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# Fractal sets defined by reflections on the complex projective space $\mathbb{P}^1(\mathbb{C})$

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ABSTRACT

We consider fractal sets which are defined by reflections on the complex projective space $\mathbb{P}^1(\mathbb{C})$. In §3 we give the defintion of fractal sets defined by reflections and we give the estimates of the contraction ratios of reflections (Theorem I). In §4 we give classfications of the fractal sets and we choose two classes of the fractal sets, which are called "Cantor type" and "Fricke-Klein type". In §5 we give two theorems on the estimate of the Hausdorff dimension of fractal sets of Cantor type and Fricke-Klein type respecting (Theorem II,III). In §6 we give some computer simulations of fractal sets defined by reflections.

## 1 Introduction

In 1982, B.Mandelbrot ([3]) has given a concept of fractal sets and tried to describe the complexty of the objects. After introduction of a concept of fractal set, this concept has been applied not only in mathematics, but also in physics, and geography and the complexity can be described in mathematical terminology. One of the most important results is an introduction of the Hausdorff dimension which describes the complexity of the fractal sets. In the paper [2], Hutchinson has introdueed a concept of "self similar fractal sets" and developed a method of calculation of self similar fractal sets. In this case we have an explicit formula of the Hausdorff dimension. In fact, we can give the Hausdorff dimension in the following manner. We take a system of self similar mappings $\{\sigma_j | j = 1, 2, \cdots, N\}$ with the contraction ratios $\lambda_j (0 < \lambda_j < 1)$ between a compact set $K_0$:

$$\sigma_j : K_0 \to K_0 \quad (j = 1, 2, \cdots, N).$$

Here we assume the following separation condition:

$$\sigma_i(K_0^\circ) \cap \sigma_j(K_0^\circ) = \phi \quad (i \neq j).$$

Putting

$$K_n = \bigcup_{j=1}^{N} \sigma_j(K_{n-1})(n = 1, 2, 3, \cdots),$$

we define a self similar fractal set by

$$K = \bigcap_{i=1}^{\infty} K_n.$$

Then we see that the Hausdorff dimension $\dim_H K (= D)$ can be given by solving the equation:

$$\sum_{j=1}^{N} \lambda_j^D = 1.$$

We can calculate the Hausdorff dimension of the Cantor set, the Serpinski gasket, and the Koch curve. In the case of the Cantor set $C$, the formula tells us

$$\dim_H C = \frac{\log 2}{\log 3}.$$

The appearence of non interger dimension inspires many people to calculate the Hausdorff dimension of other fractal sets. Although, we have still no effective method to calculate the dimensions.
In this paper we are concerned with a fractal set which is defined by reflections on the Riemann sphere $\hat{\mathbb{C}}$. In this case we can apply the theory of a complex variable and we can discuss the fractal sets in an explicit manner. In a similar manner to that of self similar contraction mappings we can define fractal sets(see §3). When we notice that the reflection is not a self similar mapping, we see that we can not apply the formula to this case(see §3). Hence we need to estimate of the Hausdorff dimension. This can be done by use of geometric observations(Theorem I). Next we proceed to classification of fractal sets which are defined by reflections and we can choose two classes of fractal sets and prove two results (Theorem II,III). Finally we give some computer simulations of the fractal sets.

The auther would like to express his hearty thanks to Prof. M.Taniguchi for informing the result of Theorem III.

## 2 Basic materials on reflections

In this section, we recall basic materials on reflections.

For a circle $C : |z - \alpha| = r$, we define the reflection with respect to $C$ by

$$R(z) = \frac{r^2}{\bar{z} - \bar{\alpha}} + \alpha.$$

Then we have the following proposition:

**Proposition 2.1** ([4])
(1) A reflection maps a circle to a circle.
(2) $R^2(z) = R(R(z)) = z$

Next we define the anharmonic ratios:

**Definition 2.2**
For four different points $z_1, z_2, z_3, z_4$ on Riemann sphere $\hat{\mathbb{C}}$, we define the anharmonic ratios:

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3}\frac{z_2 - z_4}{z_1 - z_4}$$

Then we have the following propositions:

**Proposition 2.3** ([4])
We have the following formula for the reflection $R(z)$ with respect to $C$ and three different points $z_1, z_2, z_3$ on $C$:

$$(z_1, z_2, z_3, R(z)) = \overline{(z_1, z_2, z_3, z)} \tag{1}$$

**Proof**
At first we notice the following fact: For four different points $z_1, z_2, z_3, z_4$ on Riemann sphere $\hat{\mathbb{C}}$ and a Möbius transform $T(z) = \dfrac{az+b}{cz+d}$ $(a, b, c, d \in \mathbb{C}, ad - bc \neq 0)$, we have the following formula:

$$(T(z_1), T(z_2), T(z_3), T(z_4)) = (z_1, z_2, z_3, z_4)$$

This can be obtained by a direct calculation and its proof may be omitted.

By this, we see that the anharmonic ratio is invariant under the parallel displacement $S(z) = z + b$ and the Möbius transform $T(z) = \dfrac{r^2}{z}$. Here we take a reflection:

$$R(z) = \frac{r^2}{\bar{z} - \bar{\alpha}} + \alpha.$$

Then we have

$$\begin{aligned}
\overline{(z_1, z_2, z_3, z)} &= \overline{(z_1 - \alpha, z_2 - \alpha, z_3 - \alpha, z - \alpha)} \\
&= (\bar{z}_1 - \bar{\alpha}, \bar{z}_2 - \bar{\alpha}, \bar{z}_3 - \bar{\alpha}, \bar{z} - \bar{\alpha}) \\
&= \left(\frac{r^2}{z_1 - \alpha}, \frac{r^2}{z_2 - \alpha}, \frac{r^2}{z_3 - \alpha}, \bar{z} - \bar{\alpha}\right) \quad \left(\text{by } \bar{z}_i - \bar{\alpha} = \frac{r^2}{z_i - \alpha}\right) \\
&= \left(z_1 - \alpha, z_2 - \alpha, z_3 - \alpha, \frac{r^2}{\bar{z} - \bar{\alpha}}\right)
\end{aligned}$$

$$= \left(z_1, z_2, z_3, \frac{r^2}{\bar{z} - \bar{\alpha}} + \alpha\right)$$
$$= (z_1, z_2, z_3, R(z))$$

which proves the assertion (1).

