SUM OF THE EDGE LENGTHS OF A GEODESIC GRAPH

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ABSTRACT

Consider an embedding of a complete graph of order four into the 2-sphere such that each edge becomes a shortest geodesic connecting its endpoints. Then we show that the sum of the edge lengths is at most $4\pi$, and is bigger than $3\pi$ if the graph is not contained in any hemisphere.

はじめに


1. Results

In this article, we consider a finite graph geodesically embedded into a surface with constant curvature metric, and estimate the sum of the edge lengths. As usual, we regard a finite graph as a 1-dimensional cellular complex by setting a vertex as a 0-cell and an edge as a closed 1-cell. Given an embedding $f$ of a finite graph into a surface, its image $G$ is obviously identified with the original graph. Thus we say that the image of a vertex

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and an edge under $f$ a vertex and an edge of $G$. By $S^2$, we mean the two dimensional sphere endowed with the Riemannian metric of constant curvature $+1$. Then our result is the following.

**Theorem 1.** Let $G$ be the image of an embedding of the complete graph $K_4$ of order 4 into $S^2$. Suppose that

1. each edge of $G$ is a shortest geodesic arc on $S^2$ connecting its endpoints, and
2. $G$ is not contained in any hemisphere of $S^2$. 

Let $E$ be the sum of the length of the edges of $G$. Then $3\pi < E \leq 4\pi$ holds.

This almost follows from the result of Guddum [1]. In fact, his theorem in [1] implies the inequality $3\pi \leq E \leq 4\pi$. In the next section, we will prove the theorem above using purely elementary spherical geometry.

A generalization of this estimate to the case of any graph embedded in $n$-sphere $S^n$ will appear in [4]. Our estimate depends upon the combinatorics of the graph only.

Here we append an easy observation for more general cases. Let $F_g$ be a closed, orientable surface of genus $g \geq 2$ with a fixed Riemannian metric of constant curvature $-1$. For convenience, let $F_0$ denote $S^2$.

**Proposition 1.** Let $G$ be the image of an embedding of a graph into $F_g$ where $g \neq 1$. Suppose that

1. each edge $e$ of $G$ is a shortest geodesic arc on $F_g$ connecting its endpoints, and
2. the closure of each component of $F_g - G$ is a convex polygon on $F_g$.

Then the sum of the length of the edges of $G$ is greater than $\pi|2 - 2g|$.

**Proof.** Let $\sigma$ be a complementary face of $G$, i.e., the closure of a component of $F_g - G$. By $\text{Area}(\sigma)$ and $\text{Length}(\partial\sigma)$, we denote the area of $\sigma$ and the length of the boundary $\partial\sigma$ of $\sigma$ respectively. We consider the ratio $\text{Area}(\sigma)/\text{Length}(\partial\sigma)$. This ratio is strictly less than the corresponding ratio for the disk on $F_g$ which has equilong boundary as $\sigma$. See [2] for a survey. By elementary calculations, the ratio for such a disk is shown
to be less than 1 for any \( g \neq 1 \). This implies \( \text{Length}(\partial \sigma) > \text{Area}(\sigma) \) holds. By summing the inequalities up over all complementary faces, we have \( \sum \text{Length}(\partial \sigma) > \sum \text{Area}(\sigma) = 2\pi|2 - 2g| \), where the last equality follows from the Gauss-Bonne’s theorem. Then the sum of the length of the edges of \( G \), which is equal to the half of \( \sum \text{Length}(\partial \sigma) \), is greater than \( \pi|2 - 2g| \).

\[ \square \]

2. Proof

Let us start with recalling fundamentals of spherical geometry. Let \( u_1, u_2, u_3 \) be points on \( S^2 \) such that no two of them are antipodal and no great circle includes all the three points. Let \( \Lambda_i \) be the closed hemisphere whose boundary contains the other two points than \( u_i \) and whose interior contains \( u_i \) for \( i = 1, 2, 3 \). The spherical triangle \( \Delta \) with the vertices \( u_1, u_2, u_3 \) is defined as the intersection \( \Lambda_1 \cap \Lambda_2 \cap \Lambda_3 \). Then we have the following:

- \( \Delta \) is convex, i.e., any pair of points in \( \Delta \) is connected by a geodesic arc in \( \Delta \). Moreover the arc is shortest among the arcs connecting the points, and the length is equal to the spherical distance between the points which is strictly less than \( \pi \).
- The length of an edge of \( \Delta \) is less than the sum of the length of the other two edges (the triangle inequality).

**Proof of Theorem 1.** Let \( v_1, v_2, v_3, v_4 \) be the vertices of \( G \). Let \( e_{ij} \) denote the edge of \( G \) connecting \( v_i \) and \( v_j \) for \( 1 \leq i, j \leq 4 \). Note that the assumption (1) implies that the length of \( e_{ij} \) is equal to the spherical distance \( d_{ij} \) on \( S^2 \) between \( v_i \) and \( v_j \) for \( 1 \leq i, j \leq 4 \). Thus it suffice to show that

\[
3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi.
\]

In the following, the antipodal point of \( v_i \) is denoted by \( v_{i+4} \) for \( 1 \leq i \leq 4 \). Also \( d_{ij} \) denotes the spherical distance between \( v_i \) and \( v_j \) for \( 1 \leq i, j \leq 8 \).

First we consider the case that a couple of the vertices, say \( v_1 \) and \( v_2 \), are antipodal, equivalently, \( d_{12} = \pi \). This implies that \( d_{13} + d_{32} = d_{14} + d_{42} = \pi \) holds. Together with \( 0 < d_{34} \leq \pi \), we have \( 3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi \).

