数理解析研究所講究録 1330 巻 2003 年 149-160

Singular limit analysis of aggregating patterns in a Chemotaxis–Growth model

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1 Introduction

We consider the following chemotaxis–growth model equation [7]:

$$\begin{cases} u_t = \varepsilon^2 \Delta u - \varepsilon k \nabla(u \nabla \chi(v)) + f(u) & \\ & t > 0,\ x \in \Omega \\ v_t = \Delta v + u - \gamma v, & \end{cases} \tag{1}$$

where $\chi(v) = v$ and $f(u) = u(1-u)(u-a)$ with $0 < a < 1/2$.

We showed that for sufficiently small $\varepsilon > 0$, there exist several statistic and dynamic patterns depending on the parameter k and the form of the sensitive function $\chi(v)$ of Chemotaxis in [7]. Here, we consider the patterns which did not treat in [7] and [10]. We first show the two numerical simulations (Figure 1). They imply that the band and triple junction patterns stably exist. In Section 2.1, we study the equation which governs the motion of the simple band pattern by using the formal analysis. From the second numerical simulation, it is suggested that the 2–dimensional traveling solution with a triple junction exists in the channel domain $\Omega_L = \{(x,y) | -L < x < L, -\infty < y < \infty\}$ in $\mathbf{R}^2$. In Section 2.2, we show the dependency for the velocity and the shape of these solutions on the domain size L and intensity of the chemotaxis effect k by using the result in Section 2.1 as ε tends to zero.

2 Formal Analysis

To study the motion of the front interface of the band and the triple junction patterns, we use the formal analysis by Zykov [11], Mikhailov [6] and Zykov et al. [12].

2.1 The band pattern with two interfaces

The numerical simulations (Figure 3) show that

(S1) The distance between two interfaces of the band pattern is constant.

(S2) When the curvature is small, the interfaces of the band pattern become flat. When the curvature is large, the target pattern shrinks and finally tends to the symmertic equilibrium solution.

From (S2), we note that the motion of the interface seems as similar as one of the pattern governed by the mean curvature flow. To show that, we consider the motion of the simple arc-like band pattern in $\Omega = \mathbf{R}^2$. Using the formal analysis, we assume that

Assumption: The distance between the two interfaces of the band pattern is small as compared with the radius of curvature, that is, k is large.

Therefore, we may set up that two interfacies of the band pattern have the same mean curvature κ. Using a new variable $\xi = r + \varepsilon\hat{V}t$ with $r = |\mathbf{x}|$, we rewrite (1) in $\Omega = \mathbf{R}^2$ as

$$\begin{cases} 0 = \varepsilon^2 u_{\xi\xi} - \varepsilon(\hat{V} - \varepsilon\kappa)u_\xi - \varepsilon k(u\chi'(v)v_\xi)_\xi + f(u) & \\ & ,\xi \in \mathbf{R} \\ 0 = v_{\xi\xi} - (\varepsilon\hat{V} - \kappa)v_\xi + u - \gamma v & \\ \lim\limits_{|\xi|\to\infty}(u,v)(\xi) = (0,0). & \end{cases} \qquad (2)$$

Outer Solution of (2) in R

When $\varepsilon \downarrow 0$, it follows from (2) that $f(u) = 0$. Thus, we put

$$u(\xi) = \begin{cases} 1 & \xi \in \Omega_1 \\ 0 & \xi \in \Omega_0, \end{cases}$$

where $\Omega_1 = (0, \delta)$ and $\Omega_0 = \mathbf{R} \setminus \Omega_1$.

Substituting this into (2) and putting $\varepsilon = 0$, we have

$$\begin{cases} 0 = v_{\xi\xi} + \kappa v_\xi + g_i(v), \quad \xi \in \Omega_i \ (i = 0, 1) \\ \lim\limits_{|\xi|\to\infty} v(\xi) = 0, \quad v \in C^1(\mathbf{R}), \end{cases} \qquad (3)$$

where $g_i(v) = i - \gamma v$.

