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ハンドル体の写像類群のホモロジー的アナロジーについて

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1. Introduction

A 3-dimensional handlebody H_g is an orientable 3-manifold constructed from a 3-ball by attaching g 1-handles. We denote the boundary of H_g by Σ_g, which is an orientable closed surface of genus g. Let $\mathcal{M}_g$ be the mapping class group of Σ_g and $\mathcal{H}_g$ be the mapping class group of H_g, for short, we call this group the *handlebody group*. For elements a, b and c of a group, we write $\bar{c} = c^{-1}$, and $a * b = ab\bar{a}$. Let P_g be a planar surface constructed from a 2-disk by removing g copies of disjoint 2-disks. As indicated in Figure 1, we denote the boundary components of P_g by $\gamma_0, \gamma_2, \dots, \gamma_{2g}$, and denote some properly embedded arcs of P_g by $\gamma_1, \gamma_3, \dots, \gamma_{2g+1}$, $\beta_2, \beta_4, \dots, \beta_{2g-2}$ and $\beta'_2, \beta'_4, \dots, \beta'_{2g-2}$. The 3-manifold $P_g \times [-1, 1]$ is homeomorphic to H_g. On $\partial(P_g \times [-1, 1]) = \Sigma_g$, we define $c_{2i-1} = \partial(\gamma_{2i-1} \times [-1, 1])$ $(1 \leq i \leq g+1)$, $b_{2j} = \partial(\beta_{2j} \times [-1, 1])$, $b'_{2j} = \partial(\beta'_{2j} \times [-1, 1])$ $(2 \leq j \leq g-1)$, and $c_{2k} = \gamma_{2k} \times \{0\}$ $(1 \leq k \leq g)$. In Figures 2 and 3, these circles are illustrated and oriented. For simple close curve a on Σ_g, we define the Dehn twist T_a about a as indicated in Figure 4.