**Proposition 2.4** ([1])
If there are four different points $z_1, z_2, z_3, z_4$ on one circle $C$, we see that $(z_1, z_2, z_3, z_4)$ is real munber.

**Proof**
From

$$\arg(z_1, z_2, z_3, z_4) = \arg\frac{z_1 - z_3}{z_1 - z_4} - \arg\frac{z_2 - z_3}{z_2 - z_4}, \tag{2}$$

we see that (2) is 0 or $\pm\pi$ when $z_1, z_2, z_3, z_4$ are on one circle.

## 3 Fractal sets defined by reflections

In this section, we consider fractal sets defined by reflections. At frist we give a definition of reflection configuration.

**Definition 3.1**
A set of closed discs $D_0, \cdots, D_N$ is called "reflection configuration" if it satisfies the following two conditions:

$$\text{( i ) } D_j \subsetneqq D_0 \quad (j = 1, \cdots, N) \qquad \text{( ii ) } D_i^\circ \cap D_j^\circ = \phi \quad (i \neq j).$$

We denote the configuration by $(D_1, D_2, \cdots, D_N; D_0)$.

A reflection with respect to $C_i$ is denoted by $R_i$ $(i = 1, \cdots, N)$. From a given reflection configuration, we can define a fractal set in the following manner:

**Definition 3.2**
Putting

$$K_0 = \bigcup_{i=1}^{N} K_0^{(i)} \quad \text{where} \quad K_0^{(i)} = D_i$$

and

$$K_n = \bigcup_{i=1}^{N} K_n^{(i)} \quad \text{where} \quad K_n^{(i)} = R_i(K_{n-1} \setminus K_{n-1}^{(i)}),$$

we define

$$K = \bigcap_{n=0}^{\infty} K_n$$

which we call the fractal set defined by $R_1, \cdots, R_N$.

Here we notice the following fact: Every point $z \in K$ can be expressed as follows:

$$z = \lim_{n \to \infty} R_{j_n} \circ \cdots \circ R_{j_1}(z) \tag{3}$$

Here we notice that contraction ratios of reflections are not constant:

**Example 3.3**

We take two circles:

$$C_0 : |z| = 5 \qquad C_1 : |z| = 1.$$

We denote the reflection with respect to $C_1$ by $R_1$. We see

$$R_1(C_0) : |z| = \frac{1}{5} \qquad R_1(C_1) : |z| = 1.$$

Hence we see that we can not apply the usual formula of Haursdorff dimension directly. Here we can estimate them by use of the following theorem:

**Theorem I** (Approximation theorem for contraction ratios of reflection)

For points $z_a, z_b \in D_1, \cdots, D_N$, we consider $R_{j_N} \circ \cdots \circ R_{j_1}(z_a), R_{j_N} \circ \cdots \circ R_{j_1}(z_b)$. Then we have following estimate:

$$\begin{aligned} \lambda_m^{(N)} |R_{j_{N-1}} \circ \cdots \circ R_{j_1}(z_a) - R_{j_{N-1}} \circ \cdots \circ R_{j_1}(z_b)| & \\ \leq |R_{j_N} \circ \cdots \circ R_{j_1}(z_a) - R_{j_N} \circ \cdots \circ R_{j_1}(z_b)| & \\ \leq \lambda_M^{(N)} |R_{j_{N-1}} \circ \cdots \circ R_{j_1}(z_a) - R_{j_{N-1}} \circ \cdots \circ R_{j_1}(z_b)| & \end{aligned}$$

$$\begin{cases} \lambda_m^{(N)} = \dfrac{r_{0j_N}^2}{(|\alpha_{j_{N-1}\cdots j_1 a} - \alpha_{0j_N}| + r_{j_{N-1}\cdots j_1 a})(|\alpha_{j_{N-1}\cdots j_1 b} - \alpha_{0j_N}| + r_{j_{N-1}\cdots j_1 b})} \\ \lambda_M^{(N)} = \dfrac{r_{0j_N}^2}{(|\alpha_{j_{N-1}\cdots j_1 a} - \alpha_{0j_N}| - r_{j_{N-1}\cdots j_1 a})(|\alpha_{j_{N-1}\cdots j_1 b} - \alpha_{0j_N}| - r_{j_{N-1}\cdots j_1 b})} \end{cases}$$

For the proof of theorem, we prepare the following a lemma:

**Lemma A**
Let

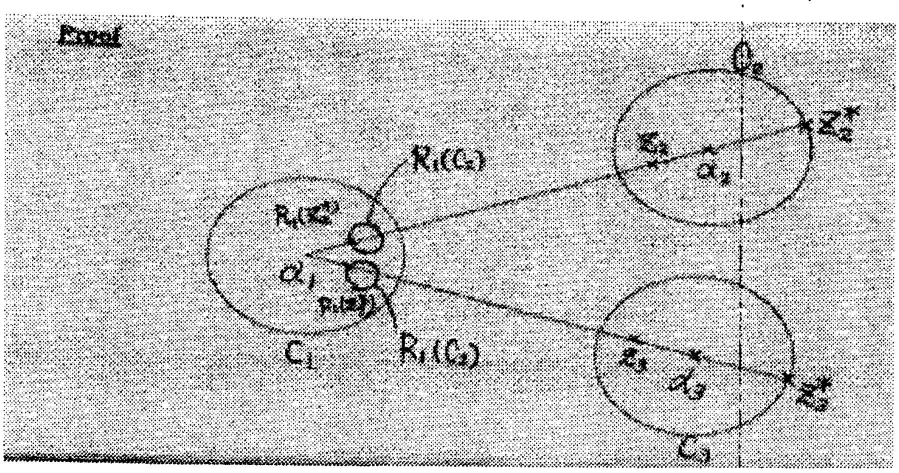
$$R_1(z) = \frac{r_1^2}{\bar{z} - \bar{\alpha}_1} + \alpha_1$$

and $D_2 = \{|z - \alpha_2| \leq r_2\}$, $D_3 = \{|z - \alpha_3| \leq r_3\}$.

