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OUTER ACTIONS OF A DISCRETE AMENABLE GROUP ON APPROXIMATELY FINITE DIMENSIONAL FACTORS

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To each factor $\mathcal{M}$ we associate an invariant $\mathrm{Ob}_m(\mathcal{M})$ to be called the *intrinsic modular obstruction* as a cohomological invariant which lives in the "third" cohomology group:

$$\mathrm{H}^{\mathrm{out}}_{\alpha,s}(\mathrm{Out}(\mathcal{M})\times\mathbb{R},\mathrm{H}^1_\theta(\mathbb{R},\mathcal{U}(\mathcal{C})),\mathcal{U}(\mathcal{C}))$$

where $\{\mathcal{C},\mathbb{R},\theta\}$ is the flow of weights on $\mathcal{M}$. If α is an outer action of a countable discrete group G on $\mathcal{M}$, then its modulus $\mathrm{mod}(\alpha)\in\mathrm{Hom}(G,\mathrm{Aut}_\theta(\mathcal{C}))$, $N=\alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$ and the pull back

$$\mathrm{Ob_m}(\alpha)=\alpha^*(\mathrm{Ob_m}(\mathcal{M}))\in\mathrm{H}^{\mathrm{out}}_{\alpha,s}(G\times\mathbb{R},N,\mathcal{U}(\mathcal{C}))$$

to be called the *modular obstruction* of α are invariants of the outer conjugacy class of the outer action α.

We prove that if the factor $\mathcal{M}$ is approximately finite dimensional and G is amenable, then the invariants uniquely determine the outer conjugacy class of α and the every invariant occurs as the invariant of an outer action α of G on $\mathcal{M}$.

§1. Introduction

Let G be a countable discrete group. A map $\alpha: g\in G\mapsto\alpha_g\in\mathrm{Aut}(\mathcal{M})$ is called an *outer* action of G on the factor $\mathcal{M}$ if α satisfies

$$\begin{aligned}&\alpha_g\circ\alpha_h\equiv\alpha_{gh}\quad \mathrm{mod}\ \mathrm{Int}(\mathcal{M}),\quad g,h\in G;\\&\alpha_1=\mathrm{id}\quad\text{and}\quad\alpha_g\not\equiv\mathrm{id}\quad\mathrm{mod}\ \mathrm{Int}(\mathcal{M})\quad\text{unless } g=1\end{aligned}$$

where $1\in G$ is the identity of G. If $\dot\alpha_g$ is the class of α_g in $\mathrm{Out}(\mathcal{M})$, then the map $\dot\alpha: g\in G\mapsto\dot\alpha_g\in\mathrm{Out}(\mathcal{M})$ is an injective homomorphism. Two outer actions α and β of G on the same factor $\mathcal{M}$ are called *outer conjugate* if there exists an automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that

$$\dot{\beta}_g = \dot{\sigma}\dot{\alpha}_g\dot{\sigma}^{-1}, \quad g \in G,$$

where $\dot{\sigma} \in \mathrm{Out}(\mathcal{M})$ is the class of σ in $\mathrm{Out}(\mathcal{M})$.

With the successful completion of the cocycle conjugacy classification of amenable discrete group actions on AFD factors by many hands over more than two decades, [C3, C6, J1, JT, O, ST1, ST2, KwST, KtST], it is only naturally to consider the outer conjugacy classification of amenable discrete group outer actions on AFD factors. In fact, the work on the program has been already started by the pioneering works of Connes, [Cnn 3, 4, 5, 6], Jones [J1] and Ocneanu [Ocn].

In this article, we complete the outer conjugacy classification of discrete amenable group outer actions on AFD factors. The cases of type I, II_1, II_∞ and III with additional technical assumption were already completed by Jones, [J1], and Ocneanu, [Ocn], so the case of type III will be mainly considered although the technical assumption in the case of type III placed in the work of Ocneanu [Ocn] must be removed.

