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CLASSIFICATION OF FINITE GROUP ACTIONS ON CLASSIFIABLE C^*-ALGEBRAS

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1. Introduction

The aim of this short note is to give an account of a classification result of finite group actions with a special property, called the *Rohlin property*, on "classifiable classes" of C^*-algebras.

Thanks to the recent progress of Elliott's classification program, there are two notable "classifiable classes" of C^*-algebras, namely, simple nuclear separable purely infinite C^*-algebras (called Kirchberg algebras) satisfying the universal coefficient theorem (abbreviated as UCT), and simple nuclear separable C^*-algebras of tracial topological rank zero satisfying the UCT [13, 15]. (An algebra in these two classes is said to be classifiable in the sequel.) I discuss finite group actions with the Rohlin property on the above two classes of C^*-algebras. Such actions are completely classified by K-theoretical invariants.

As in other classification results of group actions in operator algebras, a cohomology vanishing theorem plays a crucial role in the main results. A considerably strong cohomology vanishing theorem holds for the Tate cohomology with the K-groups as the coefficient modules.

Quasi-free actions of finite cyclic groups on the Cuntz algebras and their dual actions will be discussed from the view point of the Rohlin property.

2. The Rohlin property

For a C^*-algebra A, we define

$$c_0(A) = \{(a_n) \in \ell^\infty(\mathbb{N}, A);\ \lim_{n\to\infty} ||a_n|| = 0\},$$

$$A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(A).$$

We identify A with the C^*-subalgebra of A^∞ consisting of the equivalence classes of constant sequences, and define

$$A_\infty = A^\infty \cap A'.$$

For an automorphism α of A (or a group action α on A), we denote by α^∞ and α_∞ the automorphisms of A^∞ and A_∞ (or group actions on A^∞ and A_∞) induced by α respectively.

The notion of the Rohlin property was first introduced by Connes [3] in operator algebras for a single automorphism of finite von Neumann algebras. The notion has been generalized in various contexts later (for the case of C^*-algebras, see Kishimoto's contribution in this volume and [8]). The following definition already appeared in [12, 6, 7] with a different name.

Definition 2.1. Let α be an action of a finite group G on a unital C^*-algebra A. α is said to have the *Rohlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A_\infty$ consisting of projections satisfying

$$\alpha_{g\infty}(e_h) = e_{gh}, \quad g, h \in G.$$

Example 2.2. Let G be a finite group, and λ be the left regular representation of G. We identify $\mathbb{B}(\ell^2(G))$ with the matrix algebra $\mathbb{M}_{|G|}$ and introduce an action of G on the UHF algebra $\mathbb{M}_{|G|^\infty}$ by

$$\mu_g^G = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(\lambda(g)), \quad g \in G.$$

Then μ^G has the Rohlin property. When G is abelian, the dual action $\hat{\mu^G}$ is conjugate to μ^G under appropriate identification of G and its dual group $\hat{G}$ (see, for example, [12]).

The author naturally encountered the Rohlin property in his attempt of seeking for an equivariant version of the following remarkable result of E. Kirchberg and N. C. Phillips [11, Lemma 3.7].

Theorem 2.3. *Let A be a separable simple unital nuclear C^*-algebra. Then the following two conditions are equivalent:*

(1) *The C^*-algebra A is isomorphic to $\mathcal{O}_2$.*
(2) *There exists a C^*-subalgebra of A_∞ containing the unit of A_∞ that is isomorphic to $\mathcal{O}_2$.*

In particular, $\mathcal{O}_2 \otimes B$ is always isomorphic to $\mathcal{O}_2$ if B is a separable simple unital nuclear C^*-algebra.

In view of known results in subfactors, one can easily guess that the relative central sequence algebra $(\mathcal{O}_2^\alpha)^\infty \cap \mathcal{O}_2'$ should be a right counter part of the central sequence algebra $(\mathcal{O}_2)_\infty$ under the presence of a finite group action α.