Then for two points $z_2 \in D_2, z_3 \in D_3$, we have the following estimate:

$$\lambda_m^{(1)}|z_2 - z_3| \leq |R_1(z_2) - R_1(z_3)| \leq \lambda_M^{(1)}|z_2 - z_3|$$

$$\begin{cases} \lambda_m^{(1)} = \dfrac{r_1^2}{(|\alpha_2 - \alpha_1| + r_2)(|\alpha_3 - \alpha_1| + r_3)} \\ \lambda_M^{(1)} = \dfrac{r_1^2}{(|\alpha_2 - \alpha_1| - r_2)(|\alpha_3 - \alpha_1| - r_3)} \end{cases}$$




We put $z_2^*, z_3^*$ as in the figure. We take two points $z_2 \in D_2$, $z_3 \in D_3$. We put

$$|z_2^* - \alpha_1| = b^*, |z_2 - \alpha_1| = b, |z_3^* - \alpha_1| = c^*, |z_3 - \alpha_1| = c, |z_2^* - z_1^*| = a^*, |z_2 - z_1| = a.$$

Then we have

$$|\alpha_1 - R_1(z_2^*)| = \frac{r_1^2}{b^*}, |\alpha_1 - R_1(z_3^*)| = \frac{r_1^2}{c^*}, |\alpha_1 - R_1(z_2)| = \frac{r_1^2}{b}, |\alpha_1 - R_1(z_3)| = \frac{r_1^2}{c}.$$

Putting

$$|R_1(z_2^*) - R_2(z_3^*)| = x^*, |R_1(z_2) - R_2(z_3)| = x,$$

and using the formula $a^2 = b^2 + c^2 - 2bc\cos\theta$, we have

$$x^2 = \left(\frac{r_1}{b}\right)^2 + \left(\frac{r_1}{c}\right)^2 - 2\left(\frac{r_1}{c}\right)\left(\frac{r_1}{b}\right)\cos\theta. \tag{4}$$

From formula (4)

$$x = r_1^2\left(\frac{a}{bc}\right).$$

In a similmer manner, we have

$$x^* = r_1^2\left(\frac{a}{b^*c^*}\right).$$

From the condition

$$b^* > b, c^* > c.$$

Hence

$$\frac{|R(z_2^*) - R(z_3^*)|}{|z_2^* - z_3^*|} \leq \frac{|R(z_2) - R(z_3)|}{|z_2 - z_3|} \tag{5}$$

In a similar manner, we have the following estimate:

$$\frac{|R(z_2) - R(z_3)|}{|z_2 - z_3|} \leq \frac{|R(z_2^{**}) - R(z_3^{**})|}{|z_2^{**} - z_3^{**}|} \tag{6}$$

We can prove the assertion of Theorem I by successive uses of Lemma A for $R_{j_N} \circ \cdots \circ R_{j_1}$.

## 4 Classfication of reflection configuration

In this section we give a classfication of relection configurations following the numbers of connected components of the complements of $\{D_j\}$ in $D_0$. We make the definition:

**Definition 4.1**
(1) For a configuration $(D_1, \cdots .D_N; D_0)$, we put

$$n(D_1, \cdots, D_N) = \#(\hat{\mathbb{C}} \setminus \bigcup_{j=1}^{N} D_j), \tag{7}$$

we call the configuration of $n$-type.

(2) We denote each connected component in (7) by $E_k$ $(k = 1, 2, \cdots, M)$. Then we have

$$\hat{\mathbb{C}} \setminus \bigcup_{j=1}^{N} D_j = \bigcup_{k=1}^{M} E_k.$$

Then we call $E_k$ **sea** when $\infty \in E_k$ and $E_k$ **lake** when $\infty \notin E_k$.

(3) We call **corner** the interesction point of of two cicrles. We call $E_k$ $n$-corner lake (sea), when $E_k$ has $n$-corners.

Then we can prove the folowing proposition:

**Proposition 4.2**
For a given $N$, we have

$$n(D_1, \cdots, D_N) = 1, 2, \cdots, 2N - 4 \quad (N \geq 3).$$

Moveover, the maximum is attained in case where each lake is 3-corner lake.

**Proof**
We prove the assertion by induction of $N$. In the case $N = 3$, the assertion is trival. We assume that the assertion is true in the case of $N - 1$. Then we attain the maximum by putting a new disc on 3-corner lake so that it has touch with each arc of the 3-corner lake. Hence we have proved the assertion.

Next we introduce two classes of fractal sets:

**Type I** (Cantor type)
A reflection configuration $(D_1, \cdots, D_N; D_0)$ is called of "Cantor type" if it satisfies the following two conditions:

$$(\mathrm{i})\, D_i \cap D_j = \phi \;\; (i \neq j) \qquad (\mathrm{ii})\, D_j \subsetneqq D_0 \;\; (j = 1, \cdots, N). \tag{8}$$

Next we consider the reflection $R_i$ with respect to $C_i$, where $C_i = \partial D_i$ $(i = 0, 1, 2, \cdots, N)$.

**Type II** (Fricke-Klein type)
A reflection configuration $(D_1, \cdots, D_N; D_0)$ is called of "Fricke-Klein type", if finite closed discs $\{D_j\}_{j=1}^{N}$ in $\hat{\mathbb{C}}$ satisfy the following conditions:

(i) $D_i^\circ \cap D_j^\circ = \phi$ $(i \neq j)$,

(ii) $\hat{\mathbb{C}} \setminus \bigcup_{j=1}^{N} D_j$ has just two connected components,

(iii) Each circle has just two points of contact with two of the other circles.

## 5 Two theorems on $D_i^\circ$

In this section we prove two theorems on fractal sets defined by reflections.
At first we prove the following theorem for a fractal set of Cantor type:

**Theorem II**
Let $K$ be a fractal set of Cantor type $(n(D_1, D_2, D_3) = 1)$ and assume $N = 3$. Then we have $\dim_H K < 1$.