§2. Modified Huebschmann Jones Ratcliffe Exact Sequence

We recall the Huebschmann-Jones-Ratcliffe exact sequence, [Hb, J1, Rc]:

$$\begin{aligned}1 \longrightarrow \mathrm{H}^1(Q,A) \xrightarrow{\pi^*} \mathrm{H}^1(G,A) \longrightarrow \mathrm{H}^1(N,A)^G \longrightarrow \\ \longrightarrow \mathrm{H}^2(Q,A) \longrightarrow \mathrm{H}^2(G,A) \longrightarrow \Lambda(G,N,A) \xrightarrow{\delta_{\mathrm{HJR}}} \mathrm{H}^3(Q,A) \xrightarrow{\pi_G^*} \mathrm{H}^3(G,A),\end{aligned}$$

where either i) G is a separable locally compact group acting on a separable abelian von Neumann algebra $\mathcal{C}$ with $A = \mathcal{U}(\mathcal{C})$ and N a Borel normal subgroup, or ii) G is a discrete group and N a normal subgroup. In this section, we show that there is another exact sequence related with HJR-exact sequence, which we call modified HJR-exact sequence. This modified HJR-exact sequence is not only necessary to describe invariants to outer conjugacy of outer actions but also necessary to construct model outer actions with given invariants.

Let L and M be normal subgroups of a discrete group H with $H \triangleright L \triangleright M$ and $H/L \simeq Q$ and $H/M \simeq G$. Let $\Lambda(\widetilde{H}, L, A)$ be the characteristic invariant in [ST2] where $\widetilde{H} = H \times \mathbb{R}$.

Definition 2.1. *A subset* $\{[\lambda,\mu] \in \Lambda(\widetilde{H},L,A);\ [\lambda|_M,\mu|_M] \in \Lambda(H,M,\mathbb{T}),\ \lambda|_{M\times\mathbb{R}} = 1\}$ *of* $\Lambda(\widetilde{H},M,A)$ *is denoted by* $\Lambda(\widetilde{H},L,M,A)$.

A homomorphism $\mathrm{Res} : \mathrm{H}^2(H, \mathbb{T}) \longmapsto \Lambda(\widetilde{H}, L, M, A)$ *is the following composed map of canonical inclusions:*

$$\mathrm{H}^2(H,\mathbb{T}) \longrightarrow \mathrm{H}^2(\widetilde{H},\mathbb{T}) \longrightarrow \mathrm{H}^2_\alpha(\widetilde{H},A) \xrightarrow{\text{res}} \Lambda(\widetilde{H},L,A).$$

Definition 2.2. *A cocycle* $c \in \mathrm{Z}^3_\alpha(\widetilde{Q}, A)$ *will be called standard if it is of the form: for* $\tilde{p} = (p, s),\ \tilde{q} = (q, t),\ \tilde{r} = (r, u) \in \widetilde{Q}$,

$$c(\tilde{p},\tilde{q},\tilde{r}) = \alpha_p(d_c(s;q,r))c_Q(p,q,r)$$

with $c_Q \in \mathrm{Z}^3_\alpha(Q, A)$, $d_c(\cdot, q, r) \in \mathrm{Z}^1_\theta(\mathbb{R}, A)$ *satisfying* $\partial_Q(d_c) = \partial_\theta(c_Q)$.

We will concentrate on the subgroup $\mathrm{Z}^3_{\alpha,\mathrm{s}}(\widetilde{Q}, A)$ of all standard cocycles in $\mathrm{Z}^3_\alpha(\widetilde{Q}, A)$. The index "s" stands for "standard". We then set

$$\mathrm{H}^3_{\alpha,\mathrm{s}}(\widetilde{Q},A) = \mathrm{Z}^3_{\alpha,\mathrm{s}}(\widetilde{Q},A)/\partial_{\widetilde{Q}}(\mathrm{C}^2_\alpha(Q,A)).$$

The coboundary group $\mathrm{B}^3_{\alpha,\mathrm{s}}(\widetilde{Q}, A) = \partial_{\widetilde{Q}}(\mathrm{C}^2_\alpha(Q, A))$ is a subgroup of the usual third coboundary group $\mathrm{B}^3_\alpha(\widetilde{Q}, A) = \partial_{\widetilde{Q}}(\mathrm{C}^2_\alpha(\widetilde{Q}, A))$.

The fixed cross-section $\mathfrak{s} : Q \mapsto G$ allows us to consider the fiber product

$$\mathrm{H}^3_{\alpha,\mathrm{s}}(\widetilde{Q},A) *_{\mathfrak{s}} \mathrm{Hom}_G(N,\mathrm{H}^1_\theta)$$

consisting of those pairs $([c], \nu) \in \mathrm{H}^3_{\alpha,\mathrm{s}}(\widetilde{Q}, A) \times \mathrm{Hom}_G(N, \mathrm{H}^1_\theta)$ such that

$$[d_c(\cdot,q,r)] = \nu(\mathfrak{n}_N(q,r)) \quad \text{in } \mathrm{H}^1_\theta, \quad q,r \in Q$$

where $\mathfrak{s}(q)\mathfrak{s}(r) = \mathfrak{n}_N(q, r)\mathfrak{s}(qr)$. The group $\mathrm{H}^3_{\alpha,\mathrm{s}}(\widetilde{Q}, A) *_{\mathfrak{s}} \mathrm{Hom}_G(N, \mathrm{H}^1_\theta)$ will be denoted by $\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, A)$ for short. The suffix "$\mathfrak{s}$" is placed to indicate that this fiber product depends heavily on the cocycle $\mathfrak{n}_N$ hence on the cross-section $\mathfrak{s}$.