Therefore, the solution of (3) is represented by

$$v(\xi) = \begin{cases} C_- e^{k_+ \xi} & \xi \in (-\infty, 0) \\ C_1 e^{k_+ \xi} + C_2 e^{k_- \xi} + \dfrac{1}{\gamma} & \xi \in (0, \delta) \\ C_+ e^{k_-(\xi - \delta)} & \xi \in (\delta, \infty), \end{cases} \tag{4}$$

where $k_\pm = \frac{-\kappa \pm \sqrt{\kappa^2 + 4\gamma}}{2}$, $C_- = \frac{k_-(e^{-k_+\delta} - 1)}{\gamma\sqrt{\kappa^2 + 4\gamma}}$, $C_+ = \frac{k_+(1 - e^{k_-\delta})}{\gamma\sqrt{\kappa^2 + 4\gamma}}$, $C_1 = \frac{k_-}{e^{k_+\delta}\gamma\sqrt{\kappa^2 + 4\gamma}}$ and $C_2 = -\frac{k_+}{\gamma\sqrt{\kappa^2 + 4\gamma}}$.

Since the outer solution of u is not a good approximating solution, that is, it is discontinuous, we need to obtain a good approximating one in the neighborhood of $\xi = 0$ and $\xi = \delta$.

Inner Solution of (2) at $\xi = 0$ and $\xi = \delta$

To obtain the inner solution in the neighborhood of $\xi = 0$ and $\xi = \delta$, we introduce a new stretched variable $\zeta = \xi/\varepsilon$ or $\zeta = (\xi - \delta)/\varepsilon$. Then, the solution $v_-(\zeta) = v(\xi/\varepsilon)$ and $v_+(\zeta) = v((\xi - \delta)/\varepsilon)$ of (2) in each neighborhood satisfy

$$\begin{cases} 0 = v_{\pm\zeta\zeta} - \varepsilon(\varepsilon\hat{V} - \kappa)v_{\pm\zeta} + \varepsilon^2(u_\pm - \gamma v_\pm), & \zeta \in \mathbf{R} \\ \lim\limits_{\zeta \to \pm\infty} v_-(\zeta) = v(0), \ \lim\limits_{\zeta \to \pm\infty} v_+(\zeta) = v(\delta), \end{cases}$$

by using the matching conditions at $\xi = 0$ and $\xi = \delta$. As ε tends to zero, it holds that $v_-(\zeta) \equiv v(0) = C_-$ and $v_+(\zeta) \equiv v(\delta) = C_+$ where $v(\xi)$ is the solution of (3).

On the other hand, by using these solutions $v_\pm$, we have

$$\begin{cases} 0 = u_{\pm\zeta\zeta} - (\hat{V} - \varepsilon\kappa + k\chi'(v_\pm)v_{\pm\xi})u_{\pm\zeta} + f(u_\pm), & \zeta \in \mathbf{R} \\ \lim\limits_{\zeta \to -\infty} u_-(\zeta) = 0, \quad \lim\limits_{\zeta \to \infty} u_-(\zeta) = 1 \\ \lim\limits_{\zeta \to -\infty} u_+(\zeta) = 1, \quad \lim\limits_{\zeta \to \infty} u_+(\zeta) = 0, \end{cases} \tag{5}$$

where $u_-(\zeta) = u(\xi/\varepsilon)$ and $u_+(\zeta) = u((\xi - \delta)/\varepsilon)$, $v_{-\xi} = \frac{d}{d\xi}v(0)$ and $v_{+\xi} = \frac{d}{d\xi}v(\delta)$ for the solution $v(\xi)$ of (3).

Since the coefficient of $u_{\pm\zeta}$ are constant, it turns out that

$$\begin{cases} \hat{V} - \varepsilon\kappa + k\chi'(v_-)v_{-\xi} = c^* & at\ \xi = 0 \\ \hat{V} - \varepsilon\kappa + k\chi'(v_+)v_{+\xi} = -c^* & at\ \xi = \delta, \end{cases} \tag{6}$$

where c^* is the positive velocity of the traveling front solution of the following problem with the traveling coordinate $z = \zeta + c^* t$ (see Fife and McLeod [4]):

$$\begin{cases} u_t = u_{\zeta\zeta} + f(u), \ \zeta \in \mathbf{R} \\ \lim_{\zeta \to -\infty} u(\zeta, t) = 0, \ \lim_{\zeta \to \infty} u(\zeta, t) = 1. \end{cases}$$

It follows from (4) and (6), δ and $\hat{V}$ can be given as the function of (κ, k, ε).