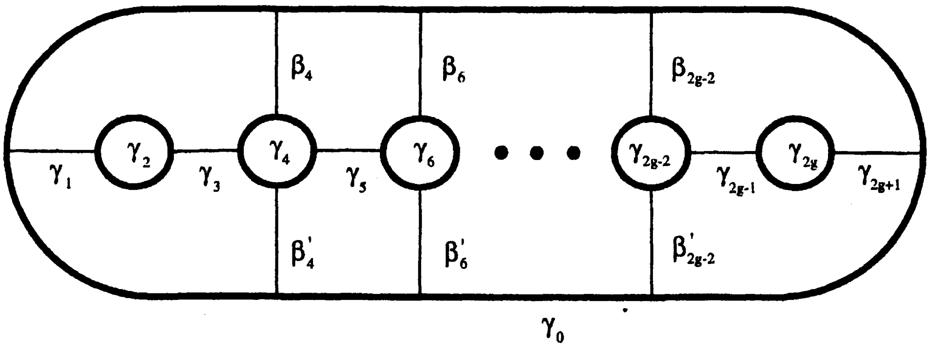

FIGURE 1

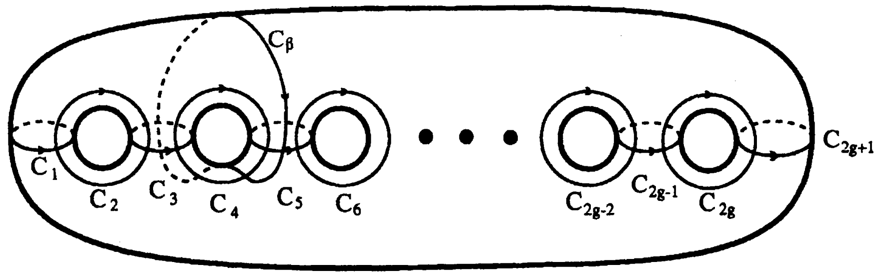

FIGURE 2

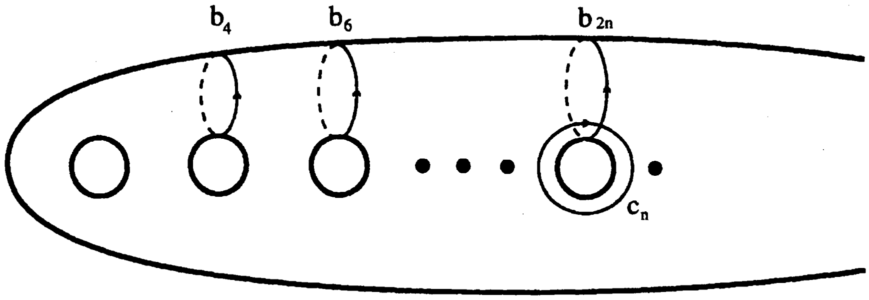

FIGURE 3

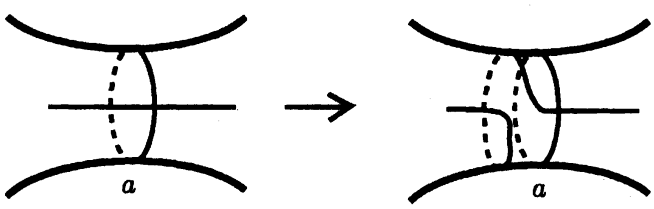

FIGURE 4

For short, we denote T_{c_i} by C_i, and $T_{b_{2i}}$ by B_{2i}. As elements of $H_1(\Sigma_g, \mathbb{Z})$, we take

$$x_1 = -c_1, \quad y_1 = -c_2$$
$$x_i = b_{2i}, \quad y_i = -c_{2i}, \text{ where } 2 \leq i \leq g-1,$$
$$x_g = -c_{2g}, \quad y_g = -c_{2g+1}.$$

Then, $\{x_1, y_1, \cdots, x_g, y_g\}$ is a basis of $H_1(\Sigma_g, \mathbb{Z})$, and satisfy $(x_i, y_j) = \delta_{i,j}$, $(x_i, x_j) = (y_i, y_j) = 0$ for the intersection form $(,)$. Let E_g be a identity $g \times g$ matrix, and

$$J = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

We define $\mathrm{Sp}(2g) = \{M \in GL(2g, \mathbb{Z}) \mid MJM' = J\}$, where M' means a transpose of M. Let p be a point on Σ_g. We can characterize the handlebody group $\mathcal{H}_g$ by the actions of each elements on the fundamental group $\pi_1(\Sigma_g, p)$. Let l_1 be an arc on Σ_g which begins from p and ends on c_1, l_i $(2 \leq i \leq g-1)$ be an arc Σ_g which begins from p and ends on b_{2i}, and l_g be an arc on Σ_g which begins from p and ends on c_{2g}. We denote $\mathcal{N}$ the normal closure of $\{l_1 c_1 \overline{l_1}\ , l_2 b_4 \overline{l_2}\ , \ldots, l_{g-1} b_{2g-2} \overline{l_{g-1}}\ , l_g c_{2g} \overline{l_g}\ \}$, then $\mathcal{H}_g = \{\phi \in \mathcal{M}_g \mid \phi(\mathcal{N}) = \mathcal{N}\}$. We define a homological analogue of $\mathcal{H}_g$. Let N be the $\mathbb{Z}$-submodule of $H_1(\Sigma_g, \mathbb{Z})$ generated by $\{x_1, \ldots, x_g\}$, and $\mathcal{HH}_g$ be a subgroup of $\mathcal{M}_g$ defined by $\mathcal{HH}_g = \{\phi \in \mathcal{M}_g \mid \phi_*(N) = N\}$. We call $\mathcal{HH}_g$ the *homological handlebody group* of genus g. For each element ϕ of $\mathcal{M}_g$, we define a $2g \times 2g$ matrix M_ϕ by

$$(\phi(x_1), \phi(x_2), \cdots, \phi(x_g), \phi(y_1), \phi(y_2), \cdots, \phi(y_g)) = (x_1, x_2, \cdots, x_g, y_1, y_2, \cdots, y_g) M_\phi.$$