**Proof**
At first we notice that

$$\begin{aligned}\left|R_2(z_1) - R_2(z_2)\right| &= \left|\left(\frac{r_2^2}{\bar{z}_1 - \bar{\alpha}_2} + \alpha_2\right) - \left(\frac{r_2^2}{\bar{z}_2 - \bar{\alpha}_2} + \alpha_2\right)\right| \\ &= \left|\frac{r_2^2(z_2 - z_1)}{(\bar{z}_1 - \bar{\alpha}_2)(\bar{z}_2 - \bar{\alpha}_2)}\right|\end{aligned}$$

Here we put

$$|z_1 - z_2| = 2r_1 \ , \ |R_2(z_1) - R(z_2)| = 2r_1'$$

$$\begin{aligned}\frac{r_1'}{r_1} &= \frac{r_2^2}{(2r_1 + b_{12} + r_2)(b_{12} + r_2)} \\ &< \frac{r_2^2}{(2r_1 + r_2)r_2} = \frac{r_2}{2r_1 + r_2}\end{aligned}$$

Putting

$$r_1'' = |R_1(R_2(z_1)) - R_1(R_2(z_2))|$$

We have

$$\frac{r_1''}{r_1} = \frac{r_2^2}{(2r_1 + b_{12}' + r_2)(b_{12}' + r_2)} \leq \frac{r_1}{2r_2 + r_1}$$

Hence

$$\frac{r_1''}{r_1} < \frac{r_2}{2r_1 + r_2} \frac{r_1}{2r_2 + r_1}$$

Next we show

$$\frac{r_1''}{r_1} < \frac{1}{6}$$

Because

$$\frac{1}{6} > \frac{r_2}{2r_1 + r_2} \frac{r_1}{2r_2 + r_1}$$

$$(2r_1 + r_2)(2r_2 + r_1) > 6r_1r_2$$

$$2t^2 - t + 2 = 2\left(t - \frac{1}{4}\right)^2 + \frac{15}{8} > 0 \quad \left(\text{by } t = \frac{r_1}{r_2}\right)$$

We consinder the process

$$K_0 \to K_2$$

$$K_2 = \bigcup_{i \neq j} g_i \circ g_j \Big( \bigcup K_0 \Big)$$

The Haursdorff dimension can be estimated by

$$D < D_M \quad , \quad \sum \lambda_{M_{ij}}^{D_M} = 1$$

From

$$\lambda_{M_{ij}} < \frac{1}{6},$$

we see that

$$D_M < 1.$$

Hence

$$\dim_K D < 1.$$

Next we prove the following theorem for a fractal set of Fricke-Klein type:

**Theorem III**

Let $K$ be a fractal set of Flicke-Klein type $(n(D_1, D_2, D_3) = 2)$ and assume $N = 3$. Then we have $\dim_H K = 1$.

**Proof**

Let $z_1, z_2, z_3$ be points of contact of circle $C_1, C_2$ and $C_3$. We assume $z_1, z_2 \in C_1$ and $z_3 \notin C_1$. Then we have $R_1(z_1) = z_1$, $R_1(z_2) = z_2$ and

$$(z_1, z_2, z_3, R_1(z_3)) = \overline{(z_1, z_2, z_3, z_3)} \quad \text{(by Proposition 2.3)}$$

$$= \overline{\frac{z_1 - z_3}{z_2 - z_3} \frac{z_2 - z_3}{z_1 - z_3}}$$

$$= 1 \in \mathbb{R}$$

Hence we see four different points $z_1, z_2, z_3, R_1(z_3)$ are on the same circle by Prop 2.4. Next we show $K = C(z_1, z_2, z_3)$.

We prepare the following propositions:

**Proposition B**

Let $z \in D_i = \{|z-\alpha_i| \leq r_i\}$, $w \in D_j = \{|z-\alpha_j| \leq r_j\}$ $(i,j=1,2,3)$, then

$$d(R(z),R(w)) \leq \rho\, d(z,w), \quad \rho = \max_{i,j=1,2,3}\left(\frac{r_i}{r_i+2r_j}\right) \quad (i \neq j)$$

Proof is easy and may be omitted.

**Proposition C**
For $z_4 \in C(z_1,z_2,z_3)$ $(z_4 \neq z_1,z_2,z_3)$, we have $R_1(z_4) \in C(z_1,z_2,z_3)$.

Proof
We take $z_4 \in C(z_1,z_2,z_3)$ with $z_4 \neq z_1,z_2,z_3$. Then we have

$$(z_1,z_2,z_3,R_1(z_4)) = \overline{(z_1,z_2,z_3,z_4)} \quad \text{(by Proposition 2.3)}$$

$$= (z_1,z_2,z_3,z_4) \in \mathbb{R} \quad \text{(by Proposition 2.4)}$$

Hence $(z_1,z_2,z_3,R_1(z_4)) \in \mathbb{R}$ and we see four different points $z_1,z_2,z_3,R_1(z_4)$ are on the same circle by Prop 2.4.

With these propositions we prove Theorem III.

$$K = C(z_1,z_2,z_3) \Longleftrightarrow K \subset C(z_1,z_2,z_3) \text{ and } K \supset C(z_1,z_2,z_3)$$

At first we notice that $K \supset C(z_1,z_2,z_3)$ is clear. Next we show $K \subset C(z_1,z_2,z_3)$.

$$K \subset C(z_1,z_2,z_3)$$

$$\Longleftrightarrow$$

$$z \in K \Rightarrow z \in C(z_1,z_2,z_3)$$

$$\Longleftrightarrow$$

$$z = \lim_{n\to\infty}(R_{j_n} \circ \cdots \circ R_{j_1}(z_0)), z_0 \in K \Rightarrow z \in C(z_1,z_2,z_3) \quad \text{(see formula (3))}$$

From

$$d(z,C(z_1,z_2,z_3)) = \lim_{n\to\infty} d(R_{j_n} \circ \cdots \circ R_{j_1}(z_0), C(z_1,z_2,z_3))$$

$$\leq \lim_{n\to\infty} d(R_{j_n} \circ \cdots \circ R_{j_1}(z_0), R_{j_n} \circ \cdots \circ R_{j_1}(w_0))$$

(by $R_{j_n} \circ \cdots \circ R_{j_1}(w_0) \in C(z_1, z_2, z_3)$, see Proposition C)

$$\leq \lim_{n\to\infty} \rho^n d(z_0, w_0) = 0 \quad (\ \rho \text{ is choosen in Proposition B})$$

