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Flows on C^*-algebras

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This is considered as the subject started by S. Sakai et al. some 30 years ago as the theory of unbounded derivations, after completion of the theory of bounded derivations. (Unbounded, or bounded, derivations include the generators of flows. My understanding is that the purpose was to gain insights into dynamics/mechanics of nature.) See [8, 9, 3, 44] for the motivations and developments made after initial impetus. This is a sort of head-on assault on the subject and I find this approach still too difficult.

Thus we are taking an easy approach through, say, back gates, where it looks we could set up numerous traps without drawing excessive reproaches.

This is a survey article on what I have been doing on flows. I suppose I made many attempts, each short-lived, to try to understand some aspects of flows and wrote in general a paper for each with whatever I got. For clarifying motivations and presentations, I also include some other results.

Contents

1 Semi-flows on Banach spaces

We mean by a semi-flow on a Banach space A a semi-group homomorphism α of $[0,\infty)$ into the bounded operators $\mathcal{B}(A)$ such that $\alpha_0 = 1$, and $\alpha_s(x) \to x$ as $s \to 0$. The generator $\delta = \delta_\alpha$ of α is defined by

$$\delta(x) = \lim_{s\to 0} \frac{\alpha_s(x) - x}{s}$$

for x in A such that the right hand limit exists. The set of such x, the domain $\mathcal{D}(\delta)$ of δ, is a dense linear subspace and δ is a closed linear operator from $\mathcal{D}(\delta)$ into A. We will call α a contraction semi-flow if $\|\alpha_s\| \leq 1$ furthermore. In this case the generator δ is dissipative; i.e., if $\|(1-\delta)(x)\| \geq \|x\|$ for any $x \in \mathcal{D}(\delta)$.

Theorem 1.1 *Let A be a Banach space and δ a linear operator in A such that the domain $\mathcal{D}(\delta)$ is dense. Then δ generates a contraction semi-flow on A if and only if δ is dissipative and the range $\mathcal{R}(1-\delta)$ of $1-\delta$ equals A.*

Proof. See, e.g., [46, 8].

Theorem 1.2 *Let (α_n) be a sequence of contraction semi-flows on a Banach space and α be a contraction semi-flow on A. Then the following conditions are equivalent:*

1. *(α_n) converges strongly to α, i.e., $\|\alpha_{n,t}(x) - \alpha(x)\| \to 0$ uniformly in t on every compact for any $x \in A$ as $n \to \infty$.*
2. *(δ_n) converges to δ_α in the graph sense, i.e., For any $x, y \in A$ it follows that $x \in \mathcal{D}(\delta_\alpha)$ and $y = \delta_\alpha(x)$ if and only if there is a sequence (x_n) in A such that $x_n \in \mathcal{D}(\delta_n)$, $\|x_n - x\| \to 0$, and $\|\delta_n(x_n) - y\| \to 0$, where $\delta_n = \delta_{\alpha_n}$.*

Proof. See, e.g., [46, 8].

2 Flows on C^*-algebras

We mean by a flow on a C^*-algebra A a *one-parameter automorphism group* of A. We always assume that a flow is strongly continuous; if α is a flow on A, then $\alpha_t(x)\rightarrow x$ in norm as $t\rightarrow 0$ for all $x \in A$. The domain $\mathcal{D}(\delta_\alpha)$ is a dense *-subalgebra of A and δ_α is a derivation from $\mathcal{D}(\delta_\alpha)$ into A, i.e., δ_α satisfies:

$$\delta_\alpha(xy) = \delta_\alpha(x)y + x\delta_\alpha(y), \quad x, y \in \mathcal{D}(\delta_\alpha)$$

and

$$\delta_\alpha(x)^* = \delta_\alpha(x^*), \quad x \in \mathcal{D}(\delta_\alpha).$$

If δ is a derivation defined everywhere on A, then it is known that δ is automatically bounded. If A is assumed to be unital and simple, then there is an $h \in A_{sa}$ such that $\delta = \operatorname{ad} ih$; such a derivation is called *inner* [45]. If A is simple but does not have a unit, there is an $h = h^*$ in the multiplier algebra of A such that $\delta = \operatorname{ad} ih$.

We call a flow α *uniformly continuous* if $\|\alpha_t - 1\|\rightarrow 0$ as $t\rightarrow 0$. If α is a uniformly continuous flow on a unital simple C^*-algebra, then δ_α is defined everywhere and hence is inner. Thus α is also inner in the sense that $\alpha_t = \operatorname{Ad} e^{iht}$ for some $h \in A_{sa}$.

Theorem 2.1 [45, 44] *Let δ be a densely-defined linear operator in the C^*-algebra A. Then δ generates a uniformly continuous flow if and only if δ is a derivation with $\mathcal{D}(\delta) = A$.*

We recall that B is a hereditary C^*-subalgebra of A if B is a C^*-subalgebra of A and $BAB \subset B$.

Definition 2.2 [22] *A flow α on a C^*-algebra A is said to be* almost uniformly continuous *if for any α-invariant closed ideal I of A the induced flow $\dot\alpha$ on the quotient A/I has a non-zero α-invariant hereditary C^*-subalgebra B such that $\dot\alpha|B$ is uniformly continuous.*

Definition 2.3 *A flow α on a C^*-algebra A is said to be* universally weakly inner *if there is a unitary flow U in the second dual A^{**} such that $\alpha_t(x) = U_t x U_t^*, \ x \in A, \ t \in \mathbf{R}$.*

Theorem 2.4 *Let α be a flow on a C^*-algebra A. Then the following conditions are equivalent:*

1. *α^* is strongly continuous on A^*.*
2. *For any pure state φ of A, $\|\varphi\alpha_t - \varphi\|\rightarrow 0$ as $t\rightarrow 0$.*
3. *α is almost uniformly continuous.*
4. *α is universally weakly inner.*
5. *There exists a net (h_ν) in A_{sa} such that $\|\operatorname{Ad} e^{ith_\nu}(x) - \alpha_t(x)\|\rightarrow 0$ and e^{ith_ν} converges to a unitary flow U_t in the weak* topology (and hence $\alpha_t(x) = U_t x U_t^*, \ x \in A$).*

Proof. See [22] for the equivalences between (1) to (4). See [13, 10] for the equivalence of (4) and (5).

Let A be a simple C^*-algebra and suppose that α^* is strongly continuous on A^*. If A is unital, then α is uniformly continuous and hence is inner. If A is not unital, then there is a unitary flow u in the multiplier algebra of A, continuous in the strict topology, such that $\alpha_t = \mathrm{Ad}\, u_t$.

Definition 2.5 *Let α be a flow on a C^*-algebra A. If there is a sequence (h_n) in A_{sa} such that $(\mathrm{Ad}\, e^{ith_n})$ converges strongly to α, i.e., $\mathrm{Ad}\, e^{ith_n}(x)$ converges to $\alpha_t(x)$ uniformly in t on every compact subset of $\mathbf{R}$ for any $x \in A$, α is called to be* approximately inner.

Any uniformly continuous flow is approximately inner. We may ask a question if there is an approximately inner flow which is not uniformly continuous. For that purpose we define a property which is not shared by uniformly continuous flows but is possessed by many examples.

Definition 2.6 *Let α be a flow on a C^*-algebra A. We say that α is* profound *if for any non-empty open subset O of $\mathbf{R}$ there exists a bounded sequence (z_n) in $A^\alpha(O)$ such that $\|[x, z_n]\|$ converges to 0 and $\lim_n \|xz_n\| = 0$ entails $x = 0$ for any $x \in A$. Here $A^\alpha(O)$ is the closure of the set of elements of the form $\int f(t)\alpha_t(x)dt$, where $x \in A$ and f is a continuous integrable function on $\mathbf{R}$ such that its Fourier transform of f has support in $-O$.*

In particular a profound flow has full spectrum and so is not uniformly continuous. It also follows that any cocycle perturbation of a profound flow is profound.

Theorem 2.7 *Let A be a separable antiliminal C^*-algebra. Then there exists an approximately inner profound flow on A.*

Proof. Since A is antiliminal and separable, there exists a (at most) countable family $\{\pi_i\}$ of irreducible representations of A such that $\pi_i(A) \cap \mathcal{K}(\mathcal{H}_{\pi_i}) = \{0\}$ and $\bigcap_i \mathrm{Ker}(\pi_i) = \{0\}$. By using this fact we can argue as in [32].

