus, the maps $g_n^{-1}$ would not offer directly an expanding sequence of embeddings from $\mathcal{N}_R(x_\infty, \mathbb{H}^3/G)$ $(n = 1, 2, \ldots)$ into $\Sigma \times (0, 1)$. We will construct an embedding of $\mathbb{H}^3/G$ into $\Sigma \times (0, 1)$ by using the fact that $\mathbb{H}^3/G$ has the structure of a block complex.

**Theorem 3.** Let $\mathcal{X}$ be any closed subset of $\Sigma \times I$ satisfying the conditions (i)-(iii) in Theorem 1. Then, there exists a geometric limit $G$ of an algebraically convergent sequence of quasi-Fuchsian representations such that $\mathbb{H}^3/G$ is homeomorphic to $\Sigma \times I - \mathcal{X}$.

A closed subset of $\Sigma \times I$ satisfying the conditions (i)-(iii) in Theorem 1 is called a crevasse in $\Sigma \times I$. We need to study crevasses from the topological point of view. This is not only necessary to prove Theorems 1 and 3, but also useful to understand topological properties of geometric limits of quasi-Fuchsian groups. As a special case, these theorems determine the topological types of $\mathbb{H}^3/G$ for geometric limits of any sequence in the Bers slice $B_{\sigma_n}$, which is naturally identified with the Teichmüller space $Teich(\Sigma)$. Then, $\mathbb{H}^3/G$ is homeomorphic to $\Sigma \times I - \mathcal{X}$ for some crevasse $\mathcal{X}$ with $\mathcal{X} \cap \Sigma \times [1/2, 1] = \emptyset$. Though the result does not imply data on the geometric structure on $\mathbb{H}^3/G$, some arguments used in the proofs of our theorems suggest implicitly that the hyperbolic structure on $\mathbb{H}^3/G$ would be controlled by those on the geometrically infinite tame ends $\mathcal{E}$ of $\mathbb{H}^3/G$ corresponding to the subsurfaces in $\mathcal{X}$ of types $B^\pm$. On the other hand, the hyperbolic structures on $\mathcal{E}$ will be determined only by their ending data if Thurston's Ending Lamination Conjecture [9] holds, where the ending data means the element of Teich$(B)$ determined by the conformal structure on the front end if $\mathcal{E}$ is geometrically finite and the ending lamination if $\mathcal{E}$ is geometrically infinite. The conjecture is proved by Minsky [5] in the case when the infimum injectivity radius of a hyperbolic 3-manifold is positive, and the project toward the complete solution is making steady progress by some people including himself. Thus, it would not be in distant future when we know all the elements of the geometric Bers boundary of Teich$(\Sigma)$.

**Problem 4.** Let $G_i$ $(i = 1, 2)$ be geometric limits of algebraically convergent sequences of quasi-Fuchsian groups with homeomorphisms $h_i : \mathbb{H}^3/G_i \to \Sigma \times I - \mathcal{X}$ for a given crevasse $\mathcal{X}$. Is $h_2^{-1} \circ h_1 : \mathbb{H}^3/G_1 \to \mathbb{H}^3/G_2$ properly homotopic to an isometry if, for any subsurface $B$ in $\mathcal{X}$ of types $B^\pm$, the corresponding ends $\mathcal{E}_i(B)$ in $\mathbb{H}^3/G_i$ have the same ending data?