Next consider the case that all the four vertices are contained in a great circle. Suppose for example that \( v_1, v_2, v_3, v_4 \) lies in a great circle \( \Gamma \) in this
order. Since $G$ is the image of an embedding, the edge $e_{13}$ is not contained in $\Gamma$. This implies that $e_{13}$ is a half of a great circle and $d_{13} = \pi$. Also we see that $d_{24} = \pi$ and so we obtain $\sum_{1 \leq i < j \leq 4} d_{ij} = 4\pi$.

Thus, in the following, we assume that $d_{ij} \neq \pi$ for $1 \leq i, j \leq 4$ and at most three vertices of $G$ lie on a great circle.

Next consider the case that three vertices are contained in a great circle. Suppose for example that $v_1$, $v_2$ and $v_3$ lie on a great circle. Then, by the triangle inequality, we have $d_{41} + d_{42} > d_{12}$, $d_{42} + d_{43} > d_{23}$ and $d_{43} + d_{41} > d_{31}$. These are added to obtain

$$2(d_{41} + d_{42} + d_{43}) > d_{12} + d_{23} + d_{31} = 2\pi.$$ 

Thus

$$\sum_{1 \leq i < j \leq 4} d_{ij} = (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} > \pi + 2\pi = 3\pi.$$ 

In the same way as above, we have $d_{4j} + d_{4j + 1} + d_{4j + 2} > \pi$. Since $d_{4j} = \pi - d_{4(j+4)}$ for $j = 1, 2, 3$,

$$\sum_{1 \leq i < j \leq 4} d_{ij} = (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \quad \quad = 3\pi - (d_{45} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \quad \quad < 3\pi - \pi + 2\pi = 4\pi$$

holds.

Finally we consider the case that the four vertices are in a general position: We assume that $d_{ij} \neq \pi$ for $1 \leq i, j \leq 4$ and at most two vertices of $G$ lie on a great circle. This means that for any three of the points there is a triangular face which includes the three points as vertices.

Then, by the triangle inequality, we have $d_{53} + d_{53} > d_{56}$, $d_{54} + d_{64} > d_{56}$, $d_{53} + d_{54} > d_{34}$ and $d_{63} + d_{64} > d_{34}$. Add these to obtain

$$d_{53} + d_{53} + d_{54} + d_{64} > d_{34} + d_{56}.$$ 

Here note that $d_{ij} = \pi - d_{(i-4)j}$ for $i = 5, 6$, $j = 1, 2, 3$, and $d_{56} = d_{12}$. These imply that

$$4\pi - (d_{13} + d_{23} + d_{14} + d_{24}) > d_{34} + d_{12}.$$
Consequently we have

\[ 4\pi > \sum_{1 \leq i < j \leq 4} d_{ij}. \]

In the following, let \( \Delta \) be the spherical triangle bounded by \( e_{12}, e_{23} \) and \( e_{31} \).

**Claim 1.** The antipodal point \( v_8 \) of \( v_4 \) is included in the interior of \( \Delta \).

*Proof.* Let \( \Gamma_i \) be the great circle including an edge of \( \Delta \) but not including \( v_i \) for \( i = 1, 2, 3 \). By the assumption above, \( v_4 \) and hence \( v_8 \) never lie on \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \). Note that \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \) decomposes \( S^2 \) into eight spherical triangles.

Assume for a contradiction that \( v_8 \) is not included in the interior of \( \Delta \). Then \( v_4 \) is included in the interior of one of the seven spherical triangles other than the antipodal image of \( \Delta \). This implies all the four points \( v_1, v_2, v_3 \) and \( v_4 \) are included in the closed hemisphere bounded by one of \( \Gamma_1, \Gamma_2 \) or \( \Gamma_3 \). Since the four vertices are assumed in a general position, there is a hemisphere which contains whole \( G \). This contradicts the assumption (2) of the theorem. \( \square \)

**Claim 2.** The inequality \( d_{12} + d_{13} > d_{82} + d_{83} \) holds.

*Proof.* Since the length of each edge is less than \( \pi \), the edge \( e_{13} \) intersects the great circle including \( v_2 \) and \( v_8 \) at just one point \( v_9 \). Let \( d_{49} \) or \( d_{94} \) denote the distance between \( v_i \) and \( v_9 \) for \( 1 \leq i \leq 9 \). The distance \( d_{19} \) is realized by a geodesic arc included in \( e_{13} \) and also is \( d_{93} \). Thus \( d_{13} = d_{19} + d_{93} \) holds.

The distance \( d_{29} \) is realized by a geodesic arc \( e_{29} \) in \( \Delta \) since \( \Delta \) is convex. In particular, the arc \( e_{29} \) contains \( v_8 \) and so \( d_{29} = d_{28} + d_{89} \) holds.

Together with the triangle inequality \( d_{12} + d_{19} > d_{29} \) and \( d_{98} + d_{93} > d_{83} \), we conclude

\[ d_{12} + d_{13} = d_{12} + d_{19} + d_{93} > d_{29} + d_{93} = d_{28} + d_{89} + d_{93} > d_{28} + d_{83}. \]

\( \square \)

In the same way, we have \( d_{21} + d_{23} > d_{81} + d_{83} \) and \( d_{31} + d_{32} > d_{81} + d_{82} \).

By adding these inequalities, we obtain

\[ d_{12} + d_{23} + d_{31} > d_{81} + d_{82} + d_{83}. \]
Together with the equations $d_{8j} = \pi - d_{4j}$ for $i = 1, 2, 3$, we conclude that
\[
\sum_{1 \leq i < j \leq 3} d_{ij} > 3\pi - \sum_{1 \leq k \leq 3} d_{k4}.
\]
This completes the proof. □

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REFERENCES


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