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Viscous shock profile for 2 × 2 systems of hyperbolic conservation laws with an umbilic point

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1 Introduction

Let us consider a 2 × 2 system of conservation laws in one space dimension:

$$U_t + F(U)_x = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+ \quad (1)$$

where $U = (u, v) \in \Omega$ for a domain $\Omega \subseteq \mathbb{R}^2$ and $F = (F_1, F_2) : \Omega \to \mathbb{R}^2$ is a smooth map. We suppose that this system of equations (1) is hyperbolic, i.e. the Jacobian matrix $F'(U)$ has real eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$. If, in particular, these eigenvalues are distinct $\lambda_1(U) < \lambda_2(U)$, the system is called strictly hyperbolic at $U$. A state $U^* \in \Omega$ is called an umbilic point, if $\lambda_1(U) = \lambda_2(U)$ and $F'(U)$ is diagonal at $U = U^*$. We suppose that the system of equations (1) is strictly hyperbolic at any $U \in \Omega \setminus \{U^*\}$ and that $U^*$ is a single umbilic point in $\Omega$. Since $U = U^*$ is an isolated umbilic point, we have the Taylor expansion of $F(U)$ near $U = U^*$:

$$F(U) = F(U^*) + \lambda^*(U - U^*) + Q(U - U^*) + O(1)|U - U^*|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbb{R}^2 \to \mathbb{R}^2$ is a homogeneous quadratic mapping. After the Galilean change of variables: $x \to x - \lambda^* t$ and $U \to U + U^*$, we observe that the system of equations (1) is reduced to

$$U_t + Q(U)_x = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+ \quad (2)$$
modulo higher order terms. Now by a change of unknown functions $V = S^{-1}U$ with a regular constant matrix $S$, we have a new system of equations $V_t + P(V)_x = 0$ where $P(V) = S^{-1}Q(SV)$. Thus we come to

**Definition 1.1** Two quadratic mappings $Q_1(U)$ and $Q_2(U)$ are said to be equivalent, if there is a constant matrix $S \in GL_2(\mathbb{R})$ such that

$$Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \mathbb{R}^2. \quad (3)$$

A general quadratic mapping $Q(U)$ has six coefficients and $GL_2(\mathbb{R})$ is a four dimensional group. Thus by the above equivalence transformations, we can eliminate four parameters. These procedures are successfully carried out by Schaeffer-Shearer [25] and they obtained the following normal forms.

Let $Q(U)$ be a hyperbolic quadratic mapping with an isolated umbilic point $U = 0$, then there exist two real parameters $a$ and $b$ with $a \neq 1 + b^2$ such that $Q(U)$ is equivalent to $\frac{1}{2}\nabla C$ where $\nabla = (\partial_u, \partial_v)$ and

$$C(U) = \frac{1}{3}au^3 + bu^2v + uv^2. \quad (4)$$

Moreover, if $(a, b) \neq (a', b')$, then the corresponding quadratic mappings: $\frac{1}{2}\nabla C$ and $\frac{1}{2}\nabla C'$ are not equivalent.

In the following argument, we shall confine ourselves to the quadratic mapping:

$$F(U) = Q(U) = \frac{1}{2}\nabla C(U) = \frac{1}{2} \left( \frac{au^2 + 2buv + v^2}{bu^2 + 2uv} \right) (a \neq 1 + b^2). \quad (5)$$

Mathematical properties of the systems of equations (1) depends on $(a, b)$. Schaeffer-Shearer classify in [25] ab-plane into four cases: Case I is $a < \frac{3}{4}b^2$; Case II is $\frac{3}{4}b^2 < a < 1 + b^2$; for $a > 1 + b^2$, the boundary between Case III and Case IV is $4\{4b^2 - 3(a - 2)\}^3 - \{16b^3 + 9(1 - 2a)b\}^2 = 0$. We notice that these $2 \times 2$ system of hyperbolic conservation laws with an isolated umbilic point is a generalization of a three phase Buckley-Leverett model for oil reservoir flow where the flux functions are represented by a quotient of polynomials of degree two. In Appendix of [25]: in collaboration with Marchesin and Paes-Leme, they show that the quadratic approximation of the flux functions is either Case I or Case II.