**Outline of the proof of Theorem 1.** If an algebraically convergent sequence of quasi-Fuchsian representations $\rho_n : \Pi \to \text{PSL}_2(\mathbb{C})$ converges geometrically to a Kleinian group $G$, then there exists a $K_n$-quasi-isometry $g_n : \mathcal{N}_R(x_n, N_n) \to \mathcal{N}_R(x_\infty, M_\infty)$ with $R_n \not\to \infty$ and $K_n \not\subseteq 1$ for the suitable choice of base points $x_n \in N_n$ and $x_\infty \in M_\infty$, where $N_n = \mathbb{H}^3/\rho_n(\Pi)$ and $M_\infty = \mathbb{H}^3/G$. Since $N_n$ is homeomorphic to $\Sigma \times (0, 1)$, $N_n$ admits a topological fibration $\mathcal{G}_n$ with fiber $\Sigma$. Then, the foliation $\hat{\mathcal{G}}_n$ on $\mathcal{N}_R(x_\infty, M_\infty)$ is induced from $\mathcal{G}_n|\mathcal{N}_R(x, N_n)$ via $g_n$. 
However, it would be difficult to define a foliation on $M_\infty$ from $\hat{S}_n$'s since we do not have geometric data to investigate relations between $\mathcal{G}_n$ and $\mathcal{G}_{n+1}$. In our proof, we will invoke a 'coarse fibration' $\mathcal{S}_n$ on the convex core $C_n$ of $N_n$ 'fibers' of which are pleated surfaces between the two components of $\partial C_n$. Then, $\mathcal{N}_{R_n}(x_\infty, M_\infty)$ has the coarse foliation $\hat{S}_n$ induced from $\mathcal{S}_n|\mathcal{N}_{R_n}(x_n, N_n)$. Let $M_{\infty,p\text{-thick}(e)}$ be the union of parabolic cusp components of the $\varepsilon$-thin part $M_{\infty,\text{thin}(e)}$ of $M_\infty$ for a sufficiently small $\varepsilon > 0$. For any $x \in M_{\infty,p\text{-thick}(e)} = M_\infty - M_{\infty,p\text{-thick}(e)}$, there exists a constant $R(x)$ independent of $n \in \mathbb{N}$, such that, for any leaf $F(n)$ of $\hat{S}_n$ passing through the 1-neighborhood of $x$ in $M_\infty$, the diameter of the component $F_0^{(n)}$ of $F(n) \cap M_{\infty,p\text{-thick}(e)}$ nearest to $x$ is less than $R(x)$. Thus, if necessary passing to a subsequence, we may assume that $\{F_0^{(n)}\}$ converges uniformly to a surface $F$, and hence in particular $F_0^{(n)}$'s are mutually properly homotopic in $M_{\infty,p\text{-thick}(e)}$. This suggests that $\{\hat{S}_n\}$ is an expanding sequence of coarse foliations in $M_{\infty,p\text{-thick}(e)}$ after slightly modifying $\hat{S}_n$ by proper homotopy in $M_{\infty,p\text{-thick}(e)}$. Though the limit $F$ may not be an embedded surface, one can replace it by an embedded surface $S$ in the homotopy class of $F$ in $M_{\infty,p\text{-thick}(e)}$ by using the least area surface theory. A maximal set $\{Q_0\}$ of these embedded surfaces which are mutually disjoint and not properly homotopic to each other in $M_{\infty,p\text{-thick}(e)}$ divides $M_{\infty,p\text{-thick}(e)}$ into (in general infinitely many) blocks $B_\varepsilon$. Here, $B_\varepsilon$ being a block means that $B_\varepsilon$ is homeomorphic to $F \times (0, 1)$ for a subsurface $F$ of $\Sigma$. We note that this block decomposition misses points $x \in M_{\infty,p\text{-thick}(e)}$ in a small neighborhood of which does not meet the $g_n$-images of pleated surfaces. This occurs when $g_n^{-1}(x)$ is $N_n - \text{Int}C_n$ for all sufficiently large $n \in \mathbb{N}$. Each component of $N_n - \text{Int}C_n$ is a geometrically finite end and hence homeomorphic to $\Sigma \times [0, \infty)$. From this fact, we know that $x$ is in a geometrically finite end of $M_\infty$ homeomorphic to $F' \times [0, \infty)$ for some subsurface $F'$ of $\Sigma$. Regarding such ends also as blocks, we have a block decomposition of $\text{Int}M_{\infty,p\text{-thick}(e)}$. The interior $\mathcal{K}_k$ of the union $\bigcup_{i=1}^{k} \overline{B_i}$ of the first $k$ blocks is contained in $\mathcal{N}_{R_n}(x_\infty, M_\infty)$ if $n$ is sufficiently large, and hence it can be embedded in $\Sigma \times I$ via $g_n^{-1}$. In general, such embeddings are not expanding sequence as was remarked above. So, we will construct an expanding sequence by splitting blocks into sub-blocks if necessary and by embedding them into $\Sigma \times I$ at the sacrifice of a global continuity. Then, the images $\mathcal{K}_k$ are expanding open submanifolds, but not homeomorphic to the original $\mathcal{K}_k$. We will restore the originals by slit-sliding operations in $\Sigma \times I$. This splitting-to-restoring process is not a meaningless round trip since the process enables us to deal with the embedding problem stepwise. We have an expanding sequence $\{\mathcal{F}_n\}$ of unions of finitely many slits in $\Sigma \times I$ such that the 3-manifold $Y_n$ obtained by sliding along $\mathcal{F}_n$ contains $\mathcal{K}_n$ as an open submanifold. Moreover, $Y_n - \mathcal{K}_n$ contains a submanifold $W_{d(n)}$ such that there exists an embedding $\Phi_n : Y_n - W_{d(n)} \rightarrow \Sigma \times I$ with $\Phi_n(\mathcal{K}_n) = \Phi_{n+1}(\mathcal{K}_{n+1}) \subset \Phi_{n+1}(\mathcal{K}_{n+1})$. Since $\mathcal{K}_\infty = \text{Int}M_{\infty,p\text{-thick}(e)}$ is homeomorphic to $M_\infty$, our desired embedding of $M_\infty$ to $\Sigma \times I$ is defined by the expanding sequence $\{\Phi_n(\mathcal{K}_n)\}$.