Remark 1 *For $\chi(v) = v$, δ satisfies*

$$\frac{k}{2\sqrt{\kappa^2 + 4\gamma}}(2 - e^{-k+\delta} - e^{k-\delta}) = c^* \tag{7}$$

and $\hat{V}$ is represented by

$$\hat{V} = \varepsilon\kappa + \frac{k(e^{-k+\delta} - e^{k-\delta})}{2\sqrt{\kappa^2 + 4\gamma}}. \tag{8}$$

Moreover, it holds that

$$\frac{\partial \hat{V}}{\partial \kappa}|_{\kappa=0} = \varepsilon + \frac{k - 2c^*\sqrt{\gamma}}{4\gamma} \log \frac{k}{k - 2c^*\sqrt{\gamma}}. \tag{9}$$

Since $\frac{\partial \hat{V}}{\partial \kappa}|_{\kappa=0} > 0$ for any k satisfying $k > 2c^*\sqrt{\gamma}$, the planar equilibrium solution, that is $\kappa = 0$, is stable with respect to some disturbances.

As the related result, we show the stability of the planar equilibrium solution of (1) in the channel domain Ω_L.

Theorem 1 (Tsujikawa [10]) *The planar equilibrium solution of* (1) *in the channel domain Ω_L with Newmann boundary conditions on $x = -L, L$ is linearly stable for sufficientlly small ε if the solution exists.*

2.2 The pattern with a triple junction

We treat the traveling solution with a triple junction (Figure 1). From the numerical simulations, it is known that

(S3) There exists the traveling solution with a triple junction and the profile of the front interface without the neighborhood of the triple junction is independent of k for fixed domain size L (Figures 1 and 3).

(S4) The curvature and velocity of the front interface decrease when L increases (Figures 3 and 4).

(S5) The velocity of the traveling solution increases when k increases (Figure 5).

We only consider the case that there is an aggregating region Ω_1 which has three branches and they connect at one region. Each branch which we call B_i ($i = 1, 2, 3$) has two interfaces Γ_i^1 and Γ_i^2 and each curvature of their interface denotes κ_i. Then, we assume that the width $\delta(\kappa_i)$ of the branch B_i satisfies (7). Define the crossing points of each interfaces by $A_{i,j}$. Then there exists a triangle which has three vertices $A_{1,2}$, $A_{2,3}$ and $A_{3,1}$. Let $\theta_{i,j}$ and δ_k^* be the angles of each vertex $A_{i,j}$ and the length of the side opposite to $A_{i,j}$ ($i, j \neq k$). Therefore, it follows from Sine formula that

$$\frac{\delta_3^*}{\sin\theta_{1,2}} = \frac{\delta_1^*}{\sin\theta_{2,3}} = \frac{\delta_2^*}{\sin\theta_{3,1}}, \tag{10}$$

where $\delta_k^* = \delta(\kappa_k)$ and $\theta_{1,2} + \theta_{2,3} + \theta_{3,1} = 2\pi$.

Since the profile of the traveling solution is symmetric with respect to y-axis, we treat the solution in the half region $\Omega_{L/2} = \{(x, y) |\ 0 < x < L,\ -\infty < y < \infty\}$. Assuming that two interfaces of the front part of the solution have same curvature, we may only consider either interface in two ones, which we denote Γ. Next, to obtain the boundary condition of the interfaces Γ on $\partial\Omega$, the tangent unit vector of the curve $\hat{\gamma}$ denotes by $T_a(\hat{\gamma})$. If the boundary condition of (1) at $x = L$ is Neumann type, then we assume that

$$T_a(\Gamma) \perp T_a(\partial\Omega_{L/2}) \quad \text{at } x = L. \tag{11}$$

Define the curve corresponding to the interface Γ as $y = \omega(x, t) = h(x) - Vt$ with a constant velocity V. Then, the boundary conditions may be given by

$$\frac{dh}{dx}(0) = \tan\alpha, \quad \frac{dh}{dx}(L) = 0 \tag{12}$$

where α is an unknown constant.

