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Kyoto University
Appearance of singularities for a system of nonlinear wave equations with different speeds

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1. INTRODUCTION

We consider the following system of wave equations

\[
\begin{align*}
\Box_{c_1} u &= f(u, v) \\
\Box_{c_2} v &= g(u, v)
\end{align*}
\]

where \(\Box_c = \frac{1}{c^2} \partial^2 / \partial t^2 - \sum_{j=1}^{n} \partial^2 / \partial x_j^2\) and \(c_1\) and \(c_2\) are positive constants. We assume that \(f(\cdot, \cdot)\) and \(g(\cdot, \cdot)\) are in \(C^\infty\). In what follows, we shall study the singularities of the solutions to (1.1) when the solutions are 'conormal distributions' to some hyperplanes.

In 1979, J. Rauch[5] has shown that \(H^s\)-singularities with \(n/2 < s < 2s - 2/n\) for semilinear wave equations propagate along the null bicharacteristic curves for the linear part of the equation. Rauch-Reed[4] has given an example which illustrates the occurrence of new singularities by nonlinear interaction. J. M. Bony[1] and Melrose-Ritter[3] has shown that no new singularities occur except on the light cone from the interaction point if singularities are conormal.

In this section, we introduce the result of the author's paper[2], which shows the same result as J. M. Bony[1] and Melrose-Ritter[3] for a system of wave equations with different speeds. In next section, we give an example which illustrates the appearance of new singularities.

Definition 1 (Conormal distributions). Let \(\Omega\) be a domain in \(\mathbb{R}^n\). Let \(L\) be a \(C^\infty\)-manifold in \(\Omega\). We call that \(u\) is in \(H^s(L, \infty)\) in \(\Omega\) if

\[M_1 \circ M_2 \circ \cdots \circ M_l u \in H^s_{\text{loc}}(\Omega) \quad \text{for } l = 0, 1, 2, \ldots ,\]

where each \(M_j\) is a \(C^\infty\) vector field which is tangent to \(L\).

We can define the space of conormal distributions not only for a \(C^\infty\)-manifold but also for a union of two hypersurfaces which intersect each other transversally.

Now we shall state the main results. Let \(\omega \in S^{n-1}\) and \(L_{ij} = \{(t, x) \in \mathbb{R}^n; c_i t + (-1)^j \omega \cdot x = 0\}\) for \(i, j = 1, 2\).
Theorem 1.1. Let $\Omega$ be a neighborhood of the origin of $\mathbb{R}^{n+1}$, $i = 1$ or $2$ and $j = 1$ or $2$. Suppose that $u, v$ are in $H^s_{\text{loc}}(\Omega)$ for $s > (n+1)/2$, $u$ and $v$ are solutions to (1.1) and

$$u, v \in H^s(L_{ij}, \infty) \text{ in } \Omega \cap \{t < 0\},$$

then

$$u, v \in H^s(L_{ij}, \infty) \text{ in } K$$

where $K$ is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 1.2. Let $\Omega$ be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and $i, i', j, j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, $u, v$ are in $H^s_{\text{loc}}(\Omega)$ for $s > (n+1)/2$, $u$ and $v$ are solutions to (1.1) and

$$u, v \in H^s(L_{ij} \cup L_{i'j'}, \infty) \text{ in } \Omega \cap \{t < 0\},$$

then

$$u, v \in H^s(L_{ij} \cup L_{i'j'} \cup L_{ij'}, \infty) \text{ in } K$$

where $K$ is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 1.3. Let $\Omega$ be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and $i, i', j, j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, $u, v$ are in $H^s_{\text{loc}}(\Omega)$ for $s > (n+1)/2$, $u$ and $v$ are solutions to (1.1) and

$$u, v \in H^s(L_{ij} \cup L_{ij'}, \infty) \text{ in } \Omega \cap \{t < 0\},$$

then

$$u, v \in H^s(L_{ij} \cup L_{ij'} \cup L_{i'j'}, \infty) \text{ in } K$$

where $K$ is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

2. Appearance of singularities

In this section, we give an example which shows the appearance of new singularities by nonlinear interaction for the following system of nonlinear wave equations with different speeds;

$$\begin{cases}
\square_{c_1} u = f(u, v), & \text{in } \Omega, \\
\square_{c_2} v = g(u, v), & \text{in } \Omega,
\end{cases}$$

(2.2)

where $\Omega \subset \mathbb{R} \times \mathbb{R}^2$, $u$ and $v$ are real valued functions on $\Omega$, $0 < c_1 < c_2$ and $f$ and $g$ are some functions appropriately determined later.
Let $\omega \in S^1$ and we put $u_1 = h(c_j t - \omega \cdot x)$ and $u_2 = h(c_j t + \omega \cdot x)$ where $h$ is the heaviside function defined by

\begin{equation}
(2.3)
    h(s) = \begin{cases}
        0 & \text{for } s < 0 \\
        1 & \text{for } s \geq 0.
    \end{cases}
\end{equation}

Then $u_j$ solves $\Box_{c_j} u_j = 0$ for $j = 1, 2$. We put $\chi(t, x) = u_1 u_2$ and we define $V_j$ as a solution of the equation;

\begin{equation}
(2.4)
    \begin{cases}
        \Box_{c_j} V_j = \chi(t, x), & \text{in } \Omega, \\
        V_j \equiv 0 & \text{for } t < 0.
    \end{cases}
\end{equation}

In the following, we write $\{t \geq 0\} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 | t \geq 0\}$ for abbreviation.

**Proposition 2.1.** For any $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, we have $V_1(t, x), V_2(t, x) \geq 0$ and

\begin{equation}
(2.5)
    \text{sing supp } V_1 = \{t \geq 0\} \cup \{c_1 t \pm \omega \cdot x = 0\} \cup \{c_2 t + \omega \cdot x = 0\}
\end{equation}

\begin{equation}
(2.6)
    \text{sing supp } V_2 = \{t \geq 0\} \cup \{c_2 t \pm \omega \cdot x = 0\} \cup \{c_1 t - \omega \cdot x = 0\}
\end{equation}

We construct an example of appearance of new singularities before the proof of this proposition to the end of our notes.

Let $F(s)$ be a $C^{\infty}$ function defined as

\begin{equation}
(2.7)
    F(s) = \begin{cases}
        0 & \text{for } s \leq 3/2 \\
        1 & \text{for } s \geq 2,
    \end{cases}
\end{equation}

and we put $u = u_1 + V_1$ and $v = u_2 + V_2$. Since $V_j$ for $j = 1, 2$ is a positive function and $\text{supp} V_j = \{-c_2 t \leq \omega \cdot x \leq c_2 t, t \geq 0\}$ for $j = 1, 2$, $u$ and $v$ solve

\begin{equation}
(2.8)
    \begin{cases}
        \Box_{c_1} u = \Box V_1 = \chi(t, x) = F(u + v), \\
        \Box_{c_2} v = \Box V_2 = \chi(t, x) = F(u + v),
    \end{cases}
\end{equation}

for $t < \delta$ with $\delta > 0$ sufficiently small and $u (v)$ has new singularities on $\{c_1 t + \omega \cdot x = 0\}$ ($\{c_2 t + \omega \cdot x = 0\}$) respectively.

**Proof.** We only prove the proposition for $V_1$. We note that $V_1$ is expressed as follows:

\[
V_1(t, x) = C \int_{C_{t,x}} \frac{\chi(t', x')}{\sqrt{c_1^2 (t - t')^2 - |x - x'|^2}} dx' dt',
\]
where $C(t,x)$ is backward light cone for $\square c_1$ starting from $(t, x)$. Evidently $V_1$ is a positive function. It is easy to see that $\text{sing supp } V_1$ includes $\{c_2t + \omega \cdot x = 0\}$ and $\{c_1t - \omega \cdot x = 0\}$. So we prove that $\text{sing supp } V_1$ includes $\{c_1t + \omega \cdot x = 0\}$.

We put $V_1 = w_1 + w_2$, where

\[
w_1 = C \int_{C_{(\iota,x)}} \frac{u_2(t', x')}{\sqrt{c_1^2(t-t')^2 - |x-x'|^2}} dx' dt',
\]

\[
w_2 = C \int_{C_{(\iota,x)}} \frac{\chi(t', x') - u_2(t', x')}{\sqrt{c_1^2(t-t)^2 - |x-x|^2}} dx' dt'.
\]

Since $\text{sing supp } w_1(t,x) = \{c_2t + \omega \cdot x = 0\}$, it suffices to show that $\text{sing supp } w_2$ includes $\{c_1t + \omega \cdot x = 0\}$. By using the same argument as in the paper of J. Rauch and M. Reed[4], we have $w_2$ cannot be twice differentiated on $\{c_1t + \omega \cdot x = 0\}$.

\[\square\]

REFERENCES