Once $M_\infty$ is realized as an open subset of $\Sigma \times (0, 1)$, the conditions (i)-(iii) are derived from fundamental properties of hyperbolic 3-manifolds, e.g. any two parabolic cusps in a hyperbolic 3-manifold are not mutually parallel. In particular, we will
show by invoking some conditions on $\mathcal{X}$ that each component of $\lambda(\Sigma^n_{1})$ corresponds to a 'hidden' parabolic cusp of $M_{\infty}$.

Outline of the proof of Theorem 3. Consider any crevasse $\mathcal{X}$ in $\Sigma \times I$. For any $y \in \gamma = q(\mathcal{X})$, the components of $\Delta(\Lambda^x_{y}) - \lambda(\Lambda^x_{y})$ are called open subsurface of type $A^\pm$ for $\varepsilon = \pm$, and each component of $\Sigma_y - X_y$ is an open subsurface of type $C$. Thus, $X_y$ consists of geodesic loops and open subsurfaces of types $A^\pm$, $B^\pm$, and $C$, see Fig. 3 again.

In general, $\Sigma \times I - \mathcal{X}$ is not of finite type, that is, $\pi_1(\Sigma \times I - \mathcal{X})$ is possibly of infinitely generated. So, we will define as follows an expanding sequence $\{\mathcal{W}_n\}$ of submanifolds of finite type in $\Sigma \times I - \mathcal{X}$ with $\bigcup_{n=1}^{\infty} \mathcal{W}_n = \Sigma \times I - \mathcal{X}$. For any $n \in \mathbb{N}$, let $\mathcal{N}_{A,n}$ be the union of small collar neighborhoods in $\Sigma \times I$ of subsurfaces of types $A^\pm$, where the depth of each collar neighborhood in $\mathcal{N}_{A,n}$ is the $1/n$ times of that in $\mathcal{N}_{A,1}$. The number of subsurfaces of types $B^\pm$ and (saturated) geodesic loop components in $\mathcal{X}$ disjoint from $\mathcal{N}_{A,n}$ is finite. Let $B_n$ (resp. $\mathcal{P}_n$) be the set of such subsurfaces (resp. geodesic loops). Then, we have $B_n \subset B_{n+1}$ and $\bigcup \mathcal{P}_n \subset \bigcup \mathcal{P}_{n+1}$. Let $M^m_{\infty}$ be a 3-manifold obtained by cutting open $\Sigma \times I - \bigcup \mathcal{P}_n$ along $B_l \subset B_n$ and gluing back the both side of $B_l$ by the $m$-th iteration $\varphi_{m}$ of a pseudo-Anosov homeomorphism $\varphi_{1} : B_l \longrightarrow B_l$. For all sufficiently large $m \in \mathbb{N}$, we may assume that $M^m_{\infty}$ is acylindrical if necessary by adding some geodesic loops contained in subsurfaces of type $A^\pm$ to $\mathcal{P}_n$. The extended set is denoted by $\tilde{\mathcal{P}}_n$. By Thurston's Uniformization Theorem, Int$M^m_{\infty}$ admits a hyperbolic structure with two geometrically finite ends corresponding to $\Sigma \cup \Sigma_{1}$ and $Z \times Z$-cusps corresponding to elements of $\tilde{\mathcal{P}}_n$. Note that Int$M^m_{\infty}$ is homeomorphic to $\Sigma \times (0, 1)$ minus finitely many geodesic loops in fibers. By Hyperbolic Dehn Surgery Theorem in [8], Int$M^m_{\infty}$ is a geometric limits of hyperbolic 3-manifolds with quasi-Fuchsian holonomies. Let $\zeta^m_{n} : \pi_1(Q_{n,0}) \longrightarrow \text{PSL}_2(\mathbb{C})$ be the restriction of the holonomy of Int$M^m_{\infty}$, where $Q_{n,0}$ is the component of $\Sigma \times I - \bigcup (B_n \cup \tilde{\mathcal{P}}_n)$ containing $\Sigma_{1/2}$. Then, for each $n \in \mathbb{N}$, one can show that $\{\zeta^m_{n}\}_{n=1}^{\infty}$ converges algebraically to a representation $\zeta^\infty_{n}$ such that $N^\infty_{n} = \mathbb{H}^3/\mathbb{C}^{\infty}_{\tau_{1}}(\pi_1(Q_{n,0}))$ is homeomorphic to $Q_{n,0}$. In turn, we would like to show that $\{\zeta^\infty_{n}\}_{n=1}^{\infty}$ converges 'algebraically' in a reasonable sense, though $Q_{n,0}$ is not a submanifold of $Q_{n+1,0}$. If we use the component $\mathcal{W}_n$ of $Q_{n,0} - \mathcal{N}_{A,n}$ containing $\Sigma_{1/2}$ instead of $Q_{n,0}$, then $\mathcal{W}_n$ is a submanifold of $W_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{W}_n = \Sigma \times I - \mathcal{X}$. Then, $\{\zeta^\infty_{n}\}_{n=1}^{\infty}$ having a subsequence $\{\xi_{n}\}$ converging to $\xi_{\infty} : \pi_1(\Sigma \times I - \mathcal{X}) \longrightarrow \text{PSL}_2(\mathbb{C})$ means that $\{\xi_{n}\}_{n=1}^{\infty}$ converges algebraically to $\xi_{\infty}|_{\pi_1(W_{n})}$ for any $n \in \mathbb{N}$. The algebraic convergence of $\{\xi_{n}|_{\pi_1(W_{n})}\}$ is reduced to those of $\{\xi_{n}|_{\pi_1(C)}\}$ for open subsurfaces $C$ of type $C$ by Relative Boundeness Theorem [11]. The convergence of $\{\xi_{n}|_{\pi_1(C)}\}$ is examined by test pleated surfaces $\tilde{f}_{C_{n}} : C(\sigma_{n}) \longrightarrow N^\infty_{n}$ given in [10]. Usually, such a convergence theorem is proved by introducing a contradiction under the contrary assumption such that $\{\sigma_{n}\}$ is unbounded in Teich($C$). Then, we would have an accumulation point $[\nu]$ of $\{\sigma_{n}\}$ in the Thurston boundary $\mathcal{P}\mathcal{L}_0(C)$. Intuitively, the projective lamination $[\nu]$ represents the part of $C(\sigma_{n})$ which is split or collapsed most rapidly as $n \rightarrow \infty$. However, in our argument, we will need to concern relatively slowly split parts of $C(\sigma_{n})$. For example, let $l_{1}$ and $l_{2}$ be mutually disjoint simple geodesics in $C$, where $C$ is supposed to have a complete hyperbolic structure of finite area. Suppose that $\sigma_{n} \in \text{Teich}(C)$ is obtained by the $n^2$-full Dehn twist along $l_{1}$.
and the $n$-full Dehn twist along $l_2$. Then, $\{\sigma_n\}$ converges to $[l_1] \in \mathcal{P}L_0(C)$ which contains no data about $l_2$. For any simple geodesic loop $l \in C$ with $l \cap l_2 \neq \emptyset$, we have $\lim_{n \to \infty} \length_{\sigma_n}(l) = \infty$ even if $l \cap l_1 = \emptyset$. Such a divergence of the lengths of certain geodesic loops or measured laminations in $C(\sigma_n)$ is crucial to obtain our desired contradiction.

Once the algebraic convergence of $\{\xi_a\}$ to $\xi_\infty$ is proved, one can show that $H_\infty = \mathbb{H}^3/G$ is homeomorphic to $\Sigma \times I - \mathcal{X}$ for $G = \xi_\infty(\Sigma \times I - \mathcal{X})$ by using the fact that $\mathbb{H}^3/G$ has an expanding sequence $\{H_n\}$ of submanifolds homeomorphic to $W_n$. This fact can be also proved by using results in Anderson-Canary-McCullogugh [1]. For any $t \in \mathbb{N}$, $\xi_\infty|_{\pi_1(W_t)}$ is algebraically approximated by $\xi_a|_{\pi_1(W_t)} = \xi_{\infty}|_{\pi_1(W_t)}$, and the latter by $\zeta_{\infty}(\pi_1(W_t))$. For all sufficiently large $n(a)$, any parabolic elements of $\xi_\infty(\pi_1(W_t))$ corresponds to those of $\zeta_{\infty}(\pi_1(W_t))$. It follows that $\xi_\infty(\pi_1(W_t))$ is a geometric limit of $\{\zeta_{n(a)}^{m(a)}(\pi_1(W_t))\}$. Note that the holonomy group $G_a$ of $\text{Int} M_{n(a)}^{m(a)}$ contains $\zeta_{n(a)}^{m(a)}(\pi_1(W_t))$. From this fact, we will show that $G$ is a geometric limit of $\{G_a\}$. In turn, each $G_a$ is the geometric limit of quasi-Fuchsian groups as we have remarked above.

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