Since $\hat{V}$ is the velocity of the normal direction to the front interface, V satisfies

$$\begin{aligned} V &= \sqrt{1 + h'(x)^2}\hat{V} \\ &= \sqrt{1 + h'(x)^2}\left\{\varepsilon\kappa + \frac{k(e^{-k+\delta} - e^{k-\delta})}{2\sqrt{\kappa^2 + 4\gamma}}\right\} \end{aligned}$$

where $\kappa = \dfrac{-h''(x)}{(1 + h'(x)^2)^{\frac{3}{2}}}$.

Here, we assume that k is large and κ is small. Then it follows from (7) that

$$\delta \cong \frac{2c^*}{k} + O\left(\left(\frac{1}{k} + \kappa\right)^2\right). \tag{13}$$

Therefore, we have

$$\begin{aligned} V &\cong \sqrt{1 + h'(x)^2}\kappa \left(\varepsilon + \frac{c^*}{\sqrt{\kappa^2 + 4\gamma}} - \frac{c^{*2}}{k}\right) \\ &\cong \sqrt{1 + h'(x)^2}\kappa \left\{\epsilon + \frac{c^*}{2\sqrt{\gamma}}\left(1 - \frac{2c^*\sqrt{\gamma}}{k}\right)\right\} \\ &= -\frac{h''(x)}{1 + h'(x)^2}\left\{\epsilon + \frac{c^*}{2\sqrt{\gamma}}\left(1 - \frac{2c^*\sqrt{\gamma}}{k}\right)\right\}. \end{aligned} \tag{14}$$

Since V is a constant, the solution $h(x)$ of (12), (14) is given by

$$h(x) = \frac{L}{\alpha}\log\left(\cos\frac{\alpha}{L}(x - L)\right) + c \tag{15}$$

with

$$V = \frac{\alpha}{L}\left\{\epsilon + \frac{c^*}{2\sqrt{\gamma}}\left(1 - \frac{2c^*\sqrt{\gamma}}{k}\right)\right\} \tag{16}$$

and any constant c.

Remark 2 *From* (16), *the velocity V decreases with respect to L and increases with respect to k if α is independent of L. Therefore, this supports the result of Figures* 4 *and* 5.

Remark 3 *It holds that*

$$\frac{\partial\kappa}{\partial L} = -\frac{\alpha}{L^2}\left\{\cos\frac{\alpha}{L}(x - L) + \alpha\sin\frac{\alpha}{L}(x - L)\right\}$$

where $\kappa = \frac{\alpha}{L}\cos\frac{\alpha}{L}(x - L)$. *Moreover,*

$$\frac{\partial\kappa}{\partial L} < 0 \quad \textit{for} \quad 0 < \alpha < \alpha^*,$$

where α^* satisfies $\tan(-\alpha^*) = -\frac{1}{\alpha^*}$ *for* $\frac{\pi}{4} < \alpha^* < \frac{\pi}{3}$, *that is, the mean curvature decreases with respect to L when α satisfies* $0 < \alpha < \alpha^*$. *It follows from* (10) *and* (13) *that* $\alpha = \frac{\pi}{6} + O((\frac{1}{k} + \kappa)^2)$ *for large k and small κ.*

2.3 Stability of the traveling solution $\omega(t,x) = h(x) - Vt$ for (14) and (12)

We note that $\omega(t,x)$ is the traveling solution of the following problem:

$$\begin{cases} w_t = \dfrac{w''}{1+w'^2}\left\{\varepsilon + \dfrac{c^*}{2\sqrt{\gamma}}\left(1 - \dfrac{2c^*\sqrt{\gamma}}{k}\right)\right\}, \ t>0, \ x \in (0,L) \\ w'(t,0) = \tan\alpha, \ w'(t,L) = 0, \ t > 0 \\ w(t,0) = w_0(x), \ x \in (0,L) \end{cases} \tag{17}$$

where $w_0(x)$ satisfies $\int_0^L (w_0 - h)\, dx = 0$.

Proposition 1 (Garcke, Nestler and Stoh [5]) *For the traveling solution $\omega(t,x)$ of* (17), *$w(t,x) - \omega(t,x)$ exponentially decays with respect to $L^2(0,L)$ as t increases. Therefore, the traveling solution $\omega(t,x)$ is stable.*

3 Concluding Remarks

In Section 2.1, we consider the motion of the interface curve of the band pattern with a constant curvature. For the general case, that is, the curvature is not constant, it is a open problem. But, there are several results for the one phase problem of Allen–Cahn equation and etc..

In Section 2.2, we assume the boundary conditions (12) of the front interface curve to study the velocity of the traveling solution. For the special case of (1), which is treated in [1], by the method shown in [9] the same boundary condition at $x = L$ will be shown. On the other hand, Bronsard and Reitich [2] considered the contact angle of the triple junction pattern for Allen–Cahn equation, Ei, Ikota and Mimura [3] for competition–diffusion system. They treated the contact angle at the meeting point of three curves. In our case, their consideration and the approach to construct the solution with the interior transition and boundary layers of Allen–Cahn equation by Owen et al. [8] are useful to obtain the contact angle. This will be our fearture work.

References

[1] A. Bonami, D. Holhorst, E. Logak and M. Mimura, *Singilar limit of a chemotaxis-growth model*, Advances in Differntial equations **6**, 1173–1212 (2001).

[2] L. Bronsard and F. Reitich, *On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation*, Arch. Rational Mech. Anal. **124**, 355–379 (1993).

[3] S.-I. Ei, R. Ikota and M. Mimura, *Segregating partition problem in competition-diffusion systems*, Interfaces and free boundaries **1**, 57–80 (1999).

[4] P. C. Fife and J. B. McLeod, *The approach of the solutions of nonlinear diffuion equations to travelling front solutions*, Arch. Rational Mech. Anal. **65**, 335–361 (1977).

[5] H. Garcke, B. Nestler and B. Stoth, *A multiphase field concept: Numerical simulations of moving phase boundaries and multiple junctions*, SIAM J. Appl. Math. **60**, 295–315 (1999).

[6] A. S. Mikhailov, *Foundation of synegrgetics I. Distributed active systems*, (Springer-Verlag, Belrin, 1990).

[7] M. Mimura and T. Tsujikawa, *Aggregating pattern dynamics in a chemotaxis model including growth*, Physica A **230**, 499–543 (1996).

[8] N. C. Owen, J. Rubinstein, P. Sternberg, *Minimizers and gradient flows for singularly perturbed bi-stable potentials with a Dirichlet condition*, Proc. R. Soc. Lond. A **429**, 505–532 (1990).

[9] J. Rubinstein, P. Sternberg and J. Keller, *Fast reaction, slow diffusion, and curve shortening*, SIAM J. Appl. Math. **49**, 116–133 (1989).

[10] T. Tsujikawa, *Singular limit analysis of planar equilibrium solutions to a chemotaxis model equation with growth*, Method and Applications of Analysis **3**, 401–431 (1996).

[11] V. S. Zykov, *Simulation of wave processes in excitable media*, (Manchester University Press, Manchester,1987).

[12] V. S. Zykov, A. S. Mikhailov and S. C. Müller, *Wave instabilities in excitable media with fast inhibitor diffusion*, Phys. Rev. Lett. **81**, 2811–2814 (1998).