The Riemann problem for (1) is the Cauchy problem with initial data of the form

$$U(x, 0) = \begin{cases} U_L \quad \text{for} \quad x < 0, \\ U_R \quad \text{for} \quad x > 0. \end{cases} \quad (6)$$
where $U_L, U_R$ are constant states in $\Omega$. A jump discontinuity defined by

$$U(x, t) = \begin{cases} U_L & \text{for } x < st, \\ U_R & \text{for } x > st \end{cases}$$

(7)
is a piecewise constant weak solution to the Riemann problem, provided these quantities satisfy the Rankine-Hugoniot condition:

$$s(U_R - U_L) = F(U_R) - F(U_L).$$

(8)

We say that the above discontinuity is a $j$-compressive shock wave ($j = 1, 2$) if it satisfies the Lax entropy conditions:

$$\lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R)$$

(9) (Lax [16], [17]). Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$. The presence of an umbilic point bring us to face with non-classical: overcompressive shocks and crossing shocks. We say that a piecewise constant weak solution (7) is a overcompressive shock if it satisfies

$$\lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L).$$

(10)

We say also that a piecewise constant weak solution (7) is a crossing shock if it satisfies

$$\lambda_1(U_R) < s < \lambda_2(U_R), \quad \lambda_1(U_L) < s < \lambda_2(U_L).$$

(11)

In this note, we shall confine ourselves to Case II of the representative quadratic mapping $F(U) = Q(U)$ defined by (5). Our aim is to show that there is no crossing shock with viscous profile on the complement of medians $M_1 \cup M_3$ hence the associated vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II. In Section 2, we introduce the vector field $X_s(U_L, U)$ which allows us to determine the existence of a viscous profile to the shock wave solutions. Then we classify the character of critical points for the vector field $X_s(U_L, U)$. In Section 3, we show that there is no crossing shock with viscous profile on the complement of $M_1 \cup M_3$. In Section 4, as conclusion, we show that the vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II.

## 2 Viscous Shock Profiles

One admissibility condition for shock wave solutions (7) to the Riemann problem (6) for a hyperbolic system of conservation laws (1) is to obtain these
solutions as limits of travelling wave solutions to an associated parabolic equation:

$$U_t + F(U)_x = \epsilon(B(U)U_x)_x, \epsilon > 0$$  \hspace{1cm} (12)

with an admissible matrix $B(U)$ in $[4, 8, 9, 21, 28, 31]$. More precisely, let $U_L$ and $U_R$ be two constant states to Riemann problem (1), (6). If there exists a shock $U(x, t)$ (7) with speed $s$ to this Riemann problem and the two constant states $U_L$ and $U_R$ are connected through a travelling wave solution $U_\epsilon(x, t) = U\left(\frac{x - st}{\epsilon}\right)$ to (12) with shock speed $s$ which converges to the shock wave $U(x, t)$ (7) as $\epsilon$ tends to 0, then we say that this shock (7) satisfies the viscosity admissibility criterion and that it has a viscous shock profile $U_\epsilon(x, t) = U\left(\frac{x - st}{\epsilon}\right)$. The travelling wave $U_\epsilon(x, t) = U\left(\frac{x - st}{\epsilon}\right)$ should satisfy, by integrating (12), the $2 \times 2$ system of nonlinear ordinary equations:

$$B(U)U_\xi = -s(U - U_L) + f(U) - f(U_L)$$  \hspace{1cm} (13)

with $\xi = \frac{x - st}{\epsilon}$ and the boundary conditions at the infinity

$$\lim_{\xi \rightarrow -\infty} U(\xi) = U_L, \lim_{\xi \rightarrow \infty} U(\xi) = U_R.$$  \hspace{1cm} (14)

The conditions (13), (14) required for the travelling wave solution imply automatically the Rankine-Hugoniot condition (8) for the Riemann problem. The existence of shock with a viscous profile is equivalent to the system of (13) with the boundary condition (14).

Let $X_s(U, U_L)$ be the vector field

$$X_s(U, U_L) = -s(U - U_L) + F(U) - F(U_L).$$  \hspace{1cm} (15)

The shock wave solution (7) has a viscous shock profile if and only if there exists an orbit along the vector-field $X_s(U, U_L)$ from the critical point $U_L$ to the critical point $U_R$ of this vector-field.

Let $p$ be a critical point of a vector field $X$. We say that $p$ is hyperbolic if $dX$ has two eigenvalues with non-zero real part at $p$. Clearly the eigenvalues of $dX_s(U, U_L)$ are $-s + \lambda_j(U)$. In particular, $dX_s(U, U_L)$ has real eigenvalues.

The critical point $U$ of $X_s$ is not hyperbolic if and only if $s = \lambda_j(U) (j = 1 \text{ or } 2)$.