Definition 2.3. *The* HJR*-map* δ_{HJR} *for a pair* (H, L) *with* $Q = H/L$ *induces a map* $\delta : \Lambda(\widetilde{H}, L, M, A) \longmapsto \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, A)$ *by*

$$\delta([\lambda,\ \mu]) = (c_{[\lambda,\mu]}, \nu_{[\lambda,\mu]})$$

where $\delta_{\mathrm{HJR}}([\lambda, \mu]) = c_{[\lambda,\mu]}$ *and* $\nu_{[\lambda,\mu]}(n) = [\lambda(\ell, \cdot)]$ *for some* ℓ *with* $\pi_G(\ell) = n$ *and the quotient map* $\pi_G : H \longmapsto G$.

For $[(c, \nu)] \in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, A)$, it is of the standard form

$$c(\tilde{p},\tilde{q},\tilde{r}) = c_Q(p,q,r)\alpha_p(d_c(s;q,r))$$

and take $\zeta_\nu(\cdot, n) \in Z^1_\theta(\mathbb{R}, A)$ with $\nu(n) = [\zeta_\nu(\cdot, n)]$ in $H^1(\mathbb{R}, A)$. By the definition of the fiber product $H^3_{\alpha,\mathfrak{s}}(\tilde{Q}, A) *_{\mathfrak{s}} \mathrm{Hom}_G(N, H^1_\theta)$, there is some $f \in C^2_\alpha(Q, A)$ such that

$$d_c(s; q, r) = \theta_s(f(q,r))f(q,r)^*\zeta_\nu(s; \mathfrak{n}_N(q,r)), \quad q, r \in Q, s \in \mathbb{R}.$$

The function $(g, h) \in G \times G \longmapsto \mathfrak{n}_N(\pi(g), \pi(h)) \in N$ is coboundary with $n_N(g) \in N$ where $n_N(g) = \mathfrak{s}(\pi(g))g^{-1}$ and π is the quotient map : $G \longmapsto Q$. Hence there exists $a(g, h) \in A$ for $g, h \in G$ such that

$$\begin{aligned}\zeta_\nu(s; \mathfrak{n}_N(\pi(g), \pi(h))) = \theta_s(a(g,h))a(g,h)^*\zeta_\nu(s; n_N(g))\\ \times \alpha_g(\zeta_\nu(s; n_N(h)))\zeta_\nu(s; n_N(gh))^*.\end{aligned}$$

So that the ergodicity of the flow θ can yields that

$$\pi^*(c_Q)\partial_G(\pi^*(f)a)^* \in Z^3(G, \mathbb{T}).$$

Definition 2.4. *The map* $\partial : H^{\mathrm{out}}_{\alpha.\mathfrak{s}}(G \times \mathbb{R}, N, A) \longmapsto H^3(G, \mathbb{T})$ *is defined by*

$$\partial([c]) = [\pi^*(c_Q)\partial_G(\pi^*(f)a)^*],$$

and the map Inf *is just the composition* $\mathrm{inf}\circ\partial$ *of* inf *and* ∂; *where* inf *is the inflation map of* $H^3(G, \mathbb{T})$ *to* $H^3(H, \mathbb{T})$.

We can show that those maps yields an exact sequence as follows.

Theorem 2.5 [Modified JHR-Exact Sequence]. *Suppose that* $\{\mathcal{C}, \mathbb{R}, \theta\}$ *is an ergodic flow and a homomorphism* α: $g \in H \mapsto \alpha_g \in \mathrm{Aut}_\theta(\mathcal{C})$, *the group of automorphisms of* $\mathcal{C}$ *commuting with* θ. *Assume the following*:

i) *a pair of normal subgroup* $M \subset L \subset H$ *is given*;
ii) *the subgroup* L, *hence* M *as well, acts trivially on* $\mathcal{C}$, *i.e.*, $L \subset \mathrm{Ker}(\alpha)$;
iii) *with* $G = H/M, N = L/M$ *and* $Q = H/L$, *let* $\pi_G : H \mapsto G$, $\pi : G \mapsto Q$ *and* $\dot{\pi} = \pi\circ\pi_G : H \mapsto Q$ *be the quotient maps such that*

$$\mathrm{Ker}(\pi_G) = M, \quad \mathrm{Ker}(\pi) = N \quad \textit{and} \quad \mathrm{Ker}(\dot{\pi}) = L;$$

iv) *Fix a cross-section* $\mathfrak{s}$ $Q \mapsto G$ *of the map* π *and choose cross-sections* $\mathfrak{s}_H$: $G \mapsto H$ *and* $\dot{\mathfrak{s}}$ $Q \mapsto H$ *in such a way that* $\dot{\mathfrak{s}} = \mathfrak{s}_H\circ\mathfrak{s}$.

Set $\widetilde{H} = H \times \mathbb{R}, \widetilde{G} = G \times \mathbb{R}$ *and* $\widetilde{Q} = Q \times \mathbb{R}$. *Let* A *denote the unitary group* $\mathcal{U}(\mathcal{C})$ *of* $\mathcal{C}$ *and* H^1_θ *be the first cohomology group* $H^1_\theta(\mathbb{R}, A)$ *of the ergodic flow.*

Under the above setting, there is a natural exact sequence which sits next to the Huebshmann-Jones-Ratcliffe *exact sequence*:

$$\begin{array}{ccccccc} \mathrm{H}^2(H,\mathbb{T}) & \xrightarrow{\mathrm{Res}} & \Lambda(\widetilde{H},L,M,A) & \xrightarrow{\delta} & \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G\times\mathbb{R},N,A) & \xrightarrow{\mathrm{Inf}} & \mathrm{H}^3(H,\mathbb{T}) \\ \| & & \downarrow & & \partial\downarrow & & \| \\ \mathrm{H}^2(H,\mathbb{T}) & \xrightarrow{\mathrm{res}} & \Lambda_\alpha(H,M,\mathbb{T}) & \xrightarrow{\delta_{\mathrm{HJR}}} & \mathrm{H}^3(G,\mathbb{T}) & \xrightarrow{\mathrm{inf}} & \mathrm{H}^3(H,\mathbb{T}) \end{array}$$

We prove the following lemma in the same way as in [J1] by using dimension shifting theorem in cohomology theory.

Lemma 2.6. *Let A denote the unitary group $\mathcal{U}(\mathcal{C})$ of an abelian separable von Neumann algebra $\mathcal{C}$ or the torus group $\mathbb{T}$. Let α be an action of a countable discrete group G on $\mathcal{C}$. To each $c\in \mathrm{Z}^3_\alpha(G,A)$, there corresponds a countable group $H=H(c)$ and a normal subgroup $M=M(c)$ such that:*

i) *the group G is identified with the quotient group H/M;*
ii) *there corresponds a characteristic cocycle*

$$(\lambda,\mu)=(\lambda_c,\mu_c)\in \mathrm{Z}_\alpha(H,M,A)$$

such that

$$[c]=\delta_{\mathrm{HJR}}([\lambda,\mu])$$

in the HJR-*exact sequence relative to $\{H,M,A\}$;*

iii) *the group M is abelian.*

Definition 2.7.. *We call the group $H(c)$ the resolution group of the cocycle $c\in \mathrm{Z}^3_\alpha(G,A)$ and the characteristic cocycle $(\lambda_c,\mu_c)\in \mathrm{Z}_\alpha(H,M,A)$ a resolution of the cocycle c. We also call the map $\pi_G: H(c)\mapsto G$ resolution map and the pair $\{H(c),\pi_G\}$ a resolution system.*

Corollary 2.8. *Let $\{\mathcal{C},\mathbb{R},\theta\}$ be an ergodic flow and G a discrete countable group acting on the flow $\{\mathcal{C},\mathbb{R},\theta\}$ via α. Let N be a normal subgroup of G such that $N\subset \mathrm{Ker}(\alpha)$. Then with $Q=G/N$ the quotient group of G by N and $\mathfrak{s}: Q\mapsto G$ a cross-section of the quotient map $\pi: G\mapsto Q$, for any pair*

$$([c],\nu)\in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G\times\mathbb{R},N,A)$$

there exist a countable discrete group H and a surjective homomorphism $\pi_G: H\mapsto G$ and $\chi\in\Lambda_{\pi_G^*(\alpha)}(H\times\mathbb{R},L,M,A)$ such that

$$([c],\nu)=\delta(\chi)$$

where $L = \pi_G^{-1}(N)$, $M = \mathrm{Ker}(\pi_G)$ *and* δ *is the modified* HJR-*map in* Theorem 2.5 *associated with the exact sequence*:

$$1 \longrightarrow M \longrightarrow L \xrightarrow{\pi_G} G \underset{\overleftarrow{\;s\;}}{\xrightarrow{\pi}} Q \longrightarrow 1.$$

Moreover, the kernel $M = \mathrm{Ker}(\pi_G)$ *is chosen to be abelian. Hence if* G *is amenable in addition, then* H *is amenable.*

§3. Outer Actions of a Discrete Group on a Factor

Let $\mathcal{M}$ be a separable factor. Associated with $\mathcal{M}$ is the characteristic square:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & A & \xrightarrow{\partial_\theta} & \mathrm{B}^1_\theta(\mathbb{R}, A) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{M}) & \longrightarrow & \widetilde{\mathcal{U}}(\mathcal{M}) & \xrightarrow{\partial_\theta} & \mathrm{Z}^1_\theta(\mathbb{R}, A) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Ad}} & & \Big\downarrow{\scriptstyle \widetilde{\mathrm{Ad}}} & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{M}) & \longrightarrow & \mathrm{Cnt_r}(\mathcal{M}) & \xrightarrow{\dot{\partial}_\theta} & \mathrm{H}^1_\theta(\mathbb{R}, A) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $A = \mathcal{U}(\mathcal{C})$ is the unitary group of the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ of weights on $\mathcal{M}$, which is $\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}$-equivariant. Applying the previous section to the groups

$$H = \mathrm{Aut}(\mathcal{M}), M = \mathrm{Int}(\mathcal{M}), G = \mathrm{Out}(\mathcal{M}), N = \mathrm{H}^1_\theta(\mathbb{R}, A)$$
$$Q = \mathrm{Out}_{\tau,\theta}(\widetilde{\mathcal{M}}) = \mathrm{Out}(\mathcal{M})/\mathrm{H}^1_\theta(\mathbb{R}, A) = \mathrm{Aut}(\mathcal{M})/\mathrm{Cnt_r}(\mathcal{M}),$$

we obtain the intrinsic invariant and the intrinsic modular obstruction:

$$\Theta(\mathcal{M}) \in \Lambda_{\mathrm{mod}\times\theta}(\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}, \mathrm{Cnt_r}(\mathcal{M}), A);$$
$$\mathrm{Ob_m}(\mathcal{M}) := \delta(\Theta(\mathcal{M})) \in \mathrm{H}^{\mathrm{out}}_{\mathrm{mod}\times\theta,\mathrm{s}}(\mathrm{Out}(\mathcal{M}) \times \mathbb{R}, \mathrm{H}^1_\theta(\mathbb{R}, A), A).$$

Choosing a cross-section $g \in \mathrm{Out}(\mathcal{M}) \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$, we obtain $\{u(g,h) \in \mathcal{U}(\mathcal{M}) : g, h \in \mathrm{Out}(\mathcal{M})\}$ such that

$$\alpha_g \circ \alpha_h = \mathrm{Ad}(u(g,h))\alpha_{gh}, \quad g, h \in \mathrm{Out}(\mathcal{M}),$$

we obtain a 3-cocycle $c \in \mathrm{Z}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T})$:

$$c(g,h,k) = \alpha_g(u(h,k))u(g,hk)\{u(g,h)u(gh,k)\}^*, \quad g,h,k \in \mathrm{Out}(\mathcal{M}).$$

Its cohomology class $[c] \in \mathrm{H}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T})$ does not depend on the choice of the cross-section $\alpha : g \in \mathrm{Out}(\mathcal{M}) \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ nor on the choice of $\{u(g,h)\}$. The *intrinsic obstruction* $\mathrm{Ob}(\mathcal{M}) = [c]$ of $\mathcal{M}$ is, by definition the cohomology class $[c] \in \mathrm{H}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T})$.

The following proposition states that the intrinsic obstruction, which is the invariant considered in [J1,Ocn], is reduced from the intrinsic modular invariant.