We see that $d(z, C(z_1, z_2, z_3)) = 0$. Therefore $z \in C(z_1, z_2, z_3)$.
From $\dim_H S^1 = 1$, we can conclude that $\dim_H K = 1$

## 6 Some computer simulations of fractal sets

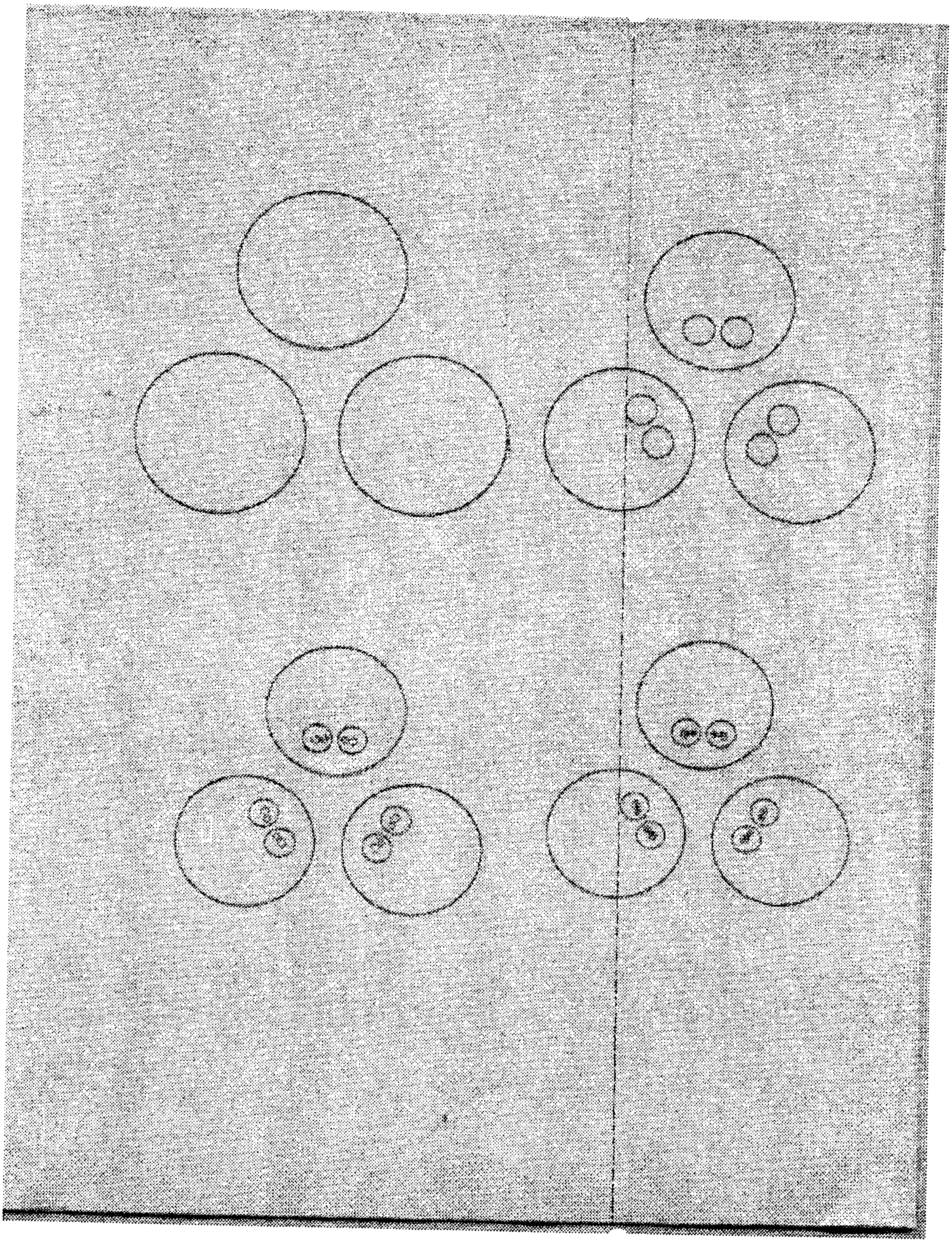
In this section we propose two problems on the topics and give some computer simulations of fractal sets defined by reflections.

