数理解析研究所講究録 1332 巻 2003 年 57-70

Quantum random walks and their boundaries

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Introduction

Random walks form an important part of classical probability theory [26, 28] and have remarkable applications to group theory, geometry and rigidity theory [16, 15, 7, 25]. Various results of the corresponding non-commutative theory can be traced back to the 70s. Notwithstanding the vast literature on quantum Markov processes and semigroups, there are important applications of random walks to subfactor theory [21, 22, 1, 10] and to product-type actions of compact groups [8]. In the early 90s Biane showed in a series of interesting papers [2, 3, 4, 5] that some of the most fundamental results for random walks on $\mathbb{Z}^d$ have analogues for duals of compact Lie groups. While it was known that the center of an algebra can often be interpreted as the Poisson boundary of a classical random walk, the boundary theory in a genuine non-commutative setting did not receive any attention until the recent works of Izumi on the Poisson boundary [11, 12]. He observed that the algebras themselves can be regarded as boundaries of certain quantum random walks. This point of view gives a convenient framework to study concrete examples, apply classical tools and look for their non-commutative analogues. In the present note we discuss a related work by the authors on the Martin boundary theory of discrete quantum groups [18]. It is worth stressing that though the theory is applicable to duals of compact Lie groups as studied by Biane, really interesting non-commutative phenomena are observed only for genuine quantum groups, e.g. for duals of q-deformations of semisimple compact Lie groups with $q \in (0,1)$.

This note is based on the talk given by the first author at the Symposium "Analysis of (Quantum) Group Actions on Operator Algebras", January 27–29, 2003, Kyoto.

1 The Martin boundary in analysis

We begin by recalling that the Dirichlet problem for a bounded domain Ω in $\mathbb{R}^n$ asks for a solution u of the equation

$$\Delta u = f, \quad u|_{\partial\Omega} = \phi,$$

for given functions f on Ω and ϕ on $\partial\Omega$. If the boundary $\partial\Omega$ and the functions ϕ and f are sufficiently regular, the problem is solved using the Green function G, which is a function $G(x,y)$ in two variables x and y that satisfies

$$\Delta G(x,\cdot) = \delta_x \text{ and } G(x,\cdot)|_{\partial\Omega} = 0$$

for all $x \in \Omega$, see e.g. [14]. In particular, a continuous function u on $\bar{\Omega}$ which is harmonic on Ω is determined by its values on the boundary according to the formula

$$u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial n_y}(x,y)d\mu(y) \tag{1.1}$$

for all $x \in \Omega$, where $d\mu(y) = u(y)dS(y)$ and n_y is the normal unit vector at the point y of the boundary $\partial\Omega$. More generally, for any positive harmonic function on Ω, there exists a measure μ such that the above formula holds. For the unit disc in $\mathbb{R}^2$ we get the usual Poisson formula with

$$\frac{\partial G}{\partial n_y}(x,y) = \frac{1-|x|^2}{2\pi|y-x|^2}.$$

It is desirable to have a representation formula analogous to (1.1) also in the case when the boundary is not regular. The problem was solved by Martin [17], who constructed an ideal boundary of Ω by looking at the asymptotic properties of the Green function. Assume that the Green function exists, fix $x_0 \in \Omega$ and consider the Martin kernel

$$K(x,y) = \frac{G(x,y)}{G(x_0,y)}.$$

If the boundary is regular, this function can be used in (1.1) instead of $\frac{\partial G}{\partial n_y}(x,y)$ as $K(x,y) = \frac{\partial G}{\partial n_y}(x,y)\frac{\partial G}{\partial n_y}(x_0,y)^{-1}$ for $y \in \partial\Omega$ by l'Hospital's rule. In the general case one considers the compactification Ω_M of Ω such that a sequence $\{y_n\}_{n=1}^{\infty}$ in Ω converges to an element in $\partial_M\Omega = \Omega_M \backslash \Omega$, if it eventually leaves any compact subset of Ω and the sequence $\{K(x,y_n)\}_{n=1}^{\infty}$ is uniformly convergent on compact subsets of Ω. Then $\partial_M\Omega$ is called the Martin boundary of Ω and provides a representation theorem stating that for any positive harmonic function u on Ω, there exists a measure μ on $\partial_M\Omega$ such that

$$u(x) = \int_{\partial_M\Omega} K(x,y)d\mu(y)$$

for any $x \in \Omega$.

2 Doob's probabilistic analogue

Suppose X is a discrete set. Let $\{p(x,y)\}_{x,y\in X}$ be a transition probability, i.e. $\sum_y p(x,y) = 1$ and $p(x,y) \geq 0$. We are particularly interested in the case when X is a discrete group and $p(x,y) = \mu(xy^{-1})$ for a probability measure μ on X. We will always suppose that the random walk is irreducible, that is, the probability of reaching any given point from another point is non-zero. In other words, for any x and y we have $p^{(n)}(x,y) > 0$ for some $n \in \mathbb{N}$, where $p^{(n)}(x,y)$ is defined by induction as $p^{(0)}(x,y) = \delta_{x,y}$ and $p^{(n)}(x,y) = \sum_{z\in X} p^{(n-1)}(x,z)p(z,y)$. We will also suppose that the random walk is transient, that is, a random path leaves eventually with probability 1 any finite subset of X. Equivalently, the expected number $g(x,y) = \sum_{n=0}^{\infty} p^{(n)}(x,y)$ of visits of a point y from a point x is finite. We will discuss this condition in more detail later.

Consider the corresponding Markov operator P on functions on X given by

$$(Pf)(x) = \sum_y p(x,y)f(y).$$

It is known that $\iota - P$ can be regarded as a discrete analogue of the Laplace operator, see e.g [28]. Thus it makes sense to say that a function f on X is harmonic if $Pf = f$. Consider the adjoint operator P^* with respect to the counting measure, so

$$(P^*f)(x) = \sum_y p(y,x)f(y).$$

Then the function $G(x,\cdot) = \sum_{n=0}^{\infty}(P^*)^n\delta_x$ is a discrete analogue of the Green function and fulfills $(\iota - P^*)G(x,\cdot) = \delta_x$. As before, fix $x_0 \in X$ and set

$$K(x,y) = \frac{G(x,y)}{G(x_0,y)}.$$

The Martin compactification X_M of X is the minimal compactification for which all the functions $y \mapsto K(x,y)$, $x \in X$, are continuous, and the Martin boundary is $\partial_M X = X_M \backslash X$. For any harmonic function f on X there exists a measure μ_f on $\partial_M X$ such that

$$f(x) = \int_{\partial_M X} K(x,y)d\mu_f(y).$$

Even though the measure μ_f is not unique, there exists a canonical one. Let μ_1 be the canonical measure representing the unit function on X. Then the Poisson boundary is by definition the measure space $(\partial_M X, \mu_1)$. It turns out, that any bounded harmonic function f on X extends to a continuous function on the Martin compactification X_M, and the canonical measure μ_f on $\partial_M X$ is absolutely continuous with respect to μ_1 with Radon-Nikodym derivative $d\mu_f/d\mu_1 = f|_{\partial_M X}$. This means in particular, that the space of bounded harmonic functions on X is isomorphic to $L^\infty(\partial_M X, \mu_1)$.

The Poisson boundary can also be described as follows. Consider the space Ω of paths starting at x_0, and let $\mathbb{P}$ be the corresponding Markov measure on Ω given by

$$\mathbb{P}(\{\underline{y} \in \Omega \,|\, y_0 = x_0, \dots, y_n = x_n\}) = p(x_0,x_1)\dots p(x_{n-1},x_n).$$

Let $\pi_n\colon \Omega \to X$ be the nth coordinate function, and ξ_n be the partition of Ω defined by saying that two elements $\underline{x}$ and $\underline{y}$ belong to the same element of the partition if and only if $x_k = y_k$ for $k \le n$. Then a bounded function f on X is harmonic if and only if $\{f\pi_n\}_n$ is a martingale with respect to the sequence of partitions ξ_n. In particular, if f is harmonic, the sequence $\{f\pi_n\}_n$ converges a.e. to a function f_∞ in $L^\infty(\Omega, \mathbb{P})$. The functions f_∞ which one gets this way are precisely the functions measurable with respect to the partition ξ defined by saying that two elements $\underline{x}$ and $\underline{y}$ belong to the same element of the partition if and only if there exist $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $x_k = y_{k+m}$ for $k \ge n$. Thus the Poisson boundary is the quotient measure space Ω/ξ.

The Poisson boundary is generally easier to compute than the Martin boundary. For example, let us give a proof of the classical Choquet-Deny theorem [6, 28].

Theorem 2.1 (Choquet-Deny) *The Poisson boundary of any abelian group is trivial.*

Proof. Let μ be the measure defining our random walk, $p(x, y) = \mu(x - y)$. The path space $(\Omega, \mathbb{P})$ is isomorphic to the measure space $(\prod_{n=1}^{\infty} X, \prod_{n=1}^{\infty} \mu)$ under the map $\gamma: \Omega \to \prod_{n=1}^{\infty} X$ given by

$$\gamma(\underline{x}) = (-x_1, x_1 - x_2, x_2 - x_3, \ldots).$$

Then $(f\pi_n\gamma^{-1})(\underline{x}) = f(-x_1 - \ldots - x_n)$. It follows that $f_\infty\gamma^{-1}$ is invariant under the canonical action of the group S_∞ of finite transpositions on $\prod_{n=1}^{\infty} X$. Hence f_∞ is a constant, and f must be constant.

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On the other hand, the computation of the Martin boundary of the abelian group $\mathbb{Z}$ is already nontrivial. The answer $\partial_M\mathbb{Z} = \{-\infty, +\infty\}$, which says that $\mathbb{Z}_M$ is the natural two-point compactification of $\mathbb{Z}$, follows from the renewal theorem. Recall that this theorem asserts that if μ is a measure on $\mathbb{Z}$ such that

$$\sum_{n\in} |n|\mu(n) < \infty \quad \text{and} \quad \lambda = \sum_{n\in} n\mu(n) > 0,$$

then the function $g(n) = \sum_{k=0}^{\infty} \mu^{*k}(n)$ converges to λ^{-1} as $n \to +\infty$ and to 0 as $n \to -\infty$. Here μ^{*k} is the measure obtained as convolution powers of the measure μ, so the potential $G(x, y)$ equals $g(x - y)$.

More generally, one has the following result [19, 28].

Theorem 2.2 (Ney-Spitzer) *Suppose the random walk on $\mathbb{Z}^d$, $d \in \mathbb{N}$, is given by a finitely supported measure μ with non-zero mean, i.e. $\sum_{n\in\ ^d} n\mu(n) \neq 0$. Then the Martin boundary of $\mathbb{Z}^d$ is homomorphic to the sphere S^{d-1}.*

Here we think of $\mathbb{Z}^d$ as sitting inside the unit ball D^d under the embedding $x \mapsto (1 + ||x||)^{-1}x$.

Note also that the Martin boundary of $\mathbb{Z}^d$ with $d \geq 3$ corresponding to a measure with zero mean is trivial.

3 Markov operators in non-commutative probability

Considering von Neumann algebras as non-commutative analogues of measure spaces, one commonly regards unital normal completely positive maps on von Neumann algebras as Markov operators. Let $P: M \to M$ be such an operator. As explained above, the algebra of bounded measurable functions on the Poisson boundary is isomorphic to the space of bounded harmonic elements. So it is natural, as suggested by Izumi [11], to call

$$H^\infty(M, P) = \{x \in M \mid Px = x\}$$

the Poisson boundary of the pair (M, P). It is a von Neumann algebra under the Choi-Effros product

$$x \cdot y = \lim_{n\to\omega} \frac{1}{n} \sum_{k=0}^{n-1} P^k(xy),$$

where ω is an arbitrary free ultrafilter on $\mathbb{N}$.

Suppose now that M is a finite discrete von Neumann algebra, so we may think of it as the algebra of bounded functions on a discrete quantum set. Let M_0 be the ideal generated by finite projections in M. We would like to construct a non-commutative analogue of the Martin boundary in this setting. It should be a unital C*-algebra A_P satisfying the following minimal requirements:

(i) A_P is a boundary, meaning that A_P is a subalgebra of M/M_0;

(ii) for A_P there is a representation theorem in the sense that harmonic elements are represented by bounded linear functionals on A_P;

(iii) there is an isomorphism $\pi_\nu(A_P)'' \cong H^\infty(M,P)$, where ν is a state representing the unit of M and $\pi_\nu(A_P)''$ is the weak closure of A_P in the associated GNS-representation.

For the moment such a construction seems to be out of reach. Even a construction of a reasonable path space in non-commutative probability, which should be more straightforward, is not altogether trivial [24]. Indeed, the obvious candidate for the path space is the algebra $\otimes_{n=0}^{\infty} M$ with the linear functional $x_0 \otimes \ldots \otimes x_n \to \varepsilon(x_0 P(x_1 P(\ldots x_{n-1} P(x_n))))$, where ε is an initial distribution. However, such an expression only makes sense in the commutative case. In fact, in order to get a workable definition one should resort to free products rather than tensor products.

In the case when the quantum set is a discrete quantum group, the classical definitions are easier to adapt thanks to the additional symmetry present. So let Γ be a discrete quantum group. The algebra of bounded functions on Γ is a finite discrete von Neumann algebra $\hat{M} = \sum_{s\in I} \oplus B(H_s)$ with comultiplication $\hat{\Delta}\colon \hat{M} \to \hat{M} \otimes \hat{M}$ (we use non-hatted notations for the dual compact quantum group). We shall consider a special class of Markov operators given by convolution with states, that is, operators of the form $P_\phi = (\phi \otimes \iota)\hat{\Delta}$, where ϕ is a normal state. Moreover, we assume that ϕ belongs to the closure C of linear combinations of q-traces. This happens precisely when the center $Z(\hat{M})$ of $\hat{M}$ is invariant under P_ϕ.

Recalling the definition of the path space of a random walk on a group used in the proof of the Choquet-Deny theorem, one immediately gets the path space of the quantum random walk. It consists of a von Neumann algebra $\hat{M}^\infty$ and a normal state ϕ^∞ given by $\otimes_{-\infty}^{-1}(\hat{M}, \phi)$. Let $j_k\colon \hat{M} \to \hat{M}^\infty$ be the unital $*$-homomorphisms given by $j_k(x) = \ldots \otimes 1 \otimes \hat{\Delta}^{k-1}(x)$ for $k \geq 1$ and $x \in \hat{M}$, and $j_0 = \hat{\varepsilon}$, where $\hat{\varepsilon}$ is the counit. Here $\hat{\Delta}^k$ is defined inductively by $\hat{\Delta}^0 = \iota$, $\hat{\Delta}^1 = \hat{\Delta}$ and $\hat{\Delta}^{k+1} = (\hat{\Delta} \otimes \iota)\hat{\Delta}^k$. The elements $j_k(x)$, $x \in \hat{M}$, are analogues of $f\pi_k$. In particular [11] the map $\theta\colon H^\infty(\hat{M}, P_\phi) \to \hat{M}^\infty$ given by $\theta(x) = s^*\text{-}\lim_{n\to\infty} j_n(x)$ defines an embedding of the von Neumann algebra $H^\infty(\hat{M}, P_\phi)$ into $\hat{M}^\infty$.