Corollary 2.8 *Let A be a separable antiliminal C^*-algebra. Let π_1 and π_2 be irreducible representations such that $\mathrm{Ker}\,\pi_1 = \mathrm{Ker}\,\pi_2$. Then there is a flow α such that $\pi_1\alpha_1$ is equivalent to π_2.*

Proof. If π_1 and π_2 are equivalent, there we may take the trivial flow id for α.

Suppose that π_1 and π_2 are disjoint. We find a profound flow α on A by the previous theorem. Since such a flow cannot be almost uniformly continuous, there is an irreducible representation π of A such that $\mathrm{Ker}\,\pi = \mathrm{Ker}\,\pi_1$ and $\pi\alpha_1$ is disjoint from π. By a stronger version of [39] (as in [16]) we have an approximately inner automorphism γ of A such that

$\pi\alpha_1\gamma$ is equivalent to π_2 and $\pi\gamma$ is to π_1. Set $\beta = \gamma^{-1}\alpha\gamma$. Then the flow β satisfies the required condition.

Without knowing the global structure of the C^*-algebra A it seems hard if not impossible to construct a flow which is not approximately inner. But for many examples of C^*-algebras we can construct such a flow.

For an approximately inner flow α there is a sequence (h_n) in A_{sa} such that $\alpha_t = \lim \mathrm{Ad}\, e^{ith_n}$; but there seems to be no canonical way to choose such (h_n). The following result is not entirely trivial (compare it with (4)$\Leftrightarrow$(5) of 2.4).

Proposition 2.9 [32] *Let A be a separable C^*-algebra and α an approximately inner flow on A. Let π be an α-covariant type I representation on a separable Hilbert space $\mathcal{H}$ such that there is a unitary flow U in $\pi(A)''$ which implements α. Then there exists a sequence (h_n) of self-adjoint elements of A such that*

$$\lim_{n\to\infty} \mathrm{Ad}\, e^{ith_n}(x) = \alpha_t(x), \quad x \in A,$$
$$\lim_{n\to\infty} \pi(e^{ith_n}) = U_t, \text{ strongly},$$

both uniformly in t on every compact subset of $\mathbf{R}$.

3 Inductive limit C^*-algebras

We sometimes consider examples of C^*-algebras, which are all obtained as inductive limit C^*-algebras and also give abundance of examples of flows. We sketch these examples.

3.1 UHF and AF C^*-algebras

UHF (uniformly hyper-finite) C^*-algebras are introduced by Glimm and AF (approximately finite dimensional) by Bratteli. A C^*-algebra is UHF if it is obtained as the inductive limit of full matrix algebras with unital homomorphisms. A C^*-algebra is AF if it is obtained as the inductive limit of finite-dimensional C^*-algebras. These algebras are classified in terms of dimension groups (or ordered K_0 groups) and look rather (technically) simple C^*-algebras but yet it seems extremely if not most difficult to get useful knowledge on flows on them. The original motivation for studying flows far from inner concerns these C^*-algebra, mainly because these C^*-algebras are the ones we often encounter in statistical mechanics. See [44].

Besides flows (i.e., time developments) coming from statistical mechanical models, there are what we will call UHF flows (on UHF C^*-algebras) and AF flows (on AF C^*-algebras), which will be defined later. (AF flows are essentially flows generated by commutative derivations in Sakai's terminology [44].) Noteworthy are quasi-free flows on the CAR algebra, which is the UHF C^*-algebra of type 2^∞, whose position is still unclear among the approximately inner flows.

3.2 Simple AT C^*-algebras of real rank zero

This class is introduced by Elliott [14] and is now only a small class among classifiable classes of stably finite C^*-algebras.

A C^*-algebra is AT if it is obtained as the inductive limit of tensor products of $C(\mathbf{T})$ and finite-dimensional C^*-algebras. AT C^*-algebras can have non-trivial K_1 contrary to the AF case above; K_1 can be an arbitrary torsion-free countable abelian group while K_0 is still a dimension group. A unital C^*-algebra A has real rank zero if any self-adjoint element of A can be approximated by self-adjoint elements of finite spectra; in particular A has so many projections that they can separate tracial states. AT C^*-algebras of real rank zero can be classified in terms of K theoretic data. This class includes the above AF C^*-algebras and all simple non-commutative tori ([15, 26] and Phillips) and allows much more wilder flows such as Rohlin flows.

An n-dimensional non-commutative torus A is generated by n unitaries $u_1, \ldots, u_n$ satisfying $u_i u_j u_i^* u_j^* \in \mathbf{C}1$ and has a natural action of the n-dimensional torus $\mathbf{T}^n$. Any one-parameter subgroup of $\mathbf{T}^n$ defines a flow on A.

3.3 Separable nuclear purely infinite simple C^*-algebras with UCT

This class is classified by Kirchberg and Phillips [19, 20] in terms of K theoretic data, K_0 and K_1 as abelian groups. If A is such a C^*-algebra with a unit, then for any non-zero $x \in A$ there are $y, z \in A$ such that $yxz = 1$. And A has real rank zero. By using their result a simple C^*-algebra is in this class if it is obtained as the inductive limit of finite direct sums of tensor products of $C(\mathbf{T})$ and a corner of a Cuntz algebra [12]. Possibly the flows in this class would be the easiest to handle.

The Cuntz algebra $\mathcal{O}_n$ belongs to this class. If $n < \infty$, then $\mathcal{O}_n$ is generated by n isometries $s_1, \ldots, s_n$ satisfying $\sum_{k=1}^n s_k s_k^* = 1$. The unitary group $U(n)$ acts on $\mathcal{O}_n$ by automorphisms and this gives many examples of flows (see [21, 17]).

4 Cocycle perturbations

Let A be a unital C^*-algebra and α a flow on A. If $h \in A_{sa}$, then $\operatorname{ad} ih : x \mapsto i[h, x]$ is an inner derivation. It follows that $\delta_\alpha + \operatorname{ad} ih$ generates a flow on A, which we call an *inner perturbation* of α and denotes by $\alpha^{(h)}$.

Definition 4.1 *Let α be a flow on a C^*-algebra A. A continuous function u on $\mathbf{R}$ into the unitary group $\mathcal{U}(A)$ of A is said to be an α-*cocycle *if $u_s \alpha_s(u_t) = u_{s+t}$ for $s, t \in \mathbf{R}$. Then $t \mapsto \operatorname{Ad} u_t \alpha_t$ is again a flow and is called a* cocycle perturbation *of α.*

Note that cocycle perturbations are more general than inner perturbations, but only slightly, see below.

Definition 4.2 *Let α and β be flows on A. α is an* approximate cocycle perturbation *of β if there is a sequence (u_n) of β-cocycles such that $\mathrm{Ad}\, u_n\beta$ converges strongly to α.*

If the cocycle u is differentiable with $ih = du_t/dt|_{t=0}$, then the generator of the cocycle perturbation is given by $\delta_\alpha + \mathrm{ad}\, ih$.

Proposition 4.3 [28] *Let u be an α-cocycle and $\epsilon > 0$. Then there is a differentiable α-cocycle w and $v \in \mathcal{U}(A)$ such that $\|v-1\| < \epsilon$ and $u_t = vw_t\alpha_t(v)^*$. Thus if $ih = dw_t/dt|_{t=0}$, then $\mathrm{Ad}\, u_t\, \alpha_t = \mathrm{Ad}\, v\alpha_t^{(h)}\mathrm{Ad}\, v^*$.*

Proof. We use the 2 by 2 trick devised by Connes. We define a flow γ on $A \otimes M_2$ by

$$\gamma_t \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \alpha_t(x_{11}) & \alpha_t(x_{12})u_t^* \\ u_t\alpha_t(x_{21}) & u_t\alpha_t(x_{22})u_t^* \end{pmatrix}.$$

Note that $\gamma_t(1 \otimes e_{21}) = u_t \otimes e_{21}$, where (e_{ij}) are the matrix units for M_2. Since $\mathcal{D}(\delta_\gamma)$ is dense and $\gamma_t(1 \otimes e_{ii}) = 1 \otimes e_{ii}$, we have a $x \in \mathcal{D}(\delta_\gamma)$ such that $\|x - 1 \otimes e_{21}\| < \epsilon$ and $x = w \otimes e_{21}$ for some $w \in A$. We may suppose that w is a unitary by functional calculus. Let $v_t = w^*u_t\alpha_t(w)$, which is an α-cocycle. Since $\gamma_t(x) = u_t\alpha_t(w) \otimes e_{21} = wv_t \otimes e_{21}$, $t \mapsto v_t$ is differentiable.

In the conclusion of the above proposition we could also require that $t \mapsto v_t$ is analytic.