Contour line (u(t, x) = a) of the solution

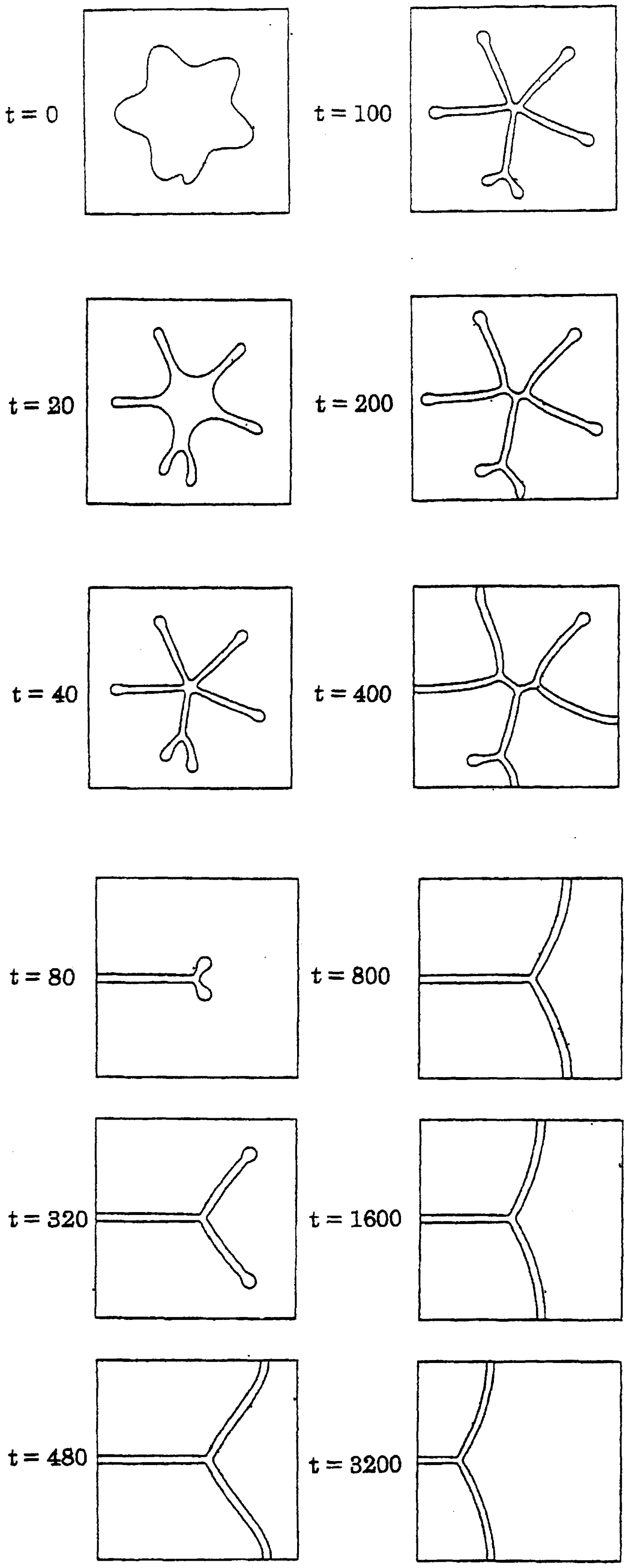


Figure 1

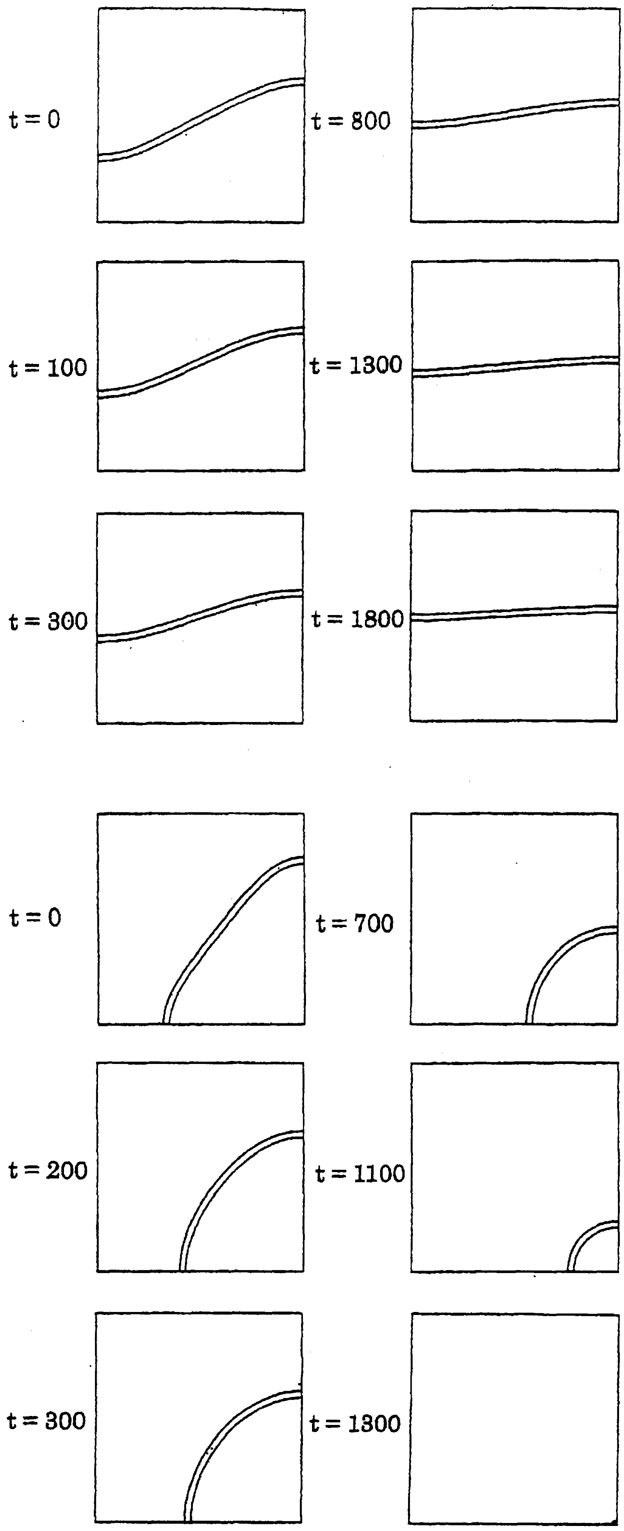


Figure 2