Proposition 3.1. *The intrinsic obstruction* $\mathrm{Ob}(\mathcal{M})$ *of the factor* $\mathcal{M}$ *is the image* $\partial(\mathrm{Ob_m}(\mathcal{M}))$ *of the intrinsic modular obstruction* $\mathrm{Ob_m}(\mathcal{M})$ *of* $\mathcal{M}$ *under the map*

$$\partial : \mathrm{H}^{\mathrm{out}}_{\mathrm{mod}\times\theta,\mathrm{s}}(\mathrm{Out}(\mathcal{M}) \times \mathbb{R}, \mathrm{H}^1_\theta, A) \mapsto \mathrm{H}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T}).$$

Let α be an outer action of G on $\mathcal{M}$ of type III. Let $[c^\alpha] \in \mathrm{H}^3(G, \mathbb{T})$ be the obstruction $\mathrm{Ob}(\alpha)$ and $c^\alpha \in \mathrm{Z}^3(G, \mathbb{T})$ represent $[c^\alpha]$ which is obtained by fixing a family $\{u(g,h) \in \mathcal{U}(\mathcal{M}) : g,h \in G\}$ such that

$$\alpha_g \circ \alpha_h = \mathrm{Ad}(u(g,h)) \circ \alpha_{gh}, \quad g,h \in G,$$

and by setting

$$c^\alpha(g,h,k) = \alpha_g(u(h,k))u(g,hk)\{u(g,h)u(gh,k)\}^* \in \mathbb{T}, \quad g,h,k \in G.$$

Then we have the obstruction $\mathrm{Ob}(\alpha) = [c^\alpha] \in \mathrm{H}^3(G, \mathbb{T})$ of α.

For the modular obstruction of α, choose $\{w(p,q) : p,q \in Q\} \subset \widetilde{\mathcal{U}}(\mathcal{M})$ so that

$$\alpha_p \circ \alpha_q = \widetilde{\mathrm{Ad}}(w(p,q)) \circ \alpha_{pq}, \quad p,q \in Q,$$

where α_p means $\tilde{\alpha}_{s(p)}$ and s is a section of the quotient map: $Q \longmapsto G$. We write $\alpha_{\tilde{p}} \in \mathrm{Aut}(\widetilde{\mathcal{M}})$ for $\alpha_p \circ \theta_s, \tilde{p} = (p,s) \in \widetilde{Q} = Q \times \mathbb{R}$. Then for each triple $\tilde{p} = (p,s), \tilde{q} = (q,t), \tilde{r} = (r,u) \in \widetilde{Q}$, the cocycle $c = c^\alpha$ representing $\mathrm{Ob_m}(\alpha)$ is given by:

$$\begin{aligned} c^\alpha(\tilde{p},\tilde{q},\tilde{r}) &= \alpha_{\tilde{p}}(w(q,r))w(p,qr)\{w(p,q)w(pq,r)\}^* \\ &= \alpha_p(\theta_s(w(q,r))w(q,r)^*)\alpha_p(w(q,r))w(p,qr)\{w(p,q)w(pq,r)\}^* \\ &= \alpha_p(d(s;q,r))c_Q(p,q,r), \end{aligned}$$

where

$$d(s;q,r) = \theta_s(w(q,r))w(q,r)^*,$$
$$c_Q(p,q,r) = \alpha_p(w(q,r))w(p,qr)\{w(p,q)w(pq,r)\}^*.$$

The G-equivariant homomorphism $\nu : N \mapsto \mathrm{H}^1_\theta(\mathbb{R}, A)$ is given by $\nu_\alpha(m) = \dot\partial_\theta(\alpha_m) \in \mathrm{H}^1_\theta(\mathbb{R}, A)$, $m \in N$. Then we obtain the modular obstruction $\mathrm{Ob_m}(\alpha)$ of α;

$$\mathrm{Ob_m}(\alpha) = \left([c_Q(p,q,r)\alpha_p(d(s;q,r))],\ \nu_\alpha\right) \in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, A).$$

where $\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, A) = \mathrm{H}^3_{\mathrm{mod}(\alpha)\times\theta,\ \mathfrak{s}}(\tilde{Q}, A) *_{\mathfrak{s}} \mathrm{Hom}_G(N, \mathrm{H}^1_\theta(\mathbb{R}, A))$.