Proposition 4.4 *Let α and β be flows on a unital separable C^*-algebra A. Then the following conditions are equivalent:*

1. *There exists a $\delta > 0$ such that $\|\alpha_t - \beta_t\| < 2$ for $t \in (-\delta, \delta)$.*
2. *α is a cocycle perturbation of β.*
3. *α is inner-conjugate to an inner perturbation of β, i.e., $\delta_\alpha = \mathrm{Ad}\, w(\delta_\beta + \mathrm{ad}\, ih)\mathrm{Ad}\, w^*$ for some $h \in A_{sa}$ and $w \in \mathcal{U}(A)$.*

Proof. That (1)$\Leftrightarrow$(2) is shown in [41]. That (2)$\Leftrightarrow$(3) follows from the previous proposition.

Definition 4.5 *Let α be a flow on a C^*-algebra A. The* cocycle conjugacy *class of α is the set of all flows given as $\phi(\mathrm{Ad}\, u\, \alpha)\phi^{-1}$, where u are α-cocycles and ϕ are automorphisms of A. Note that the cocycle conjuagcy class of α equals the set of all flows given as $\phi\alpha^{(h)}\phi^{-1}$, where $h \in A_{sa}$ and ϕ are automorphisms of A.*

One of the main purposes is to determine the cocycle conjugacy classes of flows. In the following sections we introduce several invariants which could be used for this purpose.

5 Invariants

5.1 Connes Spectra

There is a notion called Arveson spectrum (or simply spectrum) for a flow (which is just a closed subset of $\mathbf{R}$ containing 0, symmetric under $t \mapsto -t$); we denote by $\mathrm{Sp}(\alpha)$ the spectrum of a flow α. This is defined as follows: For a closed subset F of $\mathbf{R}$ let $A^\alpha(F)$ be the subset of $x \in A$ which satisfies that $\int f(t)\alpha_t(x)dt = 0$ for any f on $\mathbf{R}$ with $\mathrm{supp}(\hat{f}) \cap (-F) = \emptyset$. (Note that $A^\alpha(\mathbf{R}) = A$ and $A^\alpha(\emptyset) = \{0\}$.) The spectrum $\mathrm{Sp}(\alpha)$ is defined as the smallest F such that $A^\alpha(F) = A$.

Proposition 5.1 *Let α be a flow on A. Then α is uniformly continuous if and only if* $\mathrm{Sp}(\alpha)$ *is compact.*

The Connes spectrum may be called as *Essential* Arveson spectrum and is a closed subgroup of $\mathbf{R}$.

Definition 5.2 [42] *Let α be a flow on a C^*-algebra A. The Connes spectrum* $\mathbf{R}(\alpha)$ *of α is defined by*

$$\mathbf{R}(\alpha) = \bigcap_B \mathrm{Sp}(\alpha|B)$$

where B runs over all non-zero α-invariant hereditary C^*-subalgebras of A.

While $\mathrm{Sp}(\alpha)$ may not be invariant under cocycle perturbations of α, the Connes spectrum $\mathbf{R}(\alpha)$ is. If a flow α is profound, then it easily follows that $\mathbf{R}(\alpha) = \mathbf{R}$. For the converse we have:

Proposition 5.3 *Let A be a separable prime C^*-algebra and α a flow on A. Then the following conditions are equivalent:*

1. $\mathbf{R}(\alpha) = \mathbf{R}$ *and there is a faithful family of α-covariant irreducible representations of A.*

2. *α is profound.*

Proof. See [23, 24].

Since $\mathbf{R}(\alpha)$ is a closed subgroup of $\mathbf{R}$, there are three cases:

1. $\mathbf{R}(\alpha) = \{0\}$.

2. $\mathbf{R}(\alpha) = \lambda\mathbf{Z}$ for some $\lambda > 0$.

3. $\mathbf{R}(\alpha) = \mathbf{R}$.

If α is uniformly continuous, then $\mathbf{R}(\alpha) = \{0\}$; but the converse does not hold.

Proposition 5.4 (8.9.7 of [42]) *Let α be a flow on a unital simple C^*-algebra such that $\mathbf{R}(\alpha) = \lambda\mathbf{Z}$ for some $\lambda > 0$. Then there is a unitary $u \in A$ such that $\alpha_{t_0} = \mathrm{Ad}\, u$ with $t_0 = 2\pi/\lambda$ and $\alpha_t(u) = u$ for all t.*

But in general there may be no cocycle perturbation α' of α such that $\alpha'_{t_0} = \mathrm{id}$.

Theorem 5.5 [42] *Let α be a flow on A. Then the following conditions are equivalent:*

1. *The crossed product $A \times_\alpha \mathbf{R}$ is prime.*
2. *A is α-prime (i.e., for two non-zero α-invariant ideals I and J it follows that $I \cap J \neq \{0\}$) and $\mathbf{R}(\alpha) = \mathbf{R}$.*

Definition 5.6 *Let α be a flow on A. We define the strong spectrum $\tilde{\mathrm{Sp}}(\alpha)$ of α as the set of $p \in \mathbf{R}$ satisfying: For any closed neighborhood F of p the closed linear span of $A^\alpha(F)^* A A^\alpha(F)$ is A.*

Definition 5.7 *Let α be a flow on A. The strong Connes spectrum $\tilde{\mathbf{R}}(\alpha)$ is defined by*

$$\tilde{\mathbf{R}}(\alpha) = \bigcap_B \tilde{\mathrm{Sp}}(\alpha|B),$$

where B runs over all non-zero α-invariant hereditary C^*-subalgebras of A.

It is known that $\tilde{\mathbf{R}}(\alpha) \subset \mathbf{R}(\alpha)$, that $\tilde{\mathbf{R}}(\alpha)$ is a closed subsemigroup of $\mathbf{R}$, and that $\tilde{\mathbf{R}}(\alpha)$ is invariant under cocycle perturbations of α.

Theorem 5.8 [21] *Let α be a flow on a C^*-algebra A. Then the following conditions are equivalent:*

1. *The crossed product $A \times_\alpha \mathbf{R}$ is simple.*
2. *A is α-simple (i.e., A has no non-trivial α-invariant ideal) and $\tilde{\mathbf{R}}(\alpha) = \mathbf{R}$.*

When A has a tracial state, say τ, it is often left invariant under the flow α. Then α induces a flow $\overline{\alpha}$ on the weak closure $\pi_\tau(A)''$ and we may compute the Connes spectrum of $\overline{\alpha}$; in general $\mathbf{R}(\overline{\alpha}) \subset \mathbf{R}(\alpha)$; and hence we have another invariant $\mathbf{R}(\overline{\alpha})$.

5.2 Symmetry

For a flow α we should define a symmetry group of α as the group of automorphisms which commute with all α_t. But since what we actually consider is the set of cocycle perturbations of α rather than α itself, we introduce the following definition:

Definition 5.9 *When α is a flow on A, the symmetry group G_α of α is defined as*

$$G_\alpha = \{\gamma \in \mathrm{Aut}(A) \mid \gamma\alpha\gamma^{-1} \text{ is a cocycle perturbation}\}.$$

The topology on G_α is defined by $\gamma_n \to \gamma$ if

1. $\|\gamma_n(x) - \gamma(x)\| \to 0$ *for all $x \in A$, and*
2. *there exists α-cocycles u_n, u such that $\gamma_n\alpha_t\gamma_n^{-1} = \mathrm{Ad}\, u_n(t)\alpha_t$, $\gamma\alpha_t\gamma^{-1} = \mathrm{Ad}\, u(t)\alpha_t$, and $\|u_n(t) - u(t)\| \to 0$ uniformly in t on every compact subset of $\mathbf{R}$.*

When $\gamma \in G_\alpha$, γ extends to an automorphism of the crossed product $A \times_\alpha \mathbf{R}$ by

$$a \mapsto \gamma(a), \quad \lambda(t) \mapsto u_t\lambda(t),$$

where λ is the canonical unitary flow in the multiplier algebra of $A \times_\alpha \mathbf{R}$ and u is an α-cocycle such that $\gamma\alpha_t\gamma^{-1} = \mathrm{Ad}\, u_t\alpha_t$. If A is simple (or has trivial center), then the extension is unique up to dual automorphisms.