Profiles of u depending on domain size and intensity of chemotaxis effect k

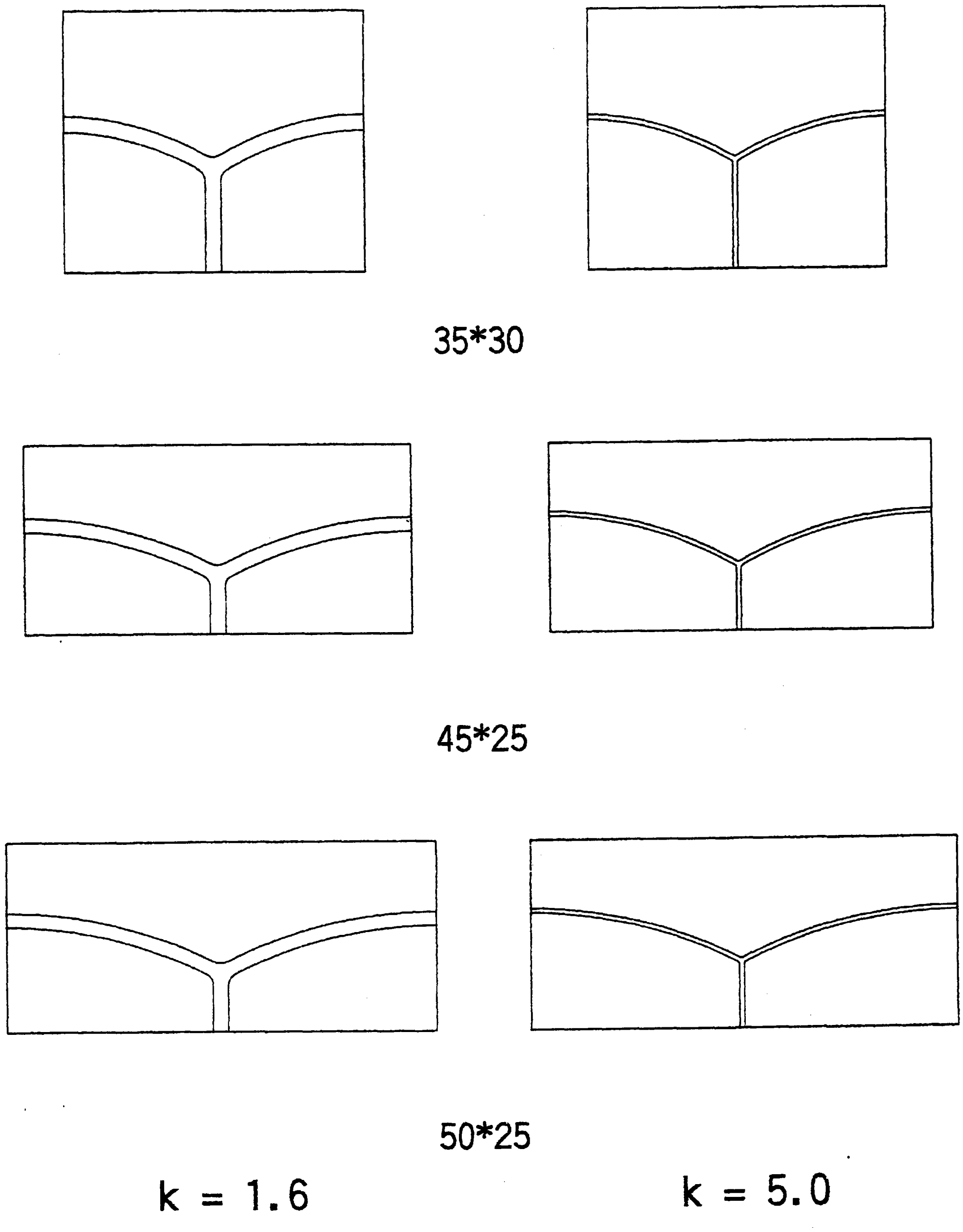
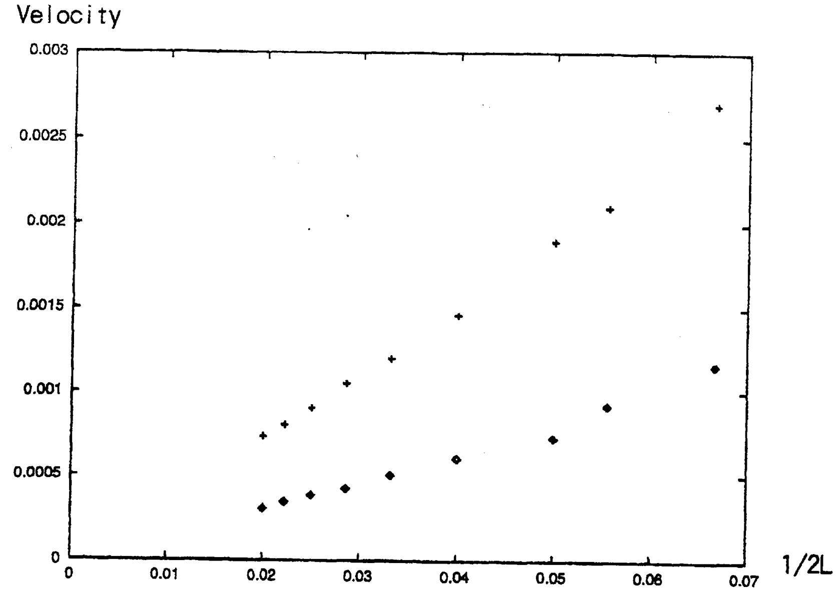


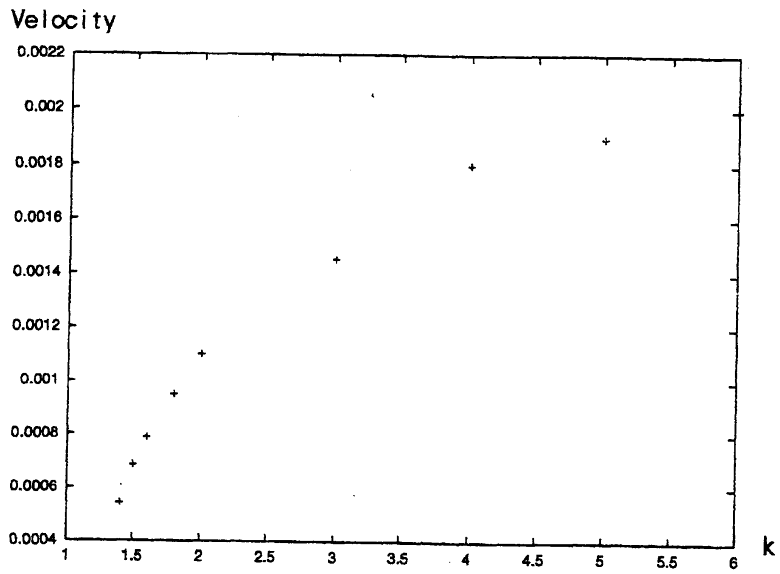
Figure 3



Velocity of traveling solution with triple junction

$X(v) = v$, $k = 1.6$, $k = 5.0$, $\varepsilon = 0.05$

Figure 4



Velocity of traveling solutions with triple junction

$X(v) = v$, $k = 5.0$, $\varepsilon = 0.05$

Figure 5