4 The Martin boundary of a discrete quantum group

Keeping the notation of the previous section, let $\hat{\mathcal{A}}$ be the algebraic direct sum of $B(H_s)$, $s \in I$, now playing the role of finitely supported functions on the discrete quantum group Γ. Any state $\phi \in C$ provides a state $\check{\phi} \in C$ uniquely determined by the condition

$$\hat{\psi}(P_\phi(x)y) = \hat{\psi}(xP_{\check{\phi}}(y)) \quad \text{for} \quad x, y \in \hat{\mathcal{A}},$$

where $\hat{\psi}$ is the right-invariant Haar weight on $\hat{M}$.

Definition 4.1 *The Martin kernel for P_ϕ is the map $K_{\check{\phi}}\colon \hat{\mathcal{A}} \to \hat{M}$ given by*

$$K_{\check{\phi}}(x) = G_{\check{\phi}}(x)G_{\check{\phi}}(I_0)^{-1},$$

where $G_{\check{\phi}} = \sum_{n=0}^{\infty} P_{\check{\phi}}^n$ and $I_0 \in \hat{M}$ is the "delta-function at the unit of Γ".

The Martin compactification of Γ with respect to P_ϕ is the C^*-algebra $\tilde{A}_\phi$ generated by the image of $K_{\check{\phi}}$ and $\hat{\mathcal{A}}$. The Martin boundary A_ϕ is the quotient C^*-algebra of $\tilde{A}_\phi$ by the norm closure $\hat{A}$ of $\hat{\mathcal{A}}$.

As in the classical case, for the definition to make sense we have to assume irreducibility and transience of the random walk. By irreducibility we mean that the corresponding classical random walk on I is irreducible, equivalently, the state $\sum_{n=1}^{\infty} 2^{-n}\phi^n$ is faithful. In this case we also say that ϕ is generating. This condition ensures that the element $G_{\check{\phi}}(I_0)$ is invertible in the algebraic multiplier algebra $M(\hat{\mathcal{A}}) = \prod_{s\in I} B(H_s)$ of $\hat{\mathcal{A}}$. Analogously, by transience we mean transience of the corresponding classical random walk, or equivalently, that the series $\sum_{n=0}^{\infty} P_{\check{\phi}}^n(x)$ converges in $M(\hat{\mathcal{A}})$ for every $x \in \hat{\mathcal{A}}$. Note that in this case the series is, in fact, convergent in strong operator topology to an element of $\hat{M}$ by complete maximum principle, see [23].

In the classical case the transience requirement is fulfilled for most random walks. This is, however, non-trivial. Recall, in particular, that if μ is a measure on $\mathbb{Z}$ with finite first moment, $\sum_{n\in} |n|\mu(n) < \infty$, then the corresponding random walk is transient if and only if $\sum_{n\in} n\mu(n) \neq 0$; and any random walk on $\mathbb{Z}^d$ for $d \geq 3$ is transient. In fact, a recurrent (that is, non-transient) random walk on a finitely generated group exists if and only if the group contains a finite index subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$, and in this case any symmetric random walk with finite second moment is recurrent [27, 23].

However, transience is automatic for all generic discrete quantum groups. Recall that there exists a canonical positive group-like element $\rho \in M(\hat{\mathcal{A}})$ which implements the square of the antipode. The number $d_s = \mathrm{Tr}(\rho I_s) = \mathrm{Tr}(\rho^{-1}I_s)$ is called the quantum dimension of s, where I_s is the unit of $B(H_s)$.

Theorem 4.2 *Suppose* $\dim H_s < d_s$ *for at least one s with $\phi(I_s) > 0$. Then ϕ is transient. Moreover, if $p_\phi^{(n)}(s,t)$ is the transition probability of the corresponding classical random walk on I defined by $P_\phi^n(I_t)I_s = p_\phi^{(n)}(s,t)I_s$, then the sequence $\{p_\phi^{(n)}(s,t)\}_{n=0}^{\infty}$ decreases exponentially.*

This result is applicable to duals of q-deformations of semisimple compact Lie groups. It also implies that for non-Kac algebras any generating state is automatically transient. In fact, transience also holds for duals of semisimple compact Lie groups, but for different reasons.

Theorem 4.3 *Suppose Γ is the dual of a simply-connected semisimple compact Lie group G, so $\hat{M}$ is the von Neumann algebra of G. Let ϕ be a generating state in C. Then ϕ is transient.*

Proof. Let $T \subset G$ be a maximal torus. Then the von Neumann algebra $W^*(T)$ is a P_ϕ-invariant Hopf-von Neumann subalgebra of $\hat{M} = W^*(G)$. Thus we get a random walk on the dual group $\hat{T}$, which can be identified with the weight lattice of the Lie algebra of the group G. Fixing a Weyl chamber C_{++} with closure C_+ we can identify the set I of equivalence classes of irreducible representations of G with $C_+ \cap \hat{T}$. Denote by μ the measure defining the random walk on $\hat{T}$, in other words the measure corresponding to the state $\phi|_{W^*(T)}$. Then by [4, 5] we get

$$p_\phi(s,t) = \frac{\dim H_t}{\dim H_s} \sum_{w \in W} \det(w)\mu(\rho + s - w(\rho + t)),$$

where W is the Weyl group and ρ is the half sum of positive roots. Since the measure μ is symmetric, it is generally recurrent when the rank of G is ≤ 2, and it is transient when the rank is ≥ 3. However, in the case when the measure is recurrent, the series $\sum_n (\mu^{*n}(0) - \mu^{*n}(\omega))$ is nevertheless convergent for any $\omega \in \hat{T}$, see [26]. As $\sum_{w\in W} \det(w) = 0$, we conclude that the series

$$\sum_{n=0}^{\infty} p_\phi^{(n)}(s,t) = \frac{\dim H_t}{\dim H_s} \sum_{n=0}^{\infty} \sum_{w \in W} \det(w)(\mu^{*n}(\rho + s - w(\rho + t)) - \mu^{*n}(0))$$

is also convergent.

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However, one should not expect exponentially fast decreasing of return probabilities anymore. For example, if we consider the random walk corresponding to the character of the fundamental representation of $SU(2)$, then the probability of return to 0 at the nth step is given by the semicircular law, that is $p_\phi^{(n)}(0,0) = \frac{2}{\pi} \int_{-1}^{1} t^n \sqrt{1-t^2}dt$.

The comultiplication $\hat{\Delta}\colon \hat{M} \to \hat{M} \otimes \hat{M}$ is the right action of Γ on itself by translations, and induces a right action of Γ on the Martin boundary given by a homomorphism $A_\phi \to M(A_\phi \otimes \hat{A})$, which we again denote by $\hat{\Delta}$.

The algebra $\hat{M}$ considered as the von Neumann algebra of the dual compact quantum group G also carries the left adjoint action of G represented by a homomorphism $\Phi\colon \hat{M} \to M \otimes \hat{M}$. This action induces a left action of G on the Martin boundary. In the classical case this action is always trivial.

Similarly we get a left action of G and a right action of Γ on the Poisson boundary.

Recall also that given a KMS-state ν on a C*-algebra A we can define an inner product on A by $(x,y)_\nu = \nu(x\sigma^\nu_{-\frac{i}{2}}(y^*))$.

Now we can state the main theorem, which justifies our definition of the Martin boundary.

Theorem 4.4 *Retain the notation above. Then*

(i) *for any superharmonic element $x \in M(\hat{A})$ (so x is positive and $P_\phi(x) \leq x$), there exists a positive linear functional ω on $\tilde{A}_\phi$ such that $(y,x)_{\hat{\psi}} = \omega K_{\check{\phi}}(y)$ for any $y \in \hat{A}$;*

(ii) *conversely, for any positive linear functional on $\tilde{A}_\phi$, there exists a unique superharmonic element $x_\omega \in M(\hat{A})$ such that $(y,x_\omega)_{\hat{\psi}} = \omega K_{\check{\phi}}(y)$ for any $y \in \hat{A}$; if x_ω is harmonic then $\omega|_{\hat{A}} = 0$;*

(iii) *if* ν *is a weak* limit point of* $\{\phi^n|_{\tilde{A}_\phi}\}_{n=1}^{\infty}$, *then* ν *is a* γ*-KMS state representing the unit, where* γ *is the dynamics obtained by restricting the modular group of* $\hat{\psi}$ *to* $\tilde{A}_\phi$;

(iv) *if the Martin kernel considered as a map from* $\hat{A}$ *to* A_ϕ *has dense range, then the state* ν *on* A_ϕ *is unique and the dual map*

$$K_{\check{\phi}}^*\colon \pi_\nu(A_\phi)'' \to H^\infty(M, P_\phi)$$

is an isomorphism which respects the actions of the dual compact quantum group G, *where*

$$(K_{\check{\phi}}^*(x), y)_{\hat{\psi}} = (x, K_{\check{\phi}}(y))_\nu$$

for $y \in \hat{A}$ *and* $x \in \pi_\nu(A_\phi)''$.

The key part of Theorem above is part (i). It is proved essentially in the same way as in the classical case by approximating superharmonic elements by potentials, that is, elements of the form $G_\phi(y)$ with $y \in \hat{A}_+$. If $x = G_\phi(y)$ is a potential, we can take $(\cdot, G_{\check{\phi}}(I_0)y)_{\hat{\psi}}$ as the linear functional representing x. Then a functional representing a general superharmonic element is obtained as a weak* limit point of functionals representing potentials. In the classical situation one usually proves that superharmonic elements can be approximated by potentials by using the lattice property of superharmonic elements. Though in our non-commutative situation self-adjoint elements do not form a lattice, quite surprisingly one can still prove that certain sets have minimal elements thanks to the following adaptation of the balayage theorem.

Theorem 4.5 *Let* X, Y *be ordered Frechet spaces and* $P\colon X \oplus Y \to X \oplus Y$ *a positive operator, and let* $E_X\colon X \oplus Y \to X$ *and* $E_Y\colon X \oplus Y \to Y$ *denote the canonical projections. Suppose that the series* $\sum_{n=0}^{\infty} P^n(x)$ *is convergent for any* $x \in X$. *Let* $x_0 \in X \oplus Y$ *be a positive* P*-superharmonic element,* $P(x_0) \le x_0$. *Then the set*

$$\{x \in (X \oplus Y)_+ \mid P(x) \le x, \quad E_X(x_0) \le E_X(x)\}$$

has a smallest element, namely $x = \sum_{n=0}^{\infty}(E_Y P)^n E_X(x_0)$, *and* $x = \sum_{n=0}^{\infty} P^n(x - P(x))$.

5 The Martin boundary of the dual of $SU_q(2)$

Consider the compact quantum group $SU_q(2)$ of Woronowicz [29] with $q \in (0,1)$. The algebra A of continuous functions on $SU_q(2)$ is the universal unital C*-algebra with generators α and γ satisfying the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1, \quad \gamma^*\gamma = \gamma\gamma^*,$$

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication Δ is determined by the formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

The standard quantum 2-sphere of Podleś [20] is defined as the quotient space $S_q^2 = SU_q(2)/\mathbb{T}$, where the inclusion $\mathbb{T} \hookrightarrow SU_q(2)$ is defined by the homomorphism $\pi\colon A \to C(\mathbb{T})$ which sends γ to 0 and α to z. Then

$$B = \{a \in A \,|\, (\iota \otimes \pi)\Delta(a) = a \otimes 1\}$$

is the algebra of continuous functions on S_q^2. The quantum sphere S_q^2 carries a left action of $SU_q(2)$ and a right action of $\widehat{SU_q(2)}$ (the latter comes from the right adjoint action of $\widehat{SU_q(2)}$ on A when we consider A as the group C*-algebra of $\widehat{SU_q(2)}$).

Theorem 5.1 *Let $\phi \in C$ be a generating state with finite first moment in the sense that*

$$\sum_{s\in I} \phi(I_s) \dim H_s < \infty.$$

Then
(i) *the Martin boundary of $SU_q(2)$, regarded as a quantum space with actions of $SU_q(2)$ and $\widehat{SU_q(2)}$, is isomorphic to the Podleś sphere S_q^2;*
(ii) *the unique $SU_q(2)$-invariant state ν on A_ϕ represents the unit, and the map $K_\phi^*\colon \pi_\nu(A_\phi)'' \to H^\infty(M, P_\phi)$ is an isomorphism which respects the actions of $SU_q(2)$ and $\widehat{SU_q(2)}$.*

Biane [5] showed that the Martin boundary of the dual of ordinary $SU(2)$ is the 2-sphere S^2. Strictly speaking, he computed the boundary for a sub-Markov operator, that is, when $\phi(1) < 1$. In fact, the boundary for a Markov operator is trivial, which means that there are no non-constant harmonic elements on $\widehat{SU(2)}$. Harmonic elements for sub-Markov operators are unbounded, so the Poisson boundary for such operators is void. Later Izumi proved [11] that the Poisson boundary of the dual of $SU_q(2)$ with respect to any finitely supported state in C is the standard 2-sphere S_q^2 of Podleś. For this Izumi studied the harmonic elements of the Markov operator associated to the q-trace of the fundamental corepresentation of $SU_q(2)$, for which he computed explicitly the Choi-Effros product. The present authors wanted to understand the connection between the works of Biane and Izumi. Having developed the Martin boundary theory we have provided a more geometric definition of the boundary, and we have shown that the explicit computations of the Choi-Effros product can be avoided.

First we want to give a different description of S_q^2, explaining in particular why it is a boundary of $\widehat{SU_q(2)}$, that is, why $B = C(S_q^2) \subset \hat{M}/\hat{A}$. The elements of the quantized universal enveloping algebra $U_q(su(2)) \subset M(\hat{A})$ of the Lie algebra $su(2)$ are affiliated with $\hat{M}$. Let ad denote the right adjoint action of $U_q(su(2))$ on itself, so $(\operatorname{ad} X)(x) = (X \otimes \iota)\Phi(x) = \sum \hat{S}(X_i)xY_i$, where $\hat{\Delta}(X) = \sum X_i \otimes Y_i$. Furthermore let $U_q^o(su(2))$ denote the elements of $U_q(su(2))$ with finite dimensional ad-orbits. Then $U_q^o(su(2))$ may be thought of as the algebra of left-invariant differential operators on $SU_q(2)$. This is indeed the algebra generated by the quantum Lie algebra of the bicovariant $4D_+$-calculus of Woronowicz [30]. As in the classical case we can talk about the order $\#x$ of a differential operator $x \in U_q^o(su(2))$, so $U_q^o(su(2))$ becomes a filtered algebra. Then $C^{-\#x}x \in \hat{M}$ for all $x \in U_q^o(su(2))$, where C is the Casimir element. The algebra Ψ generated by $\hat{A}$ and $C^{-\#x}x$, $x \in U_q^o(su(2))$, is an analogue of the algebra of left-invariant pseudo-differential operators of order 0 on $SU_q(2)$. It turns out that $\Psi/\hat{A} \cong C(S_q^2)$, and the isomorphism can be thought of as an analogue of the symbol map. Moreover, this isomorphism respects the actions of $SU_q(2)$ and $\widehat{SU_q(2)}$.

Let us now sketch a proof of Theorem 5.1. First we need to compute the boundary of the center. As in the proof of Theorem 4.3, we use the classical random walk on the dual of the maximal torus $\mathbb{T} \subset SU_q(2)$. Say it is given by a measure μ on $\mathbb{Z} = \hat{\mathbb{T}}$. By identifying the set I of equivalence classes of irreducible representations of $SU_q(2)$ with $\frac{1}{2}\mathbb{Z}_+$, we then get

$$p_\phi(s,0) = \frac{q^{2s}}{d_s}(\mu(-2s)) - q^2\mu(-2s-2))$$

for $s \in \frac{1}{2}\mathbb{Z}_+$. Set

$$g_\phi(s,0) = \sum_{n=0}^{\infty} p_\phi^{(n)}(s,0) = \frac{q^{2s}}{d_s}(g(-2s)) - q^2 g(-2s-2)),$$

where $g = \sum_{n=0}^{\infty} \mu^{*n}$. As $d_s = (q^{2s+1} - q^{-2s-1})(q-q^{-1})^{-1}$, the renewal theorem implies that the function $g_\phi(s,0)$, and more generally the function $g_\phi(s,t)$ for $t \in \frac{1}{2}\mathbb{Z}_+$, behaves like q^{4s} as $s \to +\infty$. It follows that the Martin boundary of $I = \frac{1}{2}\mathbb{Z}_+$ consists of one point.

Next let $X \subset \hat{A}$ be an ad-irreducible submodule. Then there exists a unique copy $\tilde{X}$ of X in $\Psi/\hat{A}$. The map $K_{\check{\phi}}\colon \hat{A} \to \hat{M}$ respects the adjoint action. It follows that $K_{\check{\phi}}(X) = c_X \tilde{X} \bmod \hat{A}$ for a unique (up to a scalar) element $c_X \in Z(\hat{M})/Z(\hat{A})$. As c_X reflects certain properties of the random walk, it is natural to expect that c_X belongs to the Martin boundary of the center. This is indeed the case, and as the Martin boundary of the center is trivial, the element c_X is a scalar. Thus $K_{\check{\phi}}(X) \subset \Psi/\hat{A}$. Hence $A_\phi \subset \Psi/\hat{A}$. One can furthermore show that $c_X \neq 0$, which is enough to conclude that $A_\phi = \Psi/\hat{A}$.

It is worth noting that in this case the image of $K_{\check{\phi}}\colon \hat{A} \to \hat{M}/\hat{A}$ is a subalgebra, so one can apply Theorem 4.4(iv) to compute the Poisson boundary. Since the map $K_{\check{\phi}}\colon \hat{A} \to \hat{M}/\hat{A}$ does not depend on ϕ, neither does $H^\infty(\hat{M}, P_\phi) \subset \hat{M}$.

The case of $SU_q(2)$, which we have just discussed, serves as a test for our theory. We want to make several remarks concerning the more general case of $SU_q(n)$, with a detailed study to appear elsewhere. In this case the first step, which is the computation of the Martin boundary of the center, is not much more difficult than the case of $SU_q(2)$, but the result is more interesting. As in the proof of Theorem 4.3 we get two classical random walks: one on the dual of the maximal torus and one on the set of dominant weights. The measure defining the first random walk has non-zero mean, so the corresponding Martin boundary is the sphere S^{n-2}. The Martin boundary of the second random walk consists of the points on the sphere which lie in the closure of the set of dominant weights. Note that there is a sharp distinction between the case of $SU_q(n)$ and that of $SU(n)$. The measure on the dual of the maximal torus of $SU(n)$ is symmetric, so the Martin boundary is trivial (when $n \geq 4$). The Martin boundary of the center is also trivial [3], but this is not easy to prove using only classical tools [2].

Note also that the Poisson boundary of the center for $SU_q(n)$ is trivial [9]. Thus, if $n \geq 3$, the "Poisson integral" $K^*_{\check\phi}$, which maps the Martin boundary A_ϕ into the Poisson boundary $H^\infty(\hat{M}, P_\phi)$, has non-trivial kernel. This entails that the Martin boundary cannot any longer be a homogeneous space of $SU_q(n)$.

6 Convergence to the boundary

The representation theorem for harmonic elements is one reason for introducing the Martin boundary. Another reason is to study asymptotic properties of random walks. In this direction we have no precise results for the moment, but we wish to formulate a conjecture. To simplify the discussion we shall consider convergence in mean instead of a.e. convergence, though the latter has also non-commutative analogues, see e.g. [13].

Sustaining notation and the assumptions of Sections 3 and 4, we say that an element $x \in \hat{M}$ is *regular* if the sequence $\{j_n(x)\}_{n=1}^\infty$ is s^*-convergent in $\hat{M}^\infty$, and we denote its limit by $j_\infty(x)$. Let R_ϕ be the set of regular elements.

Proposition 6.1 *Then*
(i) *the set R_ϕ is a C^*-subalgebra of $\hat{M}$ that contains $\hat{A}$ and $H^\infty(\hat{M}, P_\phi)$;*
(ii) *the map $j_\infty \colon R_\phi \to \hat{M}^\infty$ is a $*$-homomorphism of R_ϕ onto $\theta(H^\infty(\hat{M}, P_\phi))$, and $\hat{A}$ is contained in the kernel of j_∞;*
(iii) *for any $x \in R_\phi$ we have $\theta^{-1} j_\infty(x) = s^*\text{-}\lim_{n\to\infty} P^n_\phi(x)$.*

Set $\theta_0(x) = \theta^{-1} j_\infty(x)$ for $x \in R_\phi$. We now state a conjecture asserting what the analogue of the boundary convergence should be.

Conjecture
(i) *The algebra R_ϕ contains the image of $K_{\check\phi} \colon \hat{A} \to \hat{M}$.*
(ii) *If $\nu = \lim_{n\to\infty} \phi^n|_{R_\phi} = \hat\varepsilon\theta_0$, then $\hat\psi(xh) = \nu(K_{\check\phi}(x)h)$ for any $x \in \hat{A}$ and $h \in H^\infty(\hat{M}, P_\phi)$.*

The known proofs of the corresponding classical result use stopping time arguments, and therefore do not have obvious non-commutative analogues.

Provided the conjecture is true, we get the following result, which establishes a connection between the Martin boundary and the Poisson boundary without assuming that the image of the Martin kernel is dense.

Theorem 6.2 *Suppose Conjecture holds. Then*
(i) *for any positive harmonic element $h \in H^\infty(\hat{M}, P_\phi)$, the positive linear functional $(\cdot, h)_\nu$ on $\tilde{A}_\phi$ represents h;*
(ii) *the map $K^*_{\check\phi}|_{A_\phi} \colon A_\phi \to H^\infty(\hat{M}, P_\phi)$ coincides with $\theta_0|_{A_\phi}$ and induces an isomorphism $\pi_\nu(A_\phi)'' \cong H^\infty(\hat{M}, P_\phi)$ which respects the actions of Γ and of the dual compact quantum group G.*

Proof. Part (i) is an immediate consequence of definitions and the property $\hat\psi(xh) = \nu(K_{\check\phi}(x)h)$. To show (ii), note that $\theta_0 \colon R_\phi \to H^\infty(\hat{M}, P_\phi)$ is a homomorphism which restricts to the identity map on $H^\infty(\hat{M}, P_\phi) \subset R_\phi$ and respects the actions of Γ and G. By taking $h = \theta_0(a)$ for $a \in \tilde{A}_\phi$, we therefore get $\hat{\psi}(x\theta_0(a)) = \nu(K_{\check{\phi}}(x)a)$, which implies that $K^*_{\check{\phi}}$ equals θ_0 on A_ϕ. It remains to note that the image of $K^*_{\check{\phi}}$ consists of those $h \in H^\infty(\hat{M}, P_\phi)$ that can be represented by linear functionals in the space spanned by $\eta \le \nu$. Thus (i) implies that $K^*_{\check{\phi}}$ is onto.

■

Finally remark that the state ν representing the unit is G-invariant. It is, however, not Γ-invariant. In fact, the map $K^*_{\check{\phi}}$ respects the action of Γ if and only if ν is quasi-invariant with Radon-Nikodym cocycle $y = (K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0)$. In other words, $y \in M(A_\phi \otimes \hat{\mathcal{A}})$ satisfies

$$(\nu \otimes \iota)\hat{\Delta}(a) = (\nu \otimes \iota)((a \otimes 1)(\iota \otimes \hat{S})(y))$$

for $a \in A_\phi$. The element $(K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0)$ contains complete information about the map $K_{\check{\phi}}$, thus deserves to be called the Martin kernel itself. Then $(K^*_{\check{\phi}} \otimes \iota)(K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0)$ is an analogue of the Poisson kernel. As for the case of $SU_q(2)$ considered in the previous section, if we identify the Martin boundary with $C(S^2_q)$, the Martin kernel is $W(1 \otimes \rho^{-2})W^*$, where ρ is the element introduced prior to Theorem 4.2 and W is the multiplicative unitary of the compact quantum group G.

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