Definition 5.10 *When α is a flow, the core symmetry group $G_{\alpha 0}$ of α is defined as the group of automorphisms γ which satisfy: There exists a continuous map $v : [0,\infty) \to \mathcal{U}(A)$ such that $\gamma = \lim_{s\to\infty} \mathrm{Ad}\, v_s$ and $\lim_{s\to\infty} v_s\alpha_t(v_s^*)$ exists uniformly in t on every compact subset and defines an α-cocycle u such that $\gamma\alpha_t\gamma^{-1} = \mathrm{Ad}\, u_t\, \alpha_t$.*

It follows that each $\gamma \in G_{\alpha 0}$ extends to an asymptotically inner automorphism of the crossed product $A \times_\alpha \mathbf{R}$.

Theorem 5.11 [37] *Let α be a flow on a separable antiliminal simple C^*-algebra A. Let (π_1, U_1) and (π_2, U_2) be representations of (A, α) such that π_1 and π_2 are irreducible, the Connes spectrum of the flow α is non-zero, and $\mathrm{Ker}(\pi_1 \times U_1) = \mathrm{Ker}(\pi_2 \times U_2)$. Then there exists a $\gamma \in G_{\alpha 0}$ such that $\pi_1\gamma$ is equivalent to π_2.*

The condition above in terms of Connes spectrum is made to ensure that the crossed product $A \times_\alpha \mathbf{R}$ is antiliminal.

The proof of this theorem uses techniques from [39], where it is shown that the pure state space of a separable simple C^*-algebra is homogeneous under the action of asymptotically inner automorphisms.

5.3 Orbits in the spectrum

Let α be a flow on a C^*-algebra A and let $\hat{A}$ be the set of equivalence class of irreducible representations of A. Then α acts on $\hat{A}$ by $\alpha_t^*\pi = \pi\alpha_t$. We define a representation $\overline{\pi}$ by

$$\overline{\pi} = \int^{\oplus} \pi\alpha_t dt$$

on $L^2(\mathbf{R}, \mathcal{H}_\pi)$. Define a unitary flow U by

$$(U_t\xi)(s) = \xi(t+s), \quad \xi \in L^2(\mathbf{R}, \mathcal{H}_\pi).$$

Then it follows that $\mathrm{Ad}\, U_t\overline{\pi}(x) = \overline{\pi}\alpha_t(x)$, i.e., $(\overline{\pi}, U)$ is a covariant representation of (A, α). Since $\mathrm{Ad}\, U$ acts on the center of $M = \overline{\pi}(A)''$ ergodically, M is homogeneous in the sense that M is not isomorphic to the direct sum of two non-isomorphic von Neumann algebras. Hence for such von Neumann algebras M_1 and M_2, it follows that they are either isomorphic or do not have isomorphic direct summands. We define the *type* of such a von Neumann algebra M as the set of von Neumann algebras isomorphic to M. Note that for example M is either type I, type II, or type III$_\lambda$, with $\lambda \in [0, 1]$.

Definition 5.12 [23] *Let α be a flow on A and $\pi \in \hat{A}$. The type of the orbit $\{\alpha_t^*\pi \mid t \in \mathbf{R}\}$ is the type of the von Neumann algebra $\overline{\pi}(A)''$, where $\overline{\pi} = \int^\oplus \pi\alpha_t dt$ is a representation on $L^2(\mathbf{R}, \mathcal{H}_\pi)$.*

Theorem 5.13 [23, 24] *Let α be a flow on a separable simple C^*-algebra such that the Connes spectrum of α is full. Then the following conditions are equivalent:*

1. *(A, α) has an covariant irreducible representation.*
2. *(A, α) has an anti-covariant irreducible representation π in the sense that $\overline{\pi} = \int^\oplus \pi\alpha_t dt$ is the central decomposition of $\overline{\pi}$.*

Theorem 5.14 [31] *Let α be a flow on a separable simple C^*-algebra A such that the Connes spectrum of α is non-trivial. Then if the flow α^* on $\hat{A}$ has a type I orbit, then it has orbits of type II_∞, type III_λ, $\lambda \in [0, 1]$.*

In the proof of the above theorem we use the following rèsult, which is a Glimm's type result for (A, α). This result gives representations of (A, α) through those of a very special flow on a UHF C^*-algebra.

Theorem 5.15 [31] *Let A be a separable prime C^*-algebra and let α be a flow on A with $\mathbf{R}(\alpha) \neq (0)$. Then the following conditions are equivalent:*

1. *There exists a faithful family of α-covariant irreducible representations of A.*
2. *There exists a faithful α-covariant irreducible representation of A which induces a representation of the crossed product $A \times_\alpha \mathbf{R}$ (on the same Hilbert space), whose kernel is left invariant under $\hat{\alpha}|\mathbf{R}(\alpha)$.*
3. *For any UHF C^*-algebra D and any UHF flow γ on D (i.e., $\gamma_t = \otimes_{n=1}^\infty \mathrm{Ad}\, e^{ith_n}$ on $D = \otimes_{n=1}^\infty M_{k_n}$ with $h_n = h_n^* \in M_{k_n}$) such that $\mathrm{Sp}(\gamma) \subset \mathbf{R}(\alpha)$, and any $\epsilon > 0$, there*

is a C^*-subalgebra B of A, an $h = h^* \in A$, and a closed projection q of A^{**} such that

$$\begin{aligned} \|h\| &< \epsilon, \\ \alpha_t^{(h)}(B) &= B, \\ (\alpha_t^{(h)})^{**}(q) &= q, \\ qAq &= Bq, \\ (Bq, (\alpha^{(h)})^{**}|Bq) &\cong (D, \gamma), \end{aligned}$$

where $(\alpha^{(h)})_t^{**} = (\alpha_t^{(h)})^{**}$ on A^{**}, and if $c(q)$ denotes the central support of q in A^{**}, $x = 0$ iff $xc(q) = 0$ for any $x \in A$.

Since $\mathbf{R}(\alpha) \neq (0)$, then A is automatically antiliminal (i.e., it has no abelian hereditary C^*-subalgebra), which was the standing assumption for the Glimm's theorem.

Corollary 5.16 *Let A be a separable prime C^*-algebra and let α be a flow on A. Then there is an α-covariant representation π of A such that the flow on the weak closure $\pi(A)''$ induced by α has $\mathbf{R}(\alpha)$ as the Connes spectrum.*

Proof. If $\mathbf{R}(\alpha) = \{0\}$, then there is nothing to prove. If $\mathbf{R}(\alpha) \neq \{0\}$, then we apply the previous theorem. Let γ be a UHF flow on a UHF C^*-algebra D such that $\mathrm{Sp}(\gamma) = \mathbf{R}(\gamma) = \mathbf{R}(\alpha)$ and the flow on $\pi_\tau(D)''$ induced by γ has $\mathbf{R}(\alpha)$ as the Connes spectrum, where τ is the tracial state on D. By the above theorem we find a covariant representation π of A by extending π_τ on $D \cong qAq$ in the notation there. We then check that the Connes spectrum of the induced flow on $\pi(A)''$ is the same as the Connes spectrum of an inner perturbation of it on $\pi(q)\pi(A)''\pi(q)$, which is $\mathbf{R}(\alpha)$.

5.4 Domains

Let α be a flow on a C^*-algebra A and let δ_α denote the generator of α. The domain $\mathcal{D}(\delta_\alpha)$ is a Banach *-algebra with the norm defined by

$$\|x\| = \| \begin{pmatrix} x & \delta_\alpha(x) \\ 0 & x \end{pmatrix} \|.$$

See [44] for more on domains and related topics. The domain as a Banach *-algebra is apparently an invariant for cocycle-conjugacy class. In many cases the domain actually determines the generator up to inner perturbations and constant multiples.

Theorem 5.17 *Let A be a separable prime C^*-algebra and let α be a flow on A such that $\mathbf{R}(\alpha) \neq \{0\}$. Suppose that there is an α-covariant faithful irreducible representation of A. Let δ be a derivation defined on $\mathcal{D}(\delta_\alpha)$. Then there is a constant $\lambda \in \mathbf{R}$ and a bounded derivation d on A such that $\delta = \lambda\delta_\alpha + d$. In particular if A is simple, then δ generates either a flow which is an inner perturbation of a re-scaled α (i.e., $t \mapsto \alpha_{\lambda t}$) or an inner flow.*

Proof. This follows from 3.1 and 3.6 of [7] with 5.16 above.

There are quite a few results in this direction, which all show how difficult it is to determine the domains of generators and where we actually depart from the realm of C^*-algebras. We do not know how to characterize Banach *-algebras which appear as the domains of generators. See [3] for more results.

5.5 Marginal spectra

Let α be a flow on A. We denote by $A^\alpha(0,\infty)$ the closure of the union $\bigcup_n A^\alpha[1/n,\infty)$ and by $A^\alpha(-\infty,0)$ the closure of the union $\bigcup_n A^\alpha(-\infty,1/n]$. Note that $A^\alpha(-\infty,0)^* = A^\alpha(0,\infty)$.

Definition 5.18 [28] *The* bottom marginal spectrum $\mathrm{Sp}_-(\alpha)$ *of α is defined by*

$$\mathrm{Sp}_-(\alpha) = \{p \in \mathbf{R} \mid \forall\epsilon > 0 \ A^\alpha[p-\epsilon,p+\epsilon]^* A^\alpha[p-\epsilon,p+\epsilon] \not\subset [A^\alpha(0,\infty)AA^\alpha(-\infty,0)]\}.$$

The top marginal spectrum $\mathrm{Sp}_+(\alpha)$ *is defined by*

$$\mathrm{Sp}_+(\alpha) = \{p \in \mathbf{R} \mid \forall\epsilon > 0 \ A^\alpha[p-\epsilon,p+\epsilon]^* A^\alpha[p-\epsilon,p+\epsilon] \not\subset [A^\alpha(-\infty,0)AA^\alpha(0,\infty)]\}.$$

It follows that $\mathrm{Sp}_\pm(\alpha)$ is closed and that $\mathrm{Sp}_-(\alpha) \subset [0,\infty)$ and $\mathrm{Sp}_+(\alpha) \subset (-\infty,0]$. It also follows that $\mathrm{Sp}_-(\alpha)$ is empty if and only if $A^\alpha(0,\infty)AA^\alpha(-\infty,0) = A$ and that if $\mathrm{Sp}_-(\alpha)$ is not empty then $\mathrm{Sp}_-(\alpha) \ni 0$. The bottom (resp. top) marginal spectrum is associated with the spectra of the unitary groups implementing the flow in ground state representations (resp. ceiling state representations).

Let $\ell^\infty(A)$ be the C^*-algebra of bounded sequences in A and let $\ell^\infty_\alpha(A)$ be the maximal C^*-subalgebra of $\ell^\infty(A)$ on which the action $\overline{\alpha}$ is continuous, where $\overline{\alpha}$ is the (non-continuous) flow on $\ell^\infty(A)$ defined by $\overline{\alpha}((x_n)) = (\alpha_t(x_n))$. Let $c_0(A)$ be the ideal of $\ell^\infty(A)$ consisting of $x = (x_n)$ with $\lim_{n\to\infty} \|x_n\| = 0$. We set $A^\infty_\alpha = \ell^\infty_\alpha(A)/c_0(A)$, on which $\overline{\alpha}$ induces a flow, denoted by α below.

Definition 5.19 *Let α be a flow on A. The essential bottom (resp. top) marginal spectrum $\mathbf{R}_-(\alpha)$ (resp. $\mathbf{R}_+(\alpha)$) of α is defined as* $\mathrm{Sp}_-(\alpha|A' \cap A^\infty_\alpha)$ *(resp.* $\mathrm{Sp}_+(\alpha|A' \cap A^\infty_\alpha)$*).*

Let $B = A' \cap A^\infty_\alpha$. It follows that $p \in \mathbf{R}_-(\alpha) = \mathrm{Sp}_-(\alpha|B)$ if and only if there is an $x \in B$ such that $\alpha_t(x) = e^{ipt}x$ and $x^*x \notin [B^\alpha(0,\infty)BB^\alpha(-\infty,0)]$.

The essential marginal spectra are of course invariant under cocycle perturbations.

To give some legitimacy to the above definition in terms of central sequence algebras we state:

Proposition 5.20 [24] *Let A is a separable prime C^*-algebra and α a flow on A. If there is a faithful family of α-covariant irreducible representations of A (or equivalently a faithful covariant irreducible representation), then the Connes spectrum $\mathbf{R}(\alpha)$ equals* $\mathrm{Sp}(\alpha|A' \cap A^\infty_\alpha)$.

Let τ be an α-invariant tracial state of A. We will need the following definition later.

Definition 5.21 *The τ-essential bottom marginal spectrum $\mathbf{R}_{\tau,-}(\alpha)$ is defined by*

$\{p \in \mathbf{R} \mid x \in A' \cap A_\alpha^\infty,\ \alpha_t(x) = e^{ipt}x,\ x^*x \notin [B^\alpha(0,\infty)BB^\alpha(-\infty,0)],\ \limsup \tau(x_n^*x_n) > 0\}$,

where $B = A' \cap A_\alpha^\infty$. The τ-essential top marginal spectrum $\mathbf{R}_{\tau,+}(\alpha)$ is defined in a similar way.

We note that the definition for this version of essential marginal spectra in [28] is not correct and should be understood as above.

5.6 KMS states

If the flow represents a time development of a physical system, the KMS states represents equilibrium states of that system. The set of KMS states is essentially invariant under cocycle perturbations.

Definition 5.22 *Let α be a flow on a C^*-algebra A. Let ω be a state of A and $c > 0$. If for any $x, y \in A$ there is a bounded continuous function F on $S_c = \{z \in \mathbf{C} \mid 0 \leq \Im(z) \leq c\}$ such that F is holomorphic in the interior of S_c and satisfies the boundary conditions:*

$$\begin{aligned} F(t) &= \omega(x\alpha_t(y)), \\ F(t+ic) &= \omega(\alpha_t(y)x), \end{aligned}$$

for all $t \in \mathbf{R}$, then ω is called a KMS state of (A,α) at c. If $c < 0$, the same definition applies with $S_c = \{z \in \mathbf{C} \mid 0 \geq \Im(z) \geq c\}$. If $c = 0$ and ω is an α-invariant tracial state, then ω is called a KMS state of (A,α) at 0.

It easily follows that KMS states are all α-invariant.

When $A = M_n$, a flow α on M_n is of the form $\alpha_t = \operatorname{Ad} e^{ith}$ for some $h \in (M_n)_{sa}$. In this case there is a unique KMS state ω_c of $(M_n, \operatorname{Ad} e^{ith})$ for each inverse temperature $c \in \mathbf{R}$:

$$\omega_c(x) = \frac{\operatorname{Tr}(xe^{-ch})}{\operatorname{Tr}(e^{-ch})}, \quad x \in M_n.$$

In general there may be many or no KMS states.

Let K_c^α be the set of KMS states at c of (A,α). It is known that K_c^α is a Choquet simplex in the state space $S(A)$ of A if A is unital. (Note that possibly K_c^α is empty.) Let $\tilde{K}_c^\alpha$ be the cone generated by K_c^α; $\tilde{K}_c^\alpha = \{\lambda\omega \mid \lambda \geq 0,\ \omega \in K_c^\alpha\}$, which is closed and is a lattice in the set of positive functionals.

Definition 5.23 *Let α be a flow on A. Under the above notation let*

$$K^\alpha = \{(c,\phi) \mid c \in \mathbf{R},\ \phi \in \tilde{K}_c^\alpha\},$$

which is regarded as a bundle over $\mathbf{R}$ with the base map $q : K^\alpha \to \mathbf{R}$ defined by $q(c,\phi) = c$ such that the fiber at each point is a lattice cone or possibly an empty set. We call K^α the KMS field of (A,α).

The KMS field is a closed subset of $\mathbf{R} \times A^*$.

Proposition 5.24 *Let α and β be flows on a C^*-algebra A. If α and β are cocycle-conjugate, then the KMS fields $\tilde{K}^\alpha$ and $\tilde{K}^\beta$ are isomorphic. More concretely there is a homeomorphic isomorphism ϕ of $\tilde{K}^\alpha$ onto $\tilde{K}^\beta$ which induces an affine isomorphism $\tilde{K}_c^\alpha \to \tilde{K}_c^\beta$ for each $c \in \mathbf{R}$ (where they are non-empty) such that $\phi(\omega)$ is unitarily equivalent to ω, where $\phi(c, \omega) = (c, \phi(\omega))$.*

Let α be a flow on a unital C^*-algebra A. Let $F_0 = \{c \in \mathbf{R} \mid K_c^\alpha \neq \emptyset\}$. For each $k = 1, 2, \ldots$ let F_k be the set of $c \in \mathbf{R}$ such that K_c^α has affine dimension greater than or equal to k. Then we have:

Proposition 5.25 *Suppose that α is a flow on a unital separable C^*-algebra A. Under the above notation, F_0 is closed and $(F_k)_{k=0}^\infty$ is a decreasing sequences of F_σ sets of $\mathbf{R}$.*

The property that F_k is a F_σ set follows since A is separable. What is shown in [5] is the converse: For any sequence (F_k) we can realize (A, α) such that $\dim K_c^\alpha \geq k$ if and only if $k \in F_k$. And moreover it is very likely A can be chosen to be a simple AF C^*-algebra. Thus we see that the set of possible KMS fields is quite large. See [4, 5, 6] and [33] for more.

Proposition 5.26 *Let A be a unital simple C^*-algebra, α a flow on A, and $c \in \mathbf{R} \setminus \{0\}$. Then there is a continuous homomorphism Φ of the symmetry group G_α into the homeomorphism group of K_c^α such that $\Phi(\gamma)\omega$ is unitarily equivalent to $\omega\gamma^{-1}$ for $\gamma \in G_\alpha$ and $\omega \in K_c^\alpha$. Moreover $\Phi(\gamma) = \mathrm{id}$ for any inner γ.*

Proof. If $\gamma \in G_\alpha$ and $\omega \in K_c^\alpha$, then $\omega\gamma^{-1}$ is a KMS state at c for the flow $\gamma\alpha\gamma^{-1}$. Since $\gamma\alpha_t\gamma^{-1} = \mathrm{Ad}\, u_t \alpha_t$ for some α-cocycle u, we use a perturbation theory to obtain a KMS state at c for α from $\omega\gamma^{-1}$. See [1, 44, 7].

5.7 Rotation map

Let A be a C^*-algebra and let α be a flow on A. Let T be the simplex of tracial states of A and let T^α be the closed convex set of α-invariant tracial states. Let $\mathrm{Aff}(T^\alpha)$ be the real Banach space of affine continuous functions on T^α.

Definition 5.27 *Under the above notation we define a homomorphism ϕ_α of $K_1(A)$ into* $\mathrm{Aff}(\mathrm{T}^\alpha)$ *by*

$$\phi_\alpha([u])(\tau) = \frac{1}{2\pi i}\tau(\delta_\alpha(u)u^*),$$

where $u \in \mathcal{U}(A) \cap \mathcal{D}(\delta_\alpha)$ or $u \in \mathcal{U}(M_n \otimes A) \cap M_n \otimes \mathcal{D}(\delta_\alpha)$ with appropriate modifications in the above formula. We call this the rotation map *of α.*

The above is indeed well-defined; see [4, 11, 26]. For example, if $u, v \in \mathcal{U}(A) \cap \mathcal{D}(\delta_\alpha)$, then the equality $\delta_\alpha(uv) = \delta_\alpha(u)v + u\delta_\alpha(v)$ yields

$$\tau(\delta_\alpha(uv)v^*u^*) = \tau(\delta_\alpha(u)u^*) + \tau(\delta_\alpha(v)v^*)$$

for $\tau \in T^\alpha$ and if $h = h^* \in \mathcal{D}(\delta_\alpha)$, the equality $\delta_\alpha(e^{ih}) = \int_0^1 e^{ish} i\delta_\alpha(h)e^{i(1-s)h}ds$ yields

$$\tau(\delta(e^{ih})e^{-ih}) = \tau(i\delta_\alpha(h)) = 0.$$

It follows easily that the rotation map is an invariant under cocycle perturbations. We can show:

Proposition 5.28 *If α is an approximate cocycle perturbation of another flow β on A, then $\phi_\alpha = \phi_\beta$.*

Proof. We may suppose that δ_α is the limit of $\delta_\beta + \mathrm{ad}\, ih_n$ in the graph sense for a suitable sequence (h_n) in A_{sa}. Thus for any $u \in \mathcal{U}(A) \cap \mathcal{D}(\delta_\alpha)$ there is a sequence (u_n) in $\mathcal{U}(A) \cap \mathcal{D}(\delta_\beta)$ such that $\|u - u_n\| \to 0$ and $\|\delta_\alpha(u) - (\delta_\beta + \mathrm{ad}\, ih_n)(u_n)\| \to 0$. We may suppose that $[u] = [u_n]$ for all n. Since

$$\tau((\delta_\beta + \mathrm{ad}\, ih_n)(u_n)u_n^*) = \tau(\delta_\beta(u_n)u_n^*)$$

is independent of n and converges to $\tau(\delta_\alpha(u)u^*)$, this concludes the proof.

5.8 Rohlin property

Since the Rohlin property for single automorphisms is so successful, we introduce:

Definition 5.29 *Let α be a flow on a unital C^*-algebra A. We say that α has the Rohlin property if for any $p \in \mathbf{R}$ there is a central sequence (u_n) in $\mathcal{U}(A)$ such that $\|\alpha_t(u_n) - e^{ipt}u_n\|$ converges to zero uniformly in t on every bounded interval.*

Let A be a unital C^*-algebra and let $A^\infty = \ell^\infty(A)/c_0(A)$, where $\ell^\infty(A)$ is the C^*-algebra of bounded sequences in A and $c_0(A)$ is the ideal of $\ell^\infty(A)$ consisting of those sequences converging to zero. When α is a flow on A, we define a one-parameter automorphism group $\overline{\alpha}$ of $\ell^\infty(A)$ by $\overline{\alpha}_t((x_n)) = (\alpha_t(x))$. Since $\overline{\alpha}$ is not continuous (if α is not uniformly continuous), we define a C^*-subalgebra $\ell^\infty_\alpha(A)$ of $\ell^\infty(A)$ as the maximal C^*-subalgebra on which $\overline{\alpha}$ is continuous and thus forms a flow. We let

$$A^\infty_\alpha = \ell^\infty_\alpha(A)/c_0(A),$$

on which $\overline{\alpha}$ induces a flow, which we simply denote by α. Note that A is naturally imbedded into $\ell^\infty(A)$ and in turn into A^∞_α. The Rohlin property for α on A is characterized by the property: For any $p \in \mathbf{R}$, there is a $v \in \mathcal{U}(A^\infty_\alpha \cap A')$ such that $\alpha_t(v) = e^{ipt}v$.

If $u \in \mathcal{U}(A^\infty_\alpha \cap A')$ is in the connected component of 1, we denote by $\ell(u)$ the infimum of the lengths of rectifiable paths from u to 1 in $\mathcal{U}(A^\infty_\alpha \cap A')$.

Theorem 5.30 [25] *Let A be a unital separable C^*-algebra and let α be a flow on A. Then the following conditions are equivalent:*

1. *α has the Rohlin property.*
2. *For each α-cocycle u in $A^\infty_\alpha \cap A'$ such that $\lim_{t\to\infty} \ell(u(t))/t = 0$, there exists a unitary $w \in A^\infty_\alpha \cap A'$ such that $u(t) = w\alpha_t(w^*)$.*

In this case for each α-cocycle u in A such that $\lim_{t\to\infty} \ell(u(t))/t = 0$, there is a sequence (w_n) in $\mathcal{U}(A)$ such that $\|u(t) - w_n\alpha_t(w_n^*)\| \to 0$ uniformly in t on every bounded interval.

If α has the Rohlin property, then α is not approximately inner and has no KMS states (see [25]).

Proposition 5.31 [25] *Let A be a unital separable purely infinite simple C^*-algebra and let α be a flow on A. If α has the Rohlin property, then the crossed product $A \times_\alpha \mathbf{R}$ is a purely infinite simple C^*-algebra.*

6 Flows on AF C^*-algebras

Definition 6.1 [30] *A flow α is called a* UHF flow *if it is a flow on a UHF C^*-algebra A and if it has an increasing sequence (A_n) of α-invariant finite-dimensional C^*-subalgebras of A such that $A_n \ni 1_A$, $\bigcup_n A_n$ is dense in A, and A_n is isomorphic to a full matrix algebra.*

UHF flows represent non-interacting models and must be very easy to analyze; yet I still cannot understand them. A UHF flow has a unique KMS state for any inverse temperature.

Proposition 6.2 *Let A be the UHF C^*-algebra of type 2^∞, i.e., the infinite tensor product of 2 by 2 matrices, and let τ denote the unique tracial state of A. Let α and β be UHF flows on A. If $\mathbf{R}(\alpha) = \mathbf{R} = \mathbf{R}(\beta)$, $\mathbf{R}_{\tau,-}(\alpha) = [0,\infty) = \mathbf{R}_{\tau,-}(\beta)$, and $\mathbf{R}_{\tau,+}(\alpha) = (-\infty, 0] = \mathbf{R}_{\tau,+}(\beta)$, then α and β are cocycle conjugate.*

Such a flow α can be obtained as

$$\alpha_t = \bigotimes_1^\infty \mathrm{Ad}\begin{pmatrix} e^{i\lambda_n t} & 0 \\ 0 & 1 \end{pmatrix},$$

where (λ_n) is a sequence of real numbers such that $\lambda_n \to 0$ and $\sum_n \lambda_n^2 = \infty$ (cf. [30]).

Definition 6.3 [7] *A flow is called an* AF flow *if it is a flow on an AF C^*-algebra A and if it has an increasing sequence (A_n) of α-invariant finite-dimensional C^*-subalgebras of A such that $\bigcup_n A_n$ is dense in A.*

Note that UHF flows are AF flows and that there are non-UHF AF flows on a UHF C^*-algebra. Since a flow on a finite-dimensional C^*-algebra is inner, AF flows are approximately inner.

AF flows can already give a complicated picture of KMS fields. This class is supposed to correspond to *classical statistical mechanical models*, but yet there seem to be no clear criteria by which we can distinguish classical from quantal. But we know that there are non-AF flows on an AF C^*-algebra (see below).

Theorem 6.4 [44] *Let α be a flow on an AF C^*-algebra A and δ_α its generator. Then there is an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A such that $\bigcup_n A_n$ is contained in $\mathcal{D}(\delta_\alpha)$ and is dense in A. Hence in particular the trivial flow* id *is an approximate cocycle perturbation of α.*

Proof. To show the first part we use the fact that the domain $\mathcal{D}(\delta_\alpha)$ is invariant under C^∞ functional calculus. The last part follows because there is an $h_n \in A_{sa}$ such that $\delta_\alpha | A_n = \mathrm{ad}\, ih_n | A_n$. Then it follows that $\delta_\alpha - \mathrm{ad}\, ih_n$ converges to zero on $\bigcup_n A_n$ as $n \to \infty$. Hence $\delta_\alpha - \mathrm{ad}\, ih_n$ converges to zero in the graph sense. (But of course this does not mean that $\mathrm{ad}\, ih_n$ converges to δ_α by any means.)

Theorem 6.5 [44] *Let α be a flow on an AF C^*-algebra A. Suppose that the Banach* algebra $\mathcal{D}(\delta_\alpha)$ is AF, i.e., there is an increasing sequence (A_n) of finite-dimensional *-subalgebras of $\mathcal{D}(\delta_\alpha)$ with dense union. Then α is approximately inner.*

Proof. The condition that $\mathcal{D}(\delta_\alpha)$ is AF is equivalent to saying that $\bigcup_n A_n$ is a core for δ_α. Under this condition $\mathrm{ad}\, ih_n$ converges to δ_α in the graph sense as $n \to \infty$, where $h_n \in A_{sa}$ satisfies that $\delta_\alpha | A_n = \mathrm{ad}\, ih_n | A_n$.

The Powers and Sakai conjecture [43] says that all flows on a UHF C^*-algebra are approximately inner. We still do not have a definitive answer to this, but:

Proposition 6.6 [33] *There is a unital simple C^*-algebra A and a flow α on A such that α is periodic and the fixed point algebra A^α is a simple AF C^*-algebra. In particular α is not approximately inner.*

We are still short of clear criteria for approximate innerness.

Theorem 6.7 [29] *Let α be a flow on an AF C^*-algebra. Then the following conditions are equivalent:*

1. *α is a cocycle perturbation of an AF flow.*
2. *The domain $\mathcal{D}(\delta_\alpha)$ contains a canonical AF masa C, where C is an abelian C^*-subalgebra of A such that there is an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A with dense union such that C is generated by $C \cap A_n \cap A'_{n-1}$, $n = 1, 2, \ldots$, with $A_0 = \{0\}$.*

Proof. The domain $\mathcal{D}(\delta_\alpha)$ remains unchanged under cocycle perturbations; so obviously (1) implies (2).

Suppose (2). Then by a general theory $\delta_\alpha|C$ is bounded. With $C_n = C \cap A_n \cap A'_{n-1}$, (C_n) form a central sequence of finite-dimensional abelian C^*-subalgebras which generates C. By using this we can further argue that $\delta_\alpha|C$ is inner, i.e., there is an $h \in A_{sa}$ such that $\delta_\alpha(x) = \operatorname{ad} ih(x)$, $x \in C$. Replacing δ_α by $\delta_\alpha - \operatorname{ad} ih$, we can assume that $\delta_\alpha|C = 0$. Then by a small perturbation one can conclude that δ_α generates an AF flow. See [29].

In general we expect that the continuous symmetry will not act in a non-trivial way on the set K_c of KMS states (at inverse temperature c). Recall that the symmetry group G_α is defined as the group of automorphisms γ with the property that $\gamma\alpha\gamma^{-1}$ is a cocycle perturbation of α.

Proposition 6.8 [7] *Let A be a unital simple C^*-algebra and let α be an AF flow on A. Let $(\gamma_t)_{t\in[0,1]}$ be a continuous path in G_α such that*

$$\gamma_t\delta_\alpha\gamma^{-1} = \delta_\alpha + \operatorname{ad} ib(t)$$

for some rectifiable path $(b(t))_{t\in[0,1]}$ in A_{sa}. Then it follows that $\Phi(\gamma_0)(\omega) = \Phi(\gamma_1)(\omega)$ for any extreme $\omega \in K_c$.

Proposition 6.9 *Let α be a cocycle perturbation of an AF flow. Then $(A^\infty_\alpha \cap A')^\alpha$ has real rank zero and has trivial K_1.*

Proof. Apparently we may suppose that α is an AF flow. Hence we suppose that there is an increasing sequence (A_n) of α-invariant finite-dimensional C^*-subalgebras of A with dense union.

Let $b^* = b \in (A^\infty_\alpha \cap A')^\alpha$. Then there is a sequence (b_n) in A_{sa} such that $b \sim (b_n)$ (i.e., $b = (b_n) + c_0(A)$). We may suppose that $\|\delta_\alpha(b_n)\| \to 0$ and that there are increasing sequences (k_n) and (ℓ_n) in $\mathbf{N}$ such that $k_n < \ell_n$, $k_n \to \infty$, and $b_n \in B_n \equiv A_{\ell_n} \cap A'_{k_n}$. Since B_n is α-invariant and finite-dimensional, there is a $h_n^* = h_n \in B_n$ such that $\delta_\alpha|B_n = \operatorname{ad} h_n|B_n$. Since $\|[h_n, b_n]\| \to 0$ and $h_n, b_n \in (B_n)_{sa}$, we get $h'_n, b'_n \in B_n$ such that $\|h_n - h'_n\| \to 0$, $\|b_n - b'_n\| \to 0$, and $[h'_n, b'_n] = 0$ (3.1 of [7]).

Let $\epsilon > 0$ and let F be a finite subset of the spectrum $\mathrm{Sp}(b)$ of b such that any $\lambda \in \mathrm{Sp}(b)$ has $p \in F$ such that $|\lambda - p| < \epsilon$. Then we find a $b''_n \in (B_n)_{sa}$ such that b''_n is a function of b'_n, $\limsup_n \|b'_n - b''_n\| < \epsilon$, $\mathrm{Sp}(b''_n) \subset F$. Then (b''_n) defines a self-adjoint element $c \in (A^\infty_\alpha \cap A')^\alpha$ such that $\|c - b\| < \epsilon$ and $\mathrm{Sp}(c) \subset F$, which is finite. This concludes the proof that $(A^\infty_\alpha \cap A')^\alpha$ has real rank zero.

Let u be a unitary in $(A^\infty_\alpha \cap A')^\alpha$. Then as before we may suppose that there is a sequence (u_n) in $\mathcal{U}(A)$ and increasing sequences (ℓ_n) and (k_n) in $\mathbf{N}$ such that $k_n < \ell_n$, $k_n \to \infty$, $u_n \in A_{\ell_n} \cap A'_{k_n}$, and $\|\delta_\alpha(u_n)\| \to 0$. There is an $h_n^* = h_n \in B_n \equiv A_{\ell_n} \cap A'_{k_n}$ such that $\delta_\alpha|B_n = \operatorname{ad} ih_n|B_n$. Then by using the condition that $\|[u_n, h_n]\| \to 0$, we apply 4.1 of [29].

Theorem 6.10 [44, 29] *Let A be an AF C^*-algebra. Then it follows that $C_1 \supsetneqq C_2 \supsetneqq C_2$, where C_i's are defined as:*

C_1: *the class of approximately inner flows.*

C_2: *the class of flows whose domain is AF.*

C_3: *the class of cocycle perturbations of AF flows.*

Proof. That $C_1 \supset C_2 \supset C_3$ is immediate.

To show that $C_1 \neq C_2$ we construct a flow α such that the Banach *-algebra $\mathcal{D}(\delta_\alpha)$ does not have real rank zero (i.e., $D(\delta_\alpha)$ contains $h = h^*$ which cannot be approximated by self-adjoint elements of finite spectra). This in particular implies that $\mathcal{D}(\delta_\alpha)$ is not AF.

To show that $C_2 \neq C_3$ we construct a flow α such that $(A_\alpha^\infty \cap A')^\alpha$ has real rank more than zero or has non-trivial K_1 (or both).

All the examples are given by expressing an AF C^*-algebra as the inductive limit of tensor products of $C(\mathbf{T})$ or $C(I)$ with finite-dimensional C^*-algebras, where $\mathbf{T}$ is a one-dimensional torus and I is a closed interval. Thus we have to use the recent classification result for C^*-algebras (see [14]).

7 Rohlin flows on simple AT algebras of real rank zero

When T is a convex set, we denote by $\mathrm{Ex}(T)$ the set of extreme points of T; $\tau \in T$ is extreme in T if there is no non-trivial expression of the form $\tau = \lambda\varphi_1 + (1-\lambda)\varphi_2$, where $0 < \lambda < 1$ and $\varphi_i \in T$. When T is the simplex of tracial states of a C^*-algebra A, an extreme point of T corresponds to a factorial tracial state of A. In this case there is a natural map $\phi_0 : K_0(A) \to \mathrm{Aff}(T)$ such that $\phi_0([e])(\tau) = \tau(e)$ for a projection $e \in A$.

Theorem 7.1 [26, 27] *Let A be a unital simple AT C^*-algebra of real rank zero and T the simplex of tracial states of A. Suppose that* $\mathrm{Ex}(T)$ *is closed and* $\mathrm{Ex}(T)$ *is separated by a finite subset of $K_0(A)$. Suppose further that there is a homomorphism $\phi_1 : K_1(A) \to \mathrm{Aff}(T)$ such that* $\mathrm{Ran}(\phi_1)$ *is dense, and* $\mathrm{Ex}(T)$ *is separated by a finite subset of* $\mathrm{Ran}(\phi_1)$. *Then there is a Rohlin flow α of A such that the rotation map $\phi_\alpha : K_1(A) \to \mathrm{Aff}(T)$ equals to ϕ_1, and $A \times_\alpha \mathbf{R}$ is a simple stable AT C^*-algebra of real rank zero with K_0 isomorphic to $K_1(A)$ ordered through ϕ_1.*

The conditions on T above is obviously satisfied when T is a singleton. The condition of ϕ_1 implies in particular that $K_1 \neq \{0\}, \mathbf{Z}$. See [40] for the case $K_1 = \mathbf{Z}$.

As an example let us consider irrational rotation C^*-algebras. An irrational rotation C^*-algebra A_θ with $\theta \in (0,1)$ irrational is the universal C^*-algebra generated by two unitaries u, v with

$$uv = e^{2\pi i\theta} vu$$

It is known that A_θ is simple and has a unique tracial state and that $K_1 \cong \mathbf{Z}^2$ and is generated by $[u], [v]$. It is shown in [15] that that irrational rotation C^*-algebras are simple AT C^*-algebras of real rank zero. For a $p \in \mathbf{R}$ we define a flow α^p on $A_\theta = C^*(u,v)$ by

$$\alpha_t^p(u) = e^{2\pi ipt}u, \quad \alpha_t^p(v) = e^{2\pi it}v,$$

Then the rotation map $\phi_{\alpha^p} : \mathbf{Z}^2 \to \mathbf{R}$ is given by

$$(m,n) \mapsto pm + n.$$

If 1 and p are linearly independent over $\mathbf{Z} + \theta\mathbf{Z}$, then α^p has the Rohlin property [25]. In this case, by the above theorem, there is a Rohlin flow β on A_θ such that $\phi_\beta = \phi_{\alpha^p}$ such that $A \times_\beta \mathbf{R}$ is again an AT C^*-algebra of real rank zero.

We do not have any sort of uniqueness result in this case. This problem will be discussed for different C^*-algebras in the next section.

8 Rohlin flows on separable nuclear purely infinite simple C^*-algebras with UCT

Recall that a flow α is called an approximate cocycle perturbation of another flow β if α is obtained as the limit of cocycle perturbations of β.

Theorem 8.1 [34] *Let A be a unital separable nuclear purely infinite simple C^*-algebra. If each of two Rohlin flows on A is an approximate cocycle perturbation of the other, then they are cocycle-conjugate with each other.*

Our expectation here is that there are not many cocycle conjugacy classes of Rohlin flows on such a C^*-algebra, or even there may be just one, because all the invariants we have invented so far do not distinguish them at all (or cannot be calculated in the case of generator domains); well this may only show my incompetence. An evidence for that may be found for a special class of flows on the Cuntz algebras [12].

For an integer $2 \leq m < \infty$ the Cuntz algebra $\mathcal{O}_m$ is the universal C^*-algebra generated by m isometries $s_0, s_1, \ldots, s_{m-1}$ with the relation:

$$\sum_{i=0}^{m-1} s_i s_i^* = 1.$$

A quasi-free flow α on $\mathcal{O}_m$ is a flow of the form:

$$\alpha_t(s_k) = e^{ip_k t}s_k, \quad k = 0, \ldots, m-1,$$

for some $p_k \in \mathbf{R}$. Although we do not know an exact condition on $(p_0, \dots, p_{m-1})$ for α to have the Rohlin property, we know that there are many quasi-free flows with the Rohlin property and can show:

Proposition 8.2 [34] *The Rohlin quasi-free flows on $\mathcal{O}_m$ with $m < \infty$ are cocycle conjugate with each other, i.e., if α and β are such flows, there is an automorphism ϕ of $\mathcal{O}_m$ such that $\mathrm{Ad}\, u_t\alpha_t = \phi\beta_t\phi^{-1}$ for some α-cocycle u.*

A satisfactory result in this setting was obtained only for $m = 2$:

Proposition 8.3 [34] *For $p_0, p_1 \in \mathbf{R}$ define a flow α on $\mathcal{O}_2$ by $\alpha_t(s_k) = e^{ip_k t}s_k$, $k = 0, 1$. Then the following conditions are equivalent:*

1. *p_0, p_1 are rationally independent and $p_0p_1 < 0$.*
2. *$\mathcal{O}_2 \times_\alpha \mathbf{R}$ is purely infinite and simple.*
3. *α has the Rohlin property.*

Certainly quasi-free flows are rather special. The domain of the generator of a quasi-free flow contains the commutative C^*-subalgebra D_m generated by $s_Is_I^*$, where I runs over all the finite sequences in $\{0, 1, \dots, m-1\}$ and $s_I = s_{i_1}s_{i_2}\cdots s_{i_n}$ for $I = (i_1, \dots, i_n)$. Note that D_m is a Cartan masa and this reminds me of the situation of AF flows.

If α is a flow, then α_t is homotopic to the identity and so often is approximately inner for each $t \in \mathbf{R}$. The following is defined in [35].

Definition 8.4 *Let A be a C^*-algebra and α a flow on A. Then α_t is said to be α-*invariantly approximately inner *if there is a sequence (u_n) in $\mathcal{U}(A)$ such that $\alpha_t = \lim \mathrm{Ad}\, u_n$ and $\|\alpha_s(u_n) - u_n\|$ converges to zero uniformly in s on every compact subset.*

In an attempt to generalize what was obtained for quasi-free flows, we get:

Theorem 8.5 [35, 36] *Let A be a unital separable nuclear purely infinite simple C^*-algebra satisfying UCT and let α be a flow on A. Then the following conditions are equivalent.*

1. *α has the Rohlin property.*
2. *$(A' \cap A^\omega_\alpha)^\alpha$ is purely infinite and simple, $K_0((A' \cap A^\omega_\alpha)^\alpha) \cong K_0(A' \cap A^\omega)$ induced by the embedding, and $\mathrm{Sp}(\alpha|A' \cap A^\omega_\alpha) = \mathbf{R}$.*
3. *The crossed product $A \times_\alpha \mathbf{R}$ is purely infinite and simple and the dual action $\hat{\alpha}$ has the Rohlin property.*
4. *The crossed product $A \times_\alpha \mathbf{R}$ is purely infinite and simple and each α_t is α-invariantly approximately inner.*

If the above conditions are satisfied, it also follows that $K_1((A' \cap A^\omega_\alpha)^\alpha) \cong K_1(A' \cap A^\omega)$, which is induced by the embedding.

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