Then, M_ϕ is an element of $\mathrm{Sp}(2g)$, and the map μ from $\mathcal{M}_g$ to $\mathrm{Sp}(2g)$ defined by mapping ϕ to M_ϕ is a surjection. On the other hand, $\mu|_{\mathcal{H}_g}$ is not a surjection. We define a subgroup $ur\mathrm{Sp}(2g)$ of $\mathrm{Sp}(2g)$ by

$$ur\mathrm{Sp}(2g) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Sp}(2g) \right\},$$

where A, B, and D are $g \times g$ matrices, and 0 is a $g \times g$ zero matrix. We show the following theorem

Theorem 1.1. *$\mu(\mathcal{H}_g) = urSp(2g)$.*

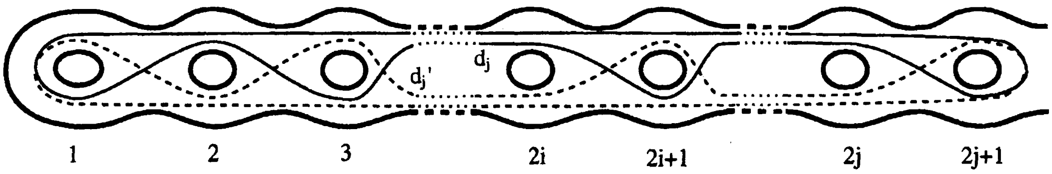

FIGURE 5

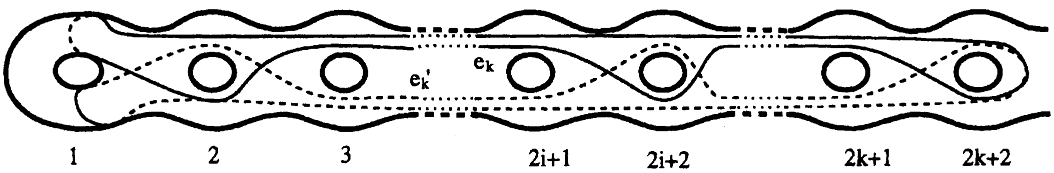

FIGURE 6

By definition, $\mathcal{HH}_g = \mu^{-1}(ur\mathrm{Sp}(2g))$. Let $[a]$ be the largest integer n which satisfies $n \leq a$, and d_j, d'_j, e_k, e'_k are indicated in Figures 5 and 6. We show

Theorem 1.2. *If $g \geq 3$, $\mathcal{HH}_g$ is generated by C_1, $C_2C_1^2C_2$, $C_2C_1C_3C_2$, $C_{2i}C_{2i-1}B_{2i}C_{2i}$, $C_{2i}C_{2i+1}B_{2i}C_{2i}$ $(2 \leq i \leq g-1)$, $C_{2g}C_{2g-1}C_{2g+1}C_{2g}$, $T_{d_j}\overline{T_{d'_j}}$ $(1 \leq j \leq [\frac{g-1}{2}])$, and $T_{e_k}\overline{T_{e'_k}}$ $(1 \leq k \leq [\frac{g-2}{2}])$.*

The author does not know whether $\mathcal{HH}_2$ is finitely generated or not. This note is a survey of a paper [1].

2. Proof of Theorem 1.1

It is easy to see that $\mu(\mathcal{HH}_g) \subset ur\mathrm{Sp}(2g)$. We show that $ur\mathrm{Sp}(2g) \subset \mu(\mathcal{HH}_g)$. Let S_0 be a $g \times g$ symmeric matrix, and U_1, U_2, U_3 be $g \times g$ unimodular matrices given by

$$S_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \ U_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, U_3 = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

By applying the argument by Hua and Reiner [2], we show

Lemma 2.1. *The group $urSp(2g)$ is generated by*

$$\left\{ \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \begin{pmatrix} U_i & 0 \\ 0 & (U_i')^{-1} \end{pmatrix}, \text{ where } i = 1, 2, 3 \right\}.$$

□

Suzuki [5] introduced elements ρ (cyclic translation of handles), ω_1 (twisting a knob), ρ_{12} (interchanging two knobs), and θ_{12} (sliding) of $\mathcal{H}_g$. In [5], their acions on the fundamental group of Σ_g were listed. With using this list, we show

$$\mu(C_1) = \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \mu(\rho) = \begin{pmatrix} U_1 & 0 \\ 0 & (U_1')^{-1} \end{pmatrix},$$

$$\mu(\rho_{12}\theta_{12}\rho_{12}^{-1}) = \begin{pmatrix} U_2 & 0 \\ 0 & (U_2')^{-1} \end{pmatrix}, \mu(\omega_1) = \begin{pmatrix} U_3 & 0 \\ 0 & (U_3')^{-1} \end{pmatrix}.$$

The above observation shows that $ur\mathrm{Sp}(2g) \subset \mu(\mathcal{H}_g)$.

3. Proof of Theorem 1.2

We denote the kernel of μ by $\mathcal{I}_g$ and call this the *Torelli group*. By Theorem 1.1, we can show that $\mathcal{HH}_g$ is generated by $\mathcal{H}_g \cup \mathcal{I}_g$. For $g \geq 3$, we find finite subsets $\mathcal{S}$ of $\mathcal{I}_g$ such that $\mathcal{H}_g \cup \mathcal{S}$ generates $\mathcal{HH}_g$. Johnson [3] showed that, when g is larger

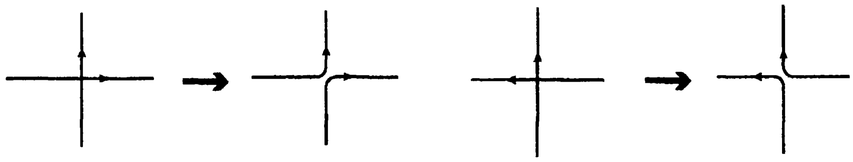

FIGURE 7

than or equal to 3, $\mathcal{I}_g$ is finitely generated. We review his result. We orient and call simple closed curves as is indicated in Figure 2, and call $(c_1, c_2, \ldots, c_{2g+1})$ and $(c_\beta, c_5, \ldots, c_{2g+1})$ as *chains*. For oriented simple closed curves d and e which mutually intersect in one point, we construct an oriented simple closed curve $d + e$ from $d \cup e$ as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 7. For a consecutive subset $\{c_i, c_{i+1}, \ldots, c_j\}$ of a chain, let $c_i + \cdots + c_j$ be the oriented simple closed curve constructed by repeated applications of the above operations. Let $(i_1, \ldots, i_{r+1})$ be a subsequence of $(1, 2, \ldots, 2g + 1)$ (Resp. $(\beta, 5, \ldots, 2g + 1)$). We construct the union of circles $\mathcal{C} = c_{i_1} + \cdots + c_{i_2-1} \cup c_{i_2} + \cdots + c_{i_3-1} \cup \cdots \cup c_{i_r} + \cdots + c_{i_{r+1}-1}$. If r is odd, the regular neighborhood of $\mathcal{C}$ is an oriented compact surface with 2 boundary components. Let ϕ be the element of $\mathcal{M}_g$ defined as the composition of the positive Dehn twist along the boundary curve to the left of $\mathcal{C}$ and the negative Dehn twist along the boundary curve to the right of $\mathcal{C}$. Then, ϕ is an element of $\mathcal{I}_g$. We denote ϕ by $[i_1, \ldots, i_{r+1}]$, and call this *the odd subchain map* of $(c_1, c_2, \ldots, c_{2g+1})$ (Resp. $(c_\beta, c_5, \ldots, c_{2g+1})$). Johnson [3] showed the following theorem:

Theorem 3.1. [3, Main Theorem] *For $g \geq 3$, the odd subchain maps of the two chains $(c_1, c_2, \ldots, c_{2g+1})$ and $(c_\beta, c_5, \ldots, c_{2g+1})$ generate $\mathcal{I}_g$.* □

By taking conjugations of odd subchain maps by elements of $\mathcal{H}_g$ and applying the following theorem by Takahashi [6], we show Theorem 1.2.

Theorem 3.2. [6] *$\mathcal{H}_g$ is generated by C_1, $C_2C_1^2C_2$, $C_2C_1C_3C_2$, $C_{2i}C_{2i-1}B_{2i}C_{2i}$, $C_{2i}C_{2i+1}B_{2i}C_{2i}$ $(2 \leq i \leq g-1)$, $C_{2g}C_{2g-1}C_{2g+1}C_{2g}$.* □

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