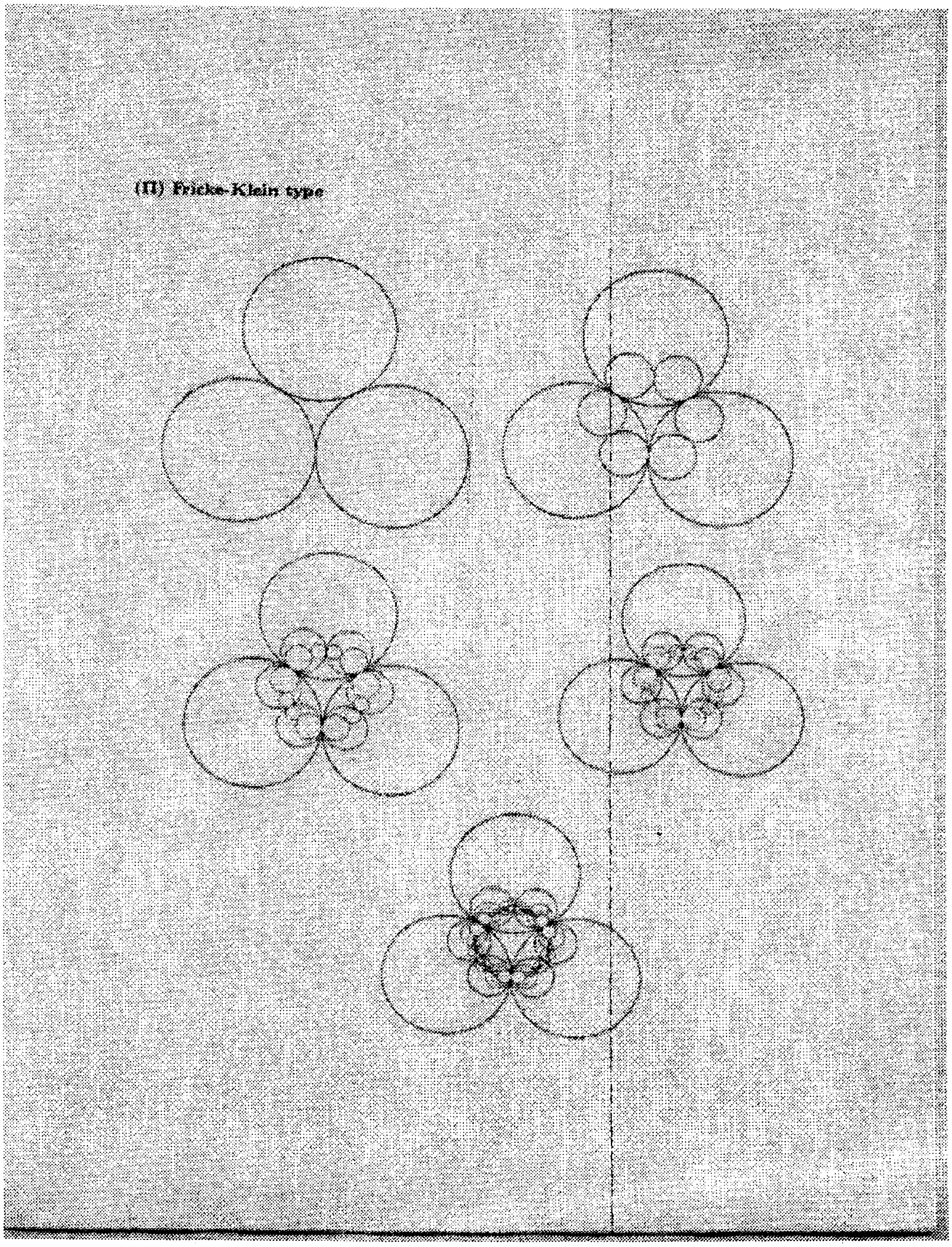
At first we propose two problems.
(1) Can we calculate the Hausdorff dimensions of fractal sets of "Cantor type" and "Fricke-Klein type"?
(2) Can we prove the following proposition?: (see §4)

$$\bigcup_{k=1}^{M} E_k = 1 \Longrightarrow \dim_H K \geq 1$$

Next we give some computer simulations of fractal sets defined by reflections.
**(I) Cantor type**
$C_1 : |z| = \frac{5}{4}, \quad C_2 : |z-3| = \frac{5}{4}$

$C_3 : \left| z - \left( \frac{3}{2} + \frac{3\sqrt{3}}{2} i \right) \right| = \frac{5}{4}$





## Appendix

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a = 5/4; b1 = 0; d1 = 0; b2 = 3; d2 = 0; b3 = 3/2; d3 = 3 √3/2;
z1 = (a*Cos[t] + b1) + (a*Sin[t] + d1) I;
z2 = (a*Cos[t] + b2) + (a*Sin[t] + d2) I;
z3 = (a*Cos[t] + b3) + (a*Sin[t] + d3) I;
c1 := {Re[z1], Im[z1]};
c2 := {Re[z2], Im[z2]};
c3 := {Re[z3], Im[z3]};

α = b1 + d1 I;
z4 = a^2/Conjugate[z2 - α] + α; z5 = a^2/Conjugate[z3 - α] + α;
c4 = {Re[z4], Im[z4]};   o1 = Re[z4] + Im[z4] I;
c5 = {Re[z5], Im[z5]};   o2 = Re[z5] + Im[z5] I;
z9 = a^2/Conjugate[o3 - α] + α; z10 = a^2/Conjugate[o4 - α] + α;
z16 = a^2/Conjugate[o5 - α] + α; z17 = a^2/Conjugate[o6 - α] + α;
c10 = {Re[z9], Im[z9]};   o10 = Re[z9] + Im[z9] I;
c11 = {Re[z10], Im[z10]};   o11 = Re[z10] + Im[z10] I;
c16 = {Re[z16], Im[z16]};   o12 = Re[z16] + Im[z16] I;
c17 = {Re[z17], Im[z17]};   o13 = Re[z17] + Im[z17] I;
z22 = a^2/Conjugate[o14 - α] + α; z23 = a^2/Conjugate[o15 - α] + α;
z24 = a^2/Conjugate[o16 - α] + α; z25 = a^2/Conjugate[o17 - α] + α;
z26 = a^2/Conjugate[o18 - α] + α; z27 = a^2/Conjugate[o19 - α] + α;
z28 = a^2/Conjugate[o20 - α] + α; z29 = a^2/Conjugate[o21 - α] + α;
c22 = {Re[z22], Im[z22]}; o22 = Re[z22] + Im[z22] I;
c23 = {Re[z23], Im[z23]}; o23 = Re[z23] + Im[z23] I;
c24 = {Re[z24], Im[z24]}; o24 = Re[z24] + Im[z24] I;
c25 = {Re[z25], Im[z25]}; o25 = Re[z25] + Im[z25] I;
c26 = {Re[z26], Im[z26]}; o26 = Re[z26] + Im[z26] I;
c27 = {Re[z27], Im[z27]}; o27 = Re[z27] + Im[z27] I;
c28 = {Re[z28], Im[z28]}; o28 = Re[z28] + Im[z28] I;
c29 = {Re[z29], Im[z29]}; o29 = Re[z29] + Im[z29] I;
z46 = a^2/Conjugate[o30 - α] + α; z47 = a^2/Conjugate[o31 - α] + α;
z48 = a^2/Conjugate[o32 - α] + α; z49 = a^2/Conjugate[o33 - α] + α;
z50 = a^2/Conjugate[o34 - α] + α; z51 = a^2/Conjugate[o35 - α] + α;
z52 = a^2/Conjugate[o36 - α] + α; z53 = a^2/Conjugate[o37 - α] + α;
z54 = a^2/Conjugate[o38 - α] + α; z55 = a^2/Conjugate[o39 - α] + α;
z56 = a^2/Conjugate[o40 - α] + α; z57 = a^2/Conjugate[o41 - α] + α;
z58 = a^2/Conjugate[o42 - α] + α; z59 = a^2/Conjugate[o43 - α] + α;
z60 = a^2/Conjugate[o44 - α] + α; z61 = a^2/Conjugate[o45 - α] + α;
c46 = {Re[z46], Im[z46]}; o46 = Re[z46] + Im[z46] I;
c47 = {Re[z47], Im[z47]}; o47 = Re[z47] + Im[z47] I;
c48 = {Re[z48], Im[z48]}; o48 = Re[z48] + Im[z48] I;
c49 = {Re[z49], Im[z49]}; o49 = Re[z49] + Im[z49] I;
c50 = {Re[z50], Im[z50]}; o50 = Re[z50] + Im[z50] I;
c51 = {Re[z51], Im[z51]}; o51 = Re[z51] + Im[z51] I;
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c52 = {Re[s52], Im[s52]}; o52 = Re[s52] + Im[s52] * I;
c53 = {Re[s53], Im[s53]}; o53 = Re[s53] + Im[s53] * I;
c54 = {Re[s54], Im[s54]}; o54 = Re[s54] + Im[s54] * I;
c55 = {Re[s55], Im[s55]}; o55 = Re[s55] + Im[s55] * I;
c56 = {Re[s56], Im[s56]}; o56 = Re[s56] + Im[s56] * I;
c57 = {Re[s57], Im[s57]}; o57 = Re[s57] + Im[s57] * I;
c58 = {Re[s58], Im[s58]}; o58 = Re[s58] + Im[s58] * I;
c59 = {Re[s59], Im[s59]}; o59 = Re[s59] + Im[s59] * I;
c60 = {Re[s60], Im[s60]}; o60 = Re[s60] + Im[s60] * I;
c61 = {Re[s61], Im[s61]}; o61 = Re[s61] + Im[s61] * I;


β = h3 + d3 * I;
s3 = a1/Conjugate[o1 - β] + β; s6 = a2/Conjugate[o2 - β] + β;
c4 = {Re[s3], Im[s3]}; o3 = Re[s3] + Im[s3] * I;
c7 = {Re[s4], Im[s4]}; o4 = Re[s4] + Im[s4] * I;
s11 = a1/Conjugate[o5 - β] + β; s12 = a2/Conjugate[o6 - β] + β;
s18 = a1/Conjugate[o1 - β] + β; s19 = a2/Conjugate[o2 - β] + β;
c12 = {Re[s11], Im[s11]}; o14 = Re[s11] + Im[s11] * I;
c13 = {Re[s12], Im[s12]}; o15 = Re[s12] + Im[s12] * I;
c18 = {Re[s18], Im[s18]}; o16 = Re[s18] + Im[s18] * I;
c19 = {Re[s19], Im[s19]}; o17 = Re[s19] + Im[s19] * I;
s30 = a1/Conjugate[o10 - β] + β; s31 = a2/Conjugate[o11 - β] + β;
s32 = a1/Conjugate[o12 - β] + β; s33 = a2/Conjugate[o13 - β] + β;
s34 = a1/Conjugate[o14 - β] + β; s35 = a2/Conjugate[o15 - β] + β;
s36 = a1/Conjugate[o20 - β] + β; s37 = a2/Conjugate[o21 - β] + β;
c30 = {Re[s30], Im[s30]}; o30 = Re[s30] + Im[s30] * I;
c31 = {Re[s31], Im[s31]}; o31 = Re[s31] + Im[s31] * I;
c32 = {Re[s32], Im[s32]}; o32 = Re[s32] + Im[s32] * I;
c33 = {Re[s33], Im[s33]}; o33 = Re[s33] + Im[s33] * I;
c34 = {Re[s34], Im[s34]}; o34 = Re[s34] + Im[s34] * I;
c35 = {Re[s35], Im[s35]}; o35 = Re[s35] + Im[s35] * I;
c36 = {Re[s36], Im[s36]}; o36 = Re[s36] + Im[s36] * I;
c37 = {Re[s37], Im[s37]}; o37 = Re[s37] + Im[s37] * I;
s62 = a1/Conjugate[o22 - β] + β; s63 = a2/Conjugate[o23 - β] + β;
s64 = a1/Conjugate[o24 - β] + β; s65 = a2/Conjugate[o25 - β] + β;
s66 = a1/Conjugate[o26 - β] + β; s67 = a2/Conjugate[o27 - β] + β;
s68 = a1/Conjugate[o28 - β] + β; s69 = a2/Conjugate[o29 - β] + β;
s70 = a1/Conjugate[o38 - β] + β; s71 = a2/Conjugate[o39 - β] + β;
s72 = a1/Conjugate[o40 - β] + β; s73 = a2/Conjugate[o41 - β] + β;
s74 = a1/Conjugate[o42 - β] + β; s75 = a2/Conjugate[o43 - β] + β;
s76 = a1/Conjugate[o44 - β] + β; s77 = a2/Conjugate[o45 - β] + β;
c62 = {Re[s62], Im[s62]}; o62 = Re[s62] + Im[s62] * I;
c63 = {Re[s63], Im[s63]}; o63 = Re[s63] + Im[s63] * I;
c64 = {Re[s64], Im[s64]}; o64 = Re[s64] + Im[s64] * I;
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c65 = {Re[z65], Im[z65]}; o65 = Re[z65] + Im[z65] * I;
c66 = {Re[z66], Im[z66]}; o66 = Re[z66] + Im[z66] * I;
c67 = {Re[z67], Im[z67]}; o67 = Re[z67] + Im[z67] * I;
c68 = {Re[z68], Im[z68]}; o68 = Re[z68] + Im[z68] * I;
c69 = {Re[z69], Im[z69]}; o69 = Re[z69] + Im[z69] * I;
c70 = {Re[z70], Im[z70]}; o70 = Re[z70] + Im[z70] * I;
c71 = {Re[z71], Im[z71]}; o71 = Re[z71] + Im[z71] * I;
c72 = {Re[z72], Im[z72]}; o72 = Re[z72] + Im[z72] * I;
c73 = {Re[z73], Im[z73]}; o73 = Re[z73] + Im[z73] * I;
c74 = {Re[z74], Im[z74]}; o74 = Re[z74] + Im[z74] * I;
c75 = {Re[z75], Im[z75]}; o75 = Re[z75] + Im[z75] * I;
c76 = {Re[z76], Im[z76]}; o76 = Re[z76] + Im[z76] * I;
c77 = {Re[z77], Im[z77]}; o77 = Re[z77] + Im[z77] * I;

γ = b2 + d2 * I;
z5 = a2/Conjugate[z1 - γ] + γ; z6 = a2/Conjugate[z2 - γ] + γ;
c5 = {Re[z5], Im[z5]}; o5 = Re[z5] + Im[z5] * I;
c6 = {Re[z6], Im[z6]}; o6 = Re[z6] + Im[z6] * I;
z13 = a2/Conjugate[o1 - γ] + γ; z14 = a2/Conjugate[o2 - γ] + γ;
z20 = a2/Conjugate[o3 - γ] + γ; z21 = a2/Conjugate[o4 - γ] + γ;
c13 = {Re[z13], Im[z13]}; o13 = Re[z13] + Im[z13] * I;
c14 = {Re[z14], Im[z14]}; o14 = Re[z14] + Im[z14] * I;
c20 = {Re[z20], Im[z20]}; o20 = Re[z20] + Im[z20] * I;
c21 = {Re[z21], Im[z21]}; o21 = Re[z21] + Im[z21] * I;
z38 = a2/Conjugate[o10 - γ] + γ; z39 = a2/Conjugate[o11 - γ] + γ;
z40 = a2/Conjugate[o12 - γ] + γ; z41 = a2/Conjugate[o13 - γ] + γ;
z42 = a2/Conjugate[o14 - γ] + γ; z43 = a2/Conjugate[o15 - γ] + γ;
z44 = a2/Conjugate[o16 - γ] + γ; z45 = a2/Conjugate[o17 - γ] + γ;
c38 = {Re[z38], Im[z38]}; o38 = Re[z38] + Im[z38] * I;
c39 = {Re[z39], Im[z39]}; o39 = Re[z39] + Im[z39] * I;
c40 = {Re[z40], Im[z40]}; o40 = Re[z40] + Im[z40] * I;
c41 = {Re[z41], Im[z41]}; o41 = Re[z41] + Im[z41] * I;
c42 = {Re[z42], Im[z42]}; o42 = Re[z42] + Im[z42] * I;
c43 = {Re[z43], Im[z43]}; o43 = Re[z43] + Im[z43] * I;
c44 = {Re[z44], Im[z44]}; o44 = Re[z44] + Im[z44] * I;
c45 = {Re[z45], Im[z45]}; o45 = Re[z45] + Im[z45] * I;
z78 = a2/Conjugate[o22 - γ] + γ; z79 = a2/Conjugate[o23 - γ] + γ;
z80 = a2/Conjugate[o24 - γ] + γ; z81 = a2/Conjugate[o25 - γ] + γ;
z82 = a2/Conjugate[o26 - γ] + γ; z83 = a2/Conjugate[o27 - γ] + γ;
z84 = a2/Conjugate[o28 - γ] + γ; z85 = a2/Conjugate[o29 - γ] + γ;
z86 = a2/Conjugate[o30 - γ] + γ; z87 = a2/Conjugate[o31 - γ] + γ;
z88 = a2/Conjugate[o32 - γ] + γ; z89 = a2/Conjugate[o33 - γ] + γ;
z90 = a2/Conjugate[o34 - γ] + γ; z91 = a2/Conjugate[o35 - γ] + γ;
z92 = a2/Conjugate[o36 - γ] + γ; z93 = a2/Conjugate[o37 - γ] + γ;
c78 = {Re[z78], Im[z78]}; o78 = Re[z78] + Im[z78] * I;
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c79 = {Re[z79], Im[z79]}; z79 = Re[z79] + Im[z79] * I;
c80 = {Re[z80], Im[z80]}; z80 = Re[z80] + Im[z80] * I;
c81 = {Re[z81], Im[z81]}; z81 = Re[z81] + Im[z81] * I;
c82 = {Re[z82], Im[z82]}; z82 = Re[z82] + Im[z82] * I;
c83 = {Re[z83], Im[z83]}; z83 = Re[z83] + Im[z83] * I;
c84 = {Re[z84], Im[z84]}; z84 = Re[z84] + Im[z84] * I;
c85 = {Re[z85], Im[z85]}; z85 = Re[z85] + Im[z85] * I;
c86 = {Re[z86], Im[z86]}; z86 = Re[z86] + Im[z86] * I;
c87 = {Re[z87], Im[z87]}; z87 = Re[z87] + Im[z87] * I;
c88 = {Re[z88], Im[z88]}; z88 = Re[z88] + Im[z88] * I;
c89 = {Re[z89], Im[z89]}; z89 = Re[z89] + Im[z89] * I;
c90 = {Re[z90], Im[z90]}; z90 = Re[z90] + Im[z90] * I;
c91 = {Re[z91], Im[z91]}; z91 = Re[z91] + Im[z91] * I;
c92 = {Re[z92], Im[z92]}; z92 = Re[z92] + Im[z92] * I;
c93 = {Re[z93], Im[z93]}; z93 = Re[z93] + Im[z93] * I;

ParametricPlot[{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16,
   c17, c18, c19, c20, c21, c22, c23, c24, c25, c26, c27, c28, c29, c30, c31, c32,
   c33, c34, c35, c36, c37, c38, c39, c40, c41, c42, c43, c44, c45, c46, c47, c48,
   c49, c50, c51, c52, c53, c54, c55, c56, c57, c58, c59, c60, c61, c62, c63, c64,
   c65, c66, c67, c68, c69, c70, c71, c72, c73, c74, c75, c76, c77, c78, c79, c80,
   c81, c82, c83, c84, c85, c86, c87, c88, c89, c90, c91, c92, c93}, {t, 0, 2 π},
  AspectRatio → Automatic, Axes → (*True*)None, (*Frame→True,*)PlotStyle →
   {{RGBColor[0, 0, 0.5]}, {RGBColor[0, 0, 0.5]}, {RGBColor[0, 0, 0.5]},
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    {RGBColor[          ]}, {RGBColor[          ]}}, PlotRange → All]];
```

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