Theorem 2.4 ([9]). *Let α be an outer action of a finite group G on $\mathcal{O}_2$. Then the following conditions are equivalent:*

(1) *Let μ^G be the action of G on $\mathbb{M}_{|G|^\infty}$ constructed in Example 2.2. Then*
$$(\mathcal{O}_2, \alpha) \cong (\mathcal{O}_2 \otimes \mathbb{M}_{|G|^\infty}, \mathrm{id} \otimes \mu^G).$$
(2) *There exists a C^*-subalgebra of $(\mathcal{O}_2^\alpha)^\infty \cap \mathcal{O}_2'$ containing the unit of $(\mathcal{O}_2^\alpha)^\infty \cap \mathcal{O}_2'$ that is isomorphic to $\mathcal{O}_2$.*
(3) *The action α has the Rohlin property.*

In particular, $(\mathcal{O}_2 \otimes B, \mathrm{id} \otimes \beta)$ is always conjugate to $(\mathcal{O}_2 \otimes \mathbb{M}_{|G|^\infty}, \mathrm{id} \otimes \mu^G)$ if B is a separable simple unital nuclear C^*-algebra and β is an outer action of G on B.

The Rohlin property is suited for a classification purpose thanks to the next theorem, which follows from a finite group version of Evans-Kishimoto's intertwining argument [4].

Theorem 2.5 ([9]). *Let A be a separable unital C^*-algebra and α and β be actions of a finite group G on A with the Rohlin property. Assume that α_g and β_g are approximately unitarily equivalent for all $g \in G$. Then there exists an approximately inner automorphism $\theta \in \mathrm{Aut}(A)$ such that*
$$\theta \cdot \alpha_g \cdot \theta^{-1} = \beta_g, \quad g \in G.$$

The Rohlin property is a so strong condition that one can easily find natural outer actions without possessing the Rohlin property. However, actions with the Rohlin property are not very rare in the sense that there are large classes of well-known finite abelian group actions whose dual actions have the Rohlin property.

Definition 2.6. Let A be a unital C^*-algebra, and α be an action of a finite abelian group G on A. The action α is said to be *approximately representable* if there exists a unitary representation u in $(A^\alpha)^\infty$ such that
$$\alpha_g(x) = u(g)xu(g)^*, \quad x \in A, \ g \in G.$$

Note that locally representable actions [5] are always approximately representable. The following is the precise relationship between the Rohlin property and approximate representability.

Lemma 2.7 ([9]). *Let A be a separable C^*-algebra, and let α be an action of a finite abelian group G on A. We denote by $\widehat{\alpha}$ the dual action of α on $A \rtimes_\alpha G$. Then*

(1) *The action α has the Rohlin property if and only if the dual action $\widehat{\alpha}$ is approximately representable.*
(2) *The action α is approximately representable if and only if the dual action $\widehat{\alpha}$ has the Rohlin property.*

Theorem 2.5 and Lemma 2.7 explain why locally representable actions are classifiable in terms of the K-theoretical invariants of the crossed products and the dual actions [5, 1].

The Rohlin property gives a very strong K-theoretical constraint.

Theorem 2.8 ([9]). *Let A be a simple unital C^*-algebra, and α be an action of a finite group G on A with the Rohlin property. Let ι be the inclusion map of A^α into A. Then $K_i(\iota)$ is injective for $i = 0, 1$. Moreover, the following equation holds in $K_i(A)$ for $i = 0, 1$:*

$$\mathrm{Im}(K_i(\iota)) = \mathrm{Im}(\sum_{g\in G} K_i(\alpha_g)) = K_i(A)^G,$$

where $K_i(A)^G = \{x \in K_i(A);\ K_i(\alpha_g)(x) = x,\ \forall g \in G\}$.

3. A cohomology vanishing theorem

Since its first appearance in operator algebras [3], the Rohlin property has always served as a tool to establish cohomology vanishing type results. However, the coefficient "modules" of the relevant cohomology are usually the unitary groups of some operator algebras. When a C^*-algebra A allows an action of a finite group G, the K-groups of A have G-module structure induced by the action. In this section, we discuss the group cohomology of G with the K-groups as coefficient modules.

Our standard reference for the group cohomology is Brown's textbook [2]. We fix a finite group G, and denote by N its norm element in the integral group ring $\mathbb{Z}G$, namely

$$N = \sum_{g\in G} g \in \mathbb{Z}G.$$

For a left G-module M, we define

$$M^G = \{m \in M;\ gm = m,\ \forall g \in G\}.$$

$$M_G = M/\langle gm - m;\ m \in M,\ g \in G\rangle \cong \mathbb{Z} \otimes_G M,$$

where $\mathbb{Z}$ is regarded as a trivial G-module. As $gm - m$ is annihilated by N for all $g \in G$ and $m \in M$, N induces a map $\overline{N} : M_G \to M^G$.

Definition 3.1. Let G be a finite group and M be a G-module.

(1) The *Tate cohomology* $\hat{H}^n(G, M)$ $(n \in \mathbb{Z})$ is defined by

$$\hat{H}^n(G,M) = \begin{cases} H^n(G,M) & n > 0 \\ \mathrm{Coker}\overline{N} & n = 0 \\ \mathrm{Ker}\overline{N} & n = -1 \\ H_{-n-1}(G,M) & n < -1 \end{cases}$$

(2) M is said to be *cohomologically trivial* (abbreviated as CT) if $\hat{H}^n(H, M) = \{0\}$ for all $n \in \mathbb{Z}$ and for all subgroups $H \subset G$.
(3) M is said to be *completely cohomologically trivial* (abbreviated as CCT) if nM, $\mathrm{Tor}(M, \mathbb{Z}/n\mathbb{Z})$, and $M \otimes \mathbb{Z}/n\mathbb{Z}$ are CT for all $n \in \mathbb{N}$ ("and" in this definition can be replaced by "or").

While the notion of CT modules is popular in group cohomology, that of CCT modules was introduced in [10] and its complete characterization was also given there with help of James Schafer. A G-module M is said to be relatively projective if M is of the form $M = \mathbb{Z}G \otimes M'$ with a trivial G-module M'.

Theorem 3.2 ([10]). *For a finite group G, a G-module is CCT if and only if it is an inductive limit of relatively projective modules.*

Example 3.3. Let p be a prime number, G be the cyclic group $\mathbb{Z}_p$ of order p, and M be a finitely generated CCT G-module. Then M has the following decomposition (each component is allowed to be zero):

$$M = M_f \oplus M(p) \oplus M(q_1) \oplus M(q_2) \oplus \cdots \oplus M(q_k),$$

where M_f is $\mathbb{Z}G$-projective, $p, q_1, q_2, \cdots, q_k$ are mutually distinct prime numbers, and $M(p)$ (respectively $M(q_i)$) is the p-component (respectively q_i-component) of M_{tor}. $M(p)$ must be an induced module, that is, $M(p) \cong \mathbb{Z}G \otimes M'$ for some trivial G-module M'. There is no restriction for $M(q_i)$. If p is less than 23, every projective $\mathbb{Z}G$-module is $\mathbb{Z}G$-free; however, it is not the case in general. The structure of projective $\mathbb{Z}G$-modules is determined by the class group of $\mathbb{Z}[e^{2\pi i/p}]$ [16]. When $p = 2$, M has the following form (each component is allowed to be zero):

$$M = (\mathbb{Z}^n \oplus \mathbb{Z}^n) \oplus (M_0(2) \oplus M_0(2)) \oplus M_+ \oplus M_-,$$

where $M_0(2)$ is a 2-group, and M_+ and M_- are odd torsion groups. G acts on $\mathbb{Z}^n \oplus \mathbb{Z}^n$ and $M_0(2) \oplus M_0(2)$ by flip of the components, and G acts on M_+ and M_- as multiplying by 1 and -1 respectively.

Theorem 2.8 shows that if an action α of a finite group on a unital simple C^*-algebra has the Rohlin property, the Tate cohomology $\hat{H}^0(G, K_i(A))$ is trivial for $i = 0, 1$. Indeed, a much stronger statement holds.

Theorem 3.4 ([10]). *Let α be an action of a finite group G on a unital simple C^*-algebra A. If α has the Rohlin property, then $K_0(A)$ and $K_1(A)$ are CCT G-modules.*

When G is the cyclic group of order n, the Tate cohomology has period 2 [2, pp. 58], and we get

$$\hat{H}^{2n+1}(G,M) = M^G/NM,$$
$$\hat{H}^{2n}(G,M) = \mathrm{Ker}(N)/(1-\sigma)M,$$

where σ denotes the generator of $\mathbb{Z}_n$. Therefore when M is CT, we get the following exact sequence.

$$0 \to M^G \to M \xrightarrow{1-\sigma} M \xrightarrow{N} M^G \to 0.$$

Using this fact and characterization of the approximately inner automorphism group of classifiable C^*-algebras in terms of M. Rørdam notion of the KL-groups [14], one can get the following two results:

Theorem 3.5 ([10]). *Let α be an action of a finite group G on a unital classifiable C^*-algebra A. Assume that α has the Rohlin property and acts on the K-groups of A trivially. Then (A, α) is conjugate to $(A \otimes \mathbb{M}_{|G|^\infty}, \mathrm{id}_A \otimes \mu^G)$.*

Corollary 3.6 ([10]). *Let A be a simple unital C^*-algebra such that either $K_0(A)$ or $K_1(A)$ is isomorphic to $\mathbb{Z}$. Then there is no non-trivial finite group action with the Rohlin property on A. In particular, there is no non-trivial finite group action with the Rohlin property on any C^*-algebra stably isomorphic to the Cuntz algebra $\mathcal{O}_\infty$.*

4. MAIN RESULTS

The K-group $K_*(A;\mathbb{Z}_p)$ for a C^*-algebra A with the coefficient module $\mathbb{Z}_p$ was introduced by C. Schochet [17], and it may be defined using the Cuntz algebra as $K_*(A;\mathbb{Z}_p) = K_*(A \otimes \mathcal{O}_{p+1})$. The entire K-group $\underline{K}(A)$ is defined by

$$\underline{K}(A) = \bigoplus_{n=0}^{\infty}(K_0(A;\mathbb{Z}_n) \oplus K_1(A;\mathbb{Z}_n)).$$

In order to classify group actions with the Rohlin property using Theorem 2.5, an invariant to detect the approximately inner automorphism group $\overline{\mathrm{Inn}}(A)$ is indispensable. It is known that for a classifiable C^*-algebra A, an automorphism α is approximately inner if and only if it acts on the entire K-group $\underline{K}(A)$ trivially [13, 15]. However, $\underline{K}(A)$ is too big as an invariant for practical use. Theorem 3.4 reduces the classification invariant $\underline{K}(A)$ to the ordinary K-groups.

Theorem 4.1 ([10]). *Let A be a unital classifiable C^*-algebra and α and β be actions of a finite group G on A with the Rohlin property. Then the following two conditions are equivalent.*

(1) $K_i(\alpha_g) = K_i(\beta_g)$ *for all* $g \in G$ *and* $i = 0, 1$.
(2) *There exists an automorphism* θ *of* A *such that* θ *acts on* $K_0(A)$ *and* $K_1(A)$ *trivially and*

$$\theta \cdot \alpha_g \cdot \theta^{-1} = \beta_g, \quad \forall g \in G.$$

Thanks to Theorem 3.2, it is also possible to construct model actions with the Rohlin property in the case of Kirchberg algebras.

Theorem 4.2 ([10]). *Let* G *be a finite group, let* M_0 *and* M_1 *be countable CCT* G*-modules, and* $a \in M_0^G$. *Then there exists a unique (up to conjugacy)* G*-action* α *with the Rohlin property on a unital simple nuclear separable purely infinite* C^**-algebra* A *satisfying the UCT such that there exist* G*-module isomorphisms from* $K_i(A)$ *onto* M_i *for* $i = 0, 1$ *that take* $[1_A] \in K_0(A)$ *to* a.

Corollary 4.3 ([10]). *Let* α *be an action of a finite group* G *with the Rohlin property on a unital simple separable nuclear* C^**-algebra* A *satisfying the UCT. Then the crossed product* $A \rtimes_\alpha G$ *satisfies the UCT too.*

5. Quasi-free Cyclic actions on the Cuntz algebras

As an application of the results mentioned so far, one can determine exactly when a quasi-free $\mathbb{Z}_{p^m}$-action is approximately representable.

A quasi-free action of a group on the Cuntz algebra $\mathcal{O}_n$ is an action induced by a unitary transformation of the canonical generators of $\mathcal{O}_n$. Let p be a prime number and $q = p^m$, $m \in \mathbb{N}$ and $\zeta_q = e^{2\pi i/q}$. Up to conjugacy, every quasi-free $\mathbb{Z}_q$-action has the following form: there exists a partition

$$\{1, 2, \cdots, n-1, n\} = \bigcup_{j=0}^{q-1} I_j$$

such that

$$\alpha(S_i) = \zeta_q^j S_i, \quad i \in I_j.$$

We denote by $|I_j|$ the cardinality of I_j.

Theorem 5.1 ([10]). *Let* p *be a prime number and* $q = p^m$, $m \in \mathbb{N}$. *We assume that* α *is a quasi-free action of* $\mathbb{Z}_q$ *on the Cuntz algebra* $\mathcal{O}_n$ *of the above form. Then,*

(1) *If* $n \not\equiv 1 \mod p$, α *is approximately representable.*
(2) *Assume* $n = p^k l + 1$ *with* l *prime to* p *and* $k > 0$. *Then* α *is approximately representable if and only if the following holds*

$$|I_0| - 1 \equiv |I_1| \equiv |I_2| \equiv \cdots \equiv |I_{q-1}| \equiv 0 \mod p^k.$$

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