We come to state the main theorem as follows:

Theorem 3.2 [Outer Conjugacy]. *Let G be a countable discrete group and $\mathcal{M}$ a separable infinite factor with flow of weights $\{\mathcal{C}, \mathbb{R}, \theta\}$. Suppose that $\alpha : g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ is an outer action of G on $\mathcal{M}$. Set $N = N(\alpha) = \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$, $Q = G/N$ and fix a cross-section $\mathfrak{s} : Q \mapsto G$ of the quotient map $\pi : G \mapsto Q$.*

i) *The modular obstruction:*

$$\mathrm{Ob_m}(\alpha) \in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, A)$$

is an invariant for the outer conjugacy class of α.

ii) *If $\mathcal{M}$ is an approximately finite dimensional factor and G is amenable, then the triplet $(N(\alpha), \mathrm{mod}(\alpha), \mathrm{Ob_m}(\alpha)\}$ is a complete invariant of the outer conjugacy class of α in the sense that if $\beta : G \mapsto \mathrm{Aut}(\mathcal{M})$ is another outer action of G on $\mathcal{M}$ such that $N(\alpha) = N(\beta)$, and there exists an automorphism $\sigma \in \mathrm{Aut}_\theta(\mathcal{C})$ with*

$$\sigma\circ\mathrm{mod}(\alpha_g)\circ\sigma^{-1} = \mathrm{mod}(\beta_g), \quad g \in G; \quad \sigma_*(\mathrm{Ob_m}(\alpha)) = \mathrm{Ob_m}(\beta),$$

then the automorphism σ of $\mathcal{C}$ can be extended to an automorphism denoted by σ again to the non-commutative flow of weights $\{\widetilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$ such that

$$\sigma\circ\alpha_g\circ\sigma^{-1} \equiv \beta_g \mod \mathrm{Int}(\mathcal{M}), \quad g \in G.$$

[Sketch of proof] Let $\pi_G : H = H(c^\alpha) \mapsto G$ be the resolution group of the cocycle $c^\alpha \in \mathrm{Z}^3(G, \mathbb{T})$ whose class is the obstruction of α and the resolution map, i.e., $\pi_G^*(c^\alpha) \in \mathrm{B}^3(H, \mathbb{T})$. Choose $b : (g,h) \in H \times H \mapsto b(g,h) \in \mathbb{T}$ such that

$$c^\alpha(\pi_G(g), \pi_G(h), \pi_G(k)) = b(h,k)b(g,hk)\{b(g,h)b(gh,k)\}^*, \quad g,h,k \in H.$$

$$\bar{u}_H(g,h) = b(g,h)^* u(\pi_G(g), \pi_G(h)), \quad g,h \in H,$$

we obtain

$$\alpha_{\pi_G(g)}(\bar{u}_H(h,k))\bar{u}_H(g,hk)\{\bar{u}_H(g,h)\bar{u}_H(gh,k)\}^* = 1.$$

Hence $\{\alpha_{\pi_G}, \bar{u}_H\}$ is a cocycle twisted action of H. Then by [St1: Theorem 4.13, page 156], there exits a family $\{v_H(g) \in \mathcal{U}(\mathcal{M}) : g \in H\}$ such that

$$\bar{u}_H(g,h) = \alpha_{\pi_G(g)}(v_H(h)^*)v_H(g)^*v_H(gh), \quad g,h \in H,$$

so that the map

$$g \in H \mapsto \bar{\alpha}_g = \mathrm{Ad}(v_H(g)) \circ \alpha_{\pi_G(g)} \in \mathrm{Aut}(\mathcal{M})$$

is an action of H on $\mathcal{M}$. It is shown that the characteristic invariant $[\lambda_{\bar{\alpha}}, \ \mu_\alpha]$ for $\bar{\alpha}$ is an element in $\Lambda_\alpha(H, L, M, A)$ such that

$$\delta([\lambda_{\bar{\alpha}}, \ \mu_\alpha]) = \mathrm{Ob_m}(\alpha)$$

where $L = \pi_G{}^{-1}(N)$, $M = \mathrm{Ker}\ \pi_G$.

Let β be another outer action of G on $\mathcal{M}$ with the conditions in (ii). The action $\bar{\beta}$ induced by β as above has a characteristic invariant $[\lambda_{\bar{\beta}}, \ \mu_\beta]$ such that

$$\delta([\lambda_{\bar{\beta}}, \ \mu_\beta]) = \mathrm{Ob_m}(\beta).$$

Since we have only to prove the outer conjugacy of α and β in the case of $\mathrm{Ob_m}(\alpha) = \mathrm{Ob_m}(\beta)$, it follows from Theorem 2.5 that $[\lambda_{\bar{\alpha}}, \ \mu_\alpha][\lambda_{\bar{\beta}}, \ \mu_\beta]^{-1}$ is the image $\mathrm{Res}[\mu]$ of $\mu \in \mathrm{H}^2(H, \mathbb{T})$. Let u be a projective unitary representation of H in $\mathcal{M}$ with 2-cocycle μ in $\mathrm{Z}^2(H, \mathbb{T})$. The action $\bar{\alpha} \otimes \mathrm{Ad}u$ of H on $\mathcal{M} \otimes \mathcal{L}(L^2(H))$ has the same characteristic invariant of $\bar{\beta}$. $\bar{\alpha} \otimes \mathrm{Ad}u$ is cocycle conjugate to $\bar{\beta}$ by [KtST]. Since $\bar{\alpha} \otimes \mathrm{Ad}u$ and $\bar{\beta}$ are the inner perturbation of each, we conclude that α and β is outer conjugate.

Theorem 3.3 [Model Construction]. *Let G be a countable discrete amenable group and N a normal subgroup. Let $\{\mathcal{C}, \mathbb{R}, \theta\}$ be an ergodic flow and α an action of G on the flow $\{\mathcal{C}, \mathbb{R}, \theta\}$ with* $\mathrm{Ker}(\alpha) \supset N$, *i.e., α is a homomorphism of G into the group $\mathrm{Aut}_\theta(\mathcal{C})$ of automorphisms commuting with the flow θ with $\alpha_m = \mathrm{id}, m \in N$. Let A denote the unitary group $\mathcal{U}(\mathcal{C})$.*

For every modular obstruction cocycle $(c,\zeta) \in \mathrm{Z}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G\times\mathbb{R},N,A)$, *there exists an amenable resolution system* $\{H,L,M,\pi_G,\lambda,\mu\}$ *with* $(\lambda,\mu)\in \mathrm{Z}_\alpha(\widetilde{H}, L,M,A)$ *and a cross-section* $\mathfrak{s}_H : G\mapsto H$ *of the map* π_G *such that*

$$\delta_{\mathfrak{s}_H}(\lambda,\mu)\equiv(c,\zeta) \mod \mathrm{B}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G\times\mathbb{R},N,A).$$

Consequently, the action $\alpha^{\lambda,\mu}$ *associated with the characteristic cocycle* (λ,μ) *gives an outer action* $\dot{\alpha}^{\lambda,\mu}=\alpha_{\mathfrak{s}_H}$ *of* G *on the approximately finite dimensional factor* $\mathcal{M}$ *with flow of weights* $\{\mathcal{C},\mathbb{R},\theta\}$ *such that*

$$\mathrm{Ob}_{\mathrm{m}}(\dot{\alpha}^{\lambda,\mu})=([c],[\zeta])\in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G\times\mathbb{R},N,A).$$

Remark 3.4. *If* $\mathcal{M}$ *is of type* III_1, *then* $\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G\times\mathbb{R},N,A)$ *is isomorphic to* $\simeq \mathrm{H}^3(G,\mathbb{T})$. *If* $\mathcal{M}$ *is of type* III_1, *then* $\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G\times\mathbb{R},N,A)$ *is also isomorphic to* $\mathrm{H}^3(Q,\mathbb{T})\times\mathrm{Hom}_G(N,\mathbb{R})$.

The other casees are relatively complicate, especially the case of type III_0.

Note that third cohomology

$$\mathrm{H}^3(\mathbb{Z},\mathbb{T})=\mathrm{H}^3(\mathbb{Z}^2,\mathbb{T})=\{1\}$$

and

$$\mathrm{H}^3(\mathbb{Z}^3,\mathbb{T})\simeq\{c(g,k,k)=\lambda^{l,m',n''} : \lambda\in\mathbb{T}\}$$

for

$$g=(l,m,n),\ h=(l',m',n'),k=(l'',m'',n'')\in\mathbb{Z}^3.$$

Example 3.5. *Let* α, β, γ *be automorphisms which induces the* Z^3*-outer action on* $\mathcal{M}$ *with the normal subgroup* $H=\{e\}$.

Then there is $\lambda\in\mathbb{T}$:

$$\alpha(u)v^*\beta(w)=\lambda w\gamma(v^*)u$$

where

$$\gamma\beta=\mathrm{Ad}u\circ\beta\gamma,\quad \alpha\gamma=\mathrm{Ad}w\circ\gamma\alpha,$$
$$\beta\alpha=\mathrm{Ad}v\circ\alpha\beta$$

This λ *induces the 3-cocycle* $\lambda^{l,m',n''}$ *and the* λ *is invariant, up to outer conjugacy, for the* $\mathbb{Z}^3$*-outer action with* $N=\{e\}$.

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