**Proposition 2.1** The shock wave (7) is
• 1-compressive shock if and only if $U_L$ is repeller and $U_R$ is saddle.
• 2-compressive shock if and only if $U_L$ is saddle and $U_R$ is attractor.
• overcompressive shock if and only if $U_L$ is repeller and $U_R$ is attractor.
• crossing shock if and only if $U_L$ and $U_R$ are saddles.

For all above shocks, both critical point $U_L$ and $U_R$ are hyperbolic. Moreover, there exists a shock wave (7) with a viscous profile if and only if there exists an orbit connecting two critical points of the vector field $X_s$.

We say, for example, repeller-saddle connection or simply R-S connection an orbit from a repeller point to a saddle point.

In Case II, we investigate the critical points of the vector-field $X_s(U, U_L)$ in the finite part of the $U-$plane and at the infinity. The Poincaré transfor-
mation [2, 9] enables us to make a one-to-one correspondence from $U-$plane including the infinity to the sphere $S_2$ by identifying two antipodal points. The line joining two antipodal points of $S_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$ intercepts the plane $P_1 = \{(u, v, -1); (u, v) \in \mathbb{R}^2\} \simeq U-$plane at one point. This mapping induces the vector field $X_s(U, U_L)$ on $U-$plane to the vector field $X_s(U, U_L)$ on the sphere $S_2$ minus the equator $\{x_3 = 0\}$. The equator $\{x_3 = 0\}$ corresponds to $\infty \times S^1$ of $U-$plane. Similarly the line joining the origin and a point on $P_2 = \{(1, w, -z); (w, z) \in \mathbb{R}^2\}$ intercepts $S_2$ at two antipodal points. By this mapping, a vector field on $P_3$ is induced to a vector field on the sphere $S_2$ minus the equator $\{x_1 = 0\}$. Therefore the composition of two mappings above transforms a point $(1, w, -z) \in P_2$ to a point $(u, v, 1) \in P_1$:

$$u = 1/z, \quad v = w/z \text{ if } z \neq 0,$$

or equivalently

$$w = v/u, \quad z = 1/u \text{ if } u \neq 0.$$ 

For $u = 0$, we take instead of the plane $P_2$ the plane $P_2' = \{(w, 1, -z); (w, z) \in \mathbb{R}^2\}$. Similarly a point $(w, 1, -z) \in P_2'$ corresponds to a point $(u, v, 1) \in P_1$:

$$w = u/v, \quad z = 1/v \text{ if } v \neq 0.$$ 

By the mapping from $P_3$ to $P_1$, the differential equation

$$\frac{dv}{du} = \frac{-sv + F_2(U)}{-su + F_1(U)}$$

of the vector field $X_s(U, U_L)$ is induced to the differential equation

$$\frac{dz}{dw} = \frac{\Psi}{\Xi}$$

(16)
where
\[
\Psi = -z\{sz(1-zu_L) + F_1(1,w) - z^2F_1(U_L)\}, \\
\Xi = -w\{sz(1-zu_L) + F_1(1,w) - z^2F_1(U_L)\} + F_2(1,w) \\
- z^2F_2(U_L) - sz(w-zv_L).
\]

The right-hand side of the differential equation (16) is well-defined also for \(\{z = 0\}\) which corresponds to the equator \(\{x_3 = 0\}\) of \(S^2\) then to the infinity of \(U-\)plane.

We consider the critical points of \(X_s(U, U_L)\) at the infinity. They satisfy \(z = 0\) then

\[ -wF_1(1,w) + F_2(1,w) = -\Phi(w) = -(w^3 + 2bw^2 + (a-2)w - b) = 0 \]

which has three distinct real roots \(\mu_1, \mu_2, \mu_3\) for \(a < 1+b^2\). The corresponding vector field of (16) is \(\dot{w} = \Xi, \dot{z} = \Psi\) and its Jacobian matrix at \(z = 0\) is

\[
\begin{pmatrix}
-F_1(1,w) - wF_1'(1,w) + F_2'(1,w) & 0 \\
0 & -F_1(1,w)
\end{pmatrix}.
\]

(17)

We have already known [3] the configuration of the roots \(\mu\) of \(\Phi(w) = 0\). For \(b > 0\),

in Case II, \(\mu_1 < -b < \mu_2 < -b/2 < 0 < \mu_3\).

(18)

Then we have

\[
-F_1(1,w) - wF_1'(1,w) + F_2'(1,w) = -\Phi'(w) \begin{cases}
< 0 & \text{for } w = \mu_1, \mu_3, \\
> 0 & \text{for } w = \mu_2
\end{cases}
\]

(19)

and

\[
-F_1(1,w) = -\frac{1}{w}(\Phi(w) + 2w + b) \begin{cases}
< 0 & \text{for } \mu_1, \mu_2, \\
> 0 & \text{for } \mu_3.
\end{cases}
\]

(20)

Therefore in Case II, \(\mu_1\) is a attractor, \(\mu_2\) is a saddle and \(\mu_3\) is a repeller. On account of the fact that, at the antipodal point, the character of a critical point is the inverse, we have

**Theorem 2.1**  The vector field \(X_s(U, U_L)\) has six singularities at infinity. In Case II, two are repellers, two are attractors and two are saddles.

We investigate critical points of \(X_s(U, U_L)\) in the bounded region of \(U-\)plane. Owing to the Poincaré-Hopf theorem, we can show
Theorem 2.2  The vector field $X_s(U, U_L)$ has two, three or four critical points in the bounded region of $U-$plane. In Case II,

(i) if the vector field $X_s(U, U_L)$ has four critical points in the bounded region of $U-$plane, then the critical points are two nodes and two saddles.

(ii) if the vector field $X_s(U, U_L)$ has three critical points in the bounded region of $U-$plane, then the critical points are one node, one saddle and one saddle-node.

(iii) if the vector field $X_s(U, U_L)$ has two critical points in the bounded region of $U-$plane, then the critical points are one node and one saddle or two saddle-nodes.

Let us recall the notion of structurally stable vector fields. Let $\chi(M^2)$ be the space of all vector fields of $C^1$ class on a 2-dimensional compact manifold $M^2$ with the $C^1$-topology.

Definition 2.1  A vector field $X \in \chi(M^2)$ is said to be structurally stable if there exists a neighborhood $N$ of $X$ in $\chi(M^2)$ such that for any $Y \in N$, there exists a homeomorphism $\rho : M^2 \rightarrow M^2$ which maps any orbit of $X$ to an orbit $Y$.

The following theorem due to Peixoto [24] gives a characterization of structurally stable vector fields.

Theorem 2.3  A vector field $X \in \chi(M^2)$ is structurally stable if and only if it satisfies the following conditions:

- there are only a finite number of critical points and all are hyperbolic,
- there are only a finite number of closed orbits and all are hyperbolic,
- the $\omega$-limit sets and $\alpha$-limit sets of any orbit consist only of critical points or closed orbits,
- there are no saddle-saddle connections.

Since both eigenvalues of $X_s(U_L, U)$ are real, we have

Proposition 2.2  The vector field $X_s(U_L, U)$ has no closed orbits, nor singular closed orbit, nor $\omega$-limit sets, nor $\alpha$-limit sets.

The most unstable connection is clearly saddle-saddle connection. We will show in the next section that there are no saddle-saddle connections on the complement of $M_1 \cup M_3$ in Case II.
3 Saddle-Saddle Connections

The aim of this section is to show that there is no crossing shock on the complement of $M_1 \cup M_3$ in the Case II.

**Theorem 3.1** A crossing shock has a viscous profile if and only if this profile comes from a saddle-saddle connection which is a straight line on the median $M_j = \{U = (u, v); v = \mu_j u\} (j = 1, 2, 3)$.

**Proof.** Suppose that there is a crossing shock. It is obvious, from Proposition 2.1 and its following remark, that the existence of a crossing shock is equivalent to the existence of a S-S connection. The next lemma is due to Chicone [6].

**Lemma 3.1** Let $X = (\Psi, \Xi)$ be a quadratic vector field on the plane where $\Psi$ and $\Xi$ are relatively prime polynomials. Then every saddle-saddle connection lies on a straight line.

To accomplish the proof of the theorem, we make of a use of a strategy of Gomes [9]. Let $U_L$ and $U_R$ be two saddle points connected by an straight orbit $L: U = (1, k) t + U_L$. Owing to the fact that the segment $\tilde{L}$ from $U_L$ to $U_R$ is invariant under the vector field $X_s$, we have $(X_s|_L, (1, k) - (k, 1)) = 0$.

Denoting $U = (u, v)$ and $U_L = (u_L, v_L)$, we have, from the above equation,

$$F_2(U) - F_2(U_L) = k (F_1(U) - F_1(U_L)),$$

i.e. $(k F_1(1, k) - F_2(1, k)) u^2 = 0$ modulo polynomial of $u$ of degree $\leq 1$. It implies that

$$k F_1(1, k) - F_2(1, k) = \Phi(k) = 0,$$

then $k = \mu_j (j = 1, 2$ or $3)$. Substituting $k = \mu_j$ into (21), we obtain

$$k^2 (bu_L + v_L) + k ((a - 1) u_L + bv_L) - (bu_L + v_L) = 0.$$

$(22) \times u_L - (23)$ gives us $(k^2 + bk - 1)(ku_L - v_L) = 0$. Because clearly $k^2 + bk - 1 \neq 0$, we have $ku_L = v_L$. Then $L$ is on a median.

Therefore the straight orbit lies on the medians and every median is invariant of the vector field $X_s$, which proves the assertion. The converse is quite clear.

In the context of the above proof, we showed
Corollary 3.1  
i) Every median $M_j$ is invariant under the vector field $X_s$ and every straight line orbit lies on a median. ii) The orbit of any saddle-saddle connection lies on a median.

Let us investigate the structure of orbits on the medians. Let $U_L = (u_L, v_L)$ be a point on a median $M = \{U = (u, v); v = \mu u\}$ where $\mu = \mu_j (1 \leq j \leq 3)$. Owing to Corollary 3.1, the orbit through $U_L$ lies on the median $M$. Then we have

$$X_s(U, U_L) = \{(a + 2b\mu + \mu^2)(u^2 - u_L^2) - s(u - u_L)\} \left(\frac{1}{\mu}\right).$$  \hspace{1cm} (24)

Let $U_1 = (u_1, v_1)$ be a point $X_s(U_1, U_L) = 0 (U_1 \neq U_L)$. Then we have $v_1 = \mu u_1$ and

$$u_1 = -u_L + \frac{\mu}{b + 2\mu}s.$$ \hspace{1cm} (25)

If $u_1 < u_L$ i.e. $u_L > \frac{\mu}{2(b + 2\mu)}s$, then both components of $X_s(U, U_L)$ are negative for $u_1 < u < u_L$ and positive for $u < u_1$ and for $u > u_L$. Hence there is an orbit from $U_L$ to $U_1$.

If $u_1 > u_L$ i.e. $u_L < \frac{\mu}{2(b + 2\mu)}s$, then both components of $X_s(U, U_L)$ are negative for $u_L < u < u_1$ and positive for $u < u_L$ and for $u > u_1$. Hence there is an orbit from $U_1$ to $U_L$.

In any case, there is an orbit between $U_L$ and $U_1$. Therefore we have

Theorem 3.2  Any point $U_L$ on a median $M_j (1 \leq j \leq 3)$ can be connected via one shock to a point $U_1$ on the common median $M_j$ and this shock has a viscous profile.

Furthermore the character of shock waves on the median $M_j (1 \leq j \leq 3)$ can be determined in Case II by the following two propositions

Proposition 3.1  Let $b \geq 0$. On the median $M_2$, there is no crossing shock in Case II.

Proof. On the median $M_2 = \{t(u, v); v = \mu_2 u\}$, the system (1) is reduced to the equation

$$v_t + \left(\frac{b}{\mu_2^2} + \frac{2}{\mu_2}\right) \left(\frac{v^2}{2}\right)_x = 0.$$  \hspace{1cm} (26)
Then the speed of shock wave joining \( U_+ = (u_+.v_+) \) and \( U_- = (u_-v_-) \) is
\[
s(U_+, U_-) = \frac{b + 2\mu_2}{2\mu_2^2}(v_+ + v_-).
\]
The Jacobian matrix \( F'(U) \) on the median \( M_2 \) is
\[
F'(U) = \begin{pmatrix} a + b\mu_2 & b + \mu_2 \\ b + \mu_2 & 1 \end{pmatrix} v.
\]
As we have already seen in Proposition 5.1 [3], the eigenvalues of \( F'(U) \) are
\[
\lambda(U) = \left( \frac{a}{\mu_2} + 2b + \mu_2 \right) v = \frac{b + 2\mu_2}{\mu_2^2} v \quad \text{and} \quad \lambda^\perp(U) = \left( \frac{1}{\mu_2} - b - \mu_2 \right) v
\]
and its eigenvectors are \( t(v, \mu_2v) \) and \( t(-\mu_2v, v) \) respectively. We can determine \( \lambda_1(U) \) and \( \lambda_2(U) \) according to the sign of \( v \) (or \( u \)). In fact, we have
\[
\lambda(U) - \lambda^\perp(U) = \frac{v}{\mu_2^2}(1 + \mu_2^2)(\mu_2 + b).
\]
On the median \( M_2 \), taking into account of (18), for \( v > 0 \), \( \lambda_1(U) = \lambda^\perp(U) \), \( \lambda_2(U) = \lambda(U) \) and, for \( v < 0 \), \( \lambda_1(U) = \lambda(U) \), \( \lambda_2(U) = \lambda^\perp(U) \).
Suppose that there is a crossing shock on the median \( M_2 \). We have four cases: i) \( v_+ \geq 0, v_- > 0 \), ii) \( v_+ > 0, v_- \leq 0 \), iii) \( v_+ < 0, v_- > 0 \), iv) \( v_+ \leq 0, v_- < 0 \). In case i), we would have
\[
s(U_+, U_-) - \lambda_2(U_+)^{} = \frac{2\mu_j + b}{\mu_j^2}(v_- - v_+) < 0,
\]
which is not possible to realize. In case ii), we would have
\[
s(U_+, U_-) - \lambda_1(U_-)^{} = \frac{2\mu_j + b}{2\mu_j^2}(v_+ - v_-) > 0 \text{ then } v_+ < v_-\]
which is not possible to realize. In case iii), we would have
\[
s(U_+, U_-) - \lambda_1(U_+) = \frac{2\mu_j + b}{2\mu_j^2}(v_- - v_+) > 0 \text{ then } v_- < v_+\]
which is not possible to realize. In case iv), we would have
\[
s(U_+, U_-) - \lambda_1(U_+)^{} = \frac{2\mu_j + b}{\mu_j^2}(v_- - v_+) < 0,
\]
\[
s(U_+, U_-) - \lambda_1(U_-) = \frac{2\mu_j + b}{\mu_j^2}(v_+ - v_-) < 0.
\]
which is not possible to realize.

Therefore there is no crossing shock on the median \( M_2 \).

**Proposition 3.2** Let \( b \geq 0 \). Suppose that \((a, b)\) belongs to Case II. On the median \( M_1 \), there is a saddle-saddle connection from \( U_- \) to \( U_+ \) if and only if \( v_- < 0 < v_+ \). On the median \( M_3 \), there is a saddle-saddle connection from \( U_- \) to \( U_+ \) if and only if \( v_+ < 0 < v_- \).

We can prove this proposition using a similar strategy as Proposition 3.1. Combining Corollary 3.1, Proposition 3.1 and Proposition 3.2, we have

**Theorem 3.3** There is no saddle-saddle connection nor crossing shock with viscous profile on the complement of \( M_1 \cup M_3 \) in Case II.

The relation \( X_s(U, U_L) = 0 \) is the intersection of two quadratic equations \( F_1(U) - F_1(U_L) - s(u - u_L) = 0 \) and \( F_2(U) - F_2(U_L) - s(v - v_L) = 0 \). Then it consists of at most four points including \( U_L \) and \( U_1 \). In fact, the others are two saddle points. More precisely

**Proposition 3.3** Let \( U_L \) be a point on a median \( M_j \) (\( 1 \leq j \leq 3 \)). The set \( X_s(U, U_L) = 0 \) consists of at most four points. The others critical points than \( U_L \) and \( U_1 \) consist only of saddle points.

**Proof.** Let \( U_L \) be a point on a median \( M_j : v_L = \mu_j u_L \). The equation \( X_s(U, U_L) = 0 \) implies that

\[
\begin{align*}
F_1(U) - F_1(U_L) - s(u - u_L) &= 0, \quad (28) \\
F_2(U) - F_2(U_L) - s(v - v_L) &= 0. \quad (29)
\end{align*}
\]

\((29) - (28) \times \mu_j\) implies that

\[
(a\mu_j - b)u^2 + 2(b\mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv + \{F_2(U_L) - \mu_j F_1(U_L)\} = 0.
\]

Here

\[
\begin{align*}
F_2(U_L) - \mu_j F_1(U_L) &= (b - a\mu_j)u_L^2 + 2(1 - b\mu_j)uv_L - \mu_j v_L^2 \\
&= u_L^2 \{ (b - a\mu_j) + 2\mu_j(1 - b\mu_j) - \mu_j^3 \} \\
&= -u_L^2 \{ \mu_j^3 + 2b\mu_j^2 + (a - 2)\mu_j - b \} \\
&= 0.
\end{align*}
\]

Hence we have

\[
\begin{align*}
0 &= (a\mu_j - b)u^2 + 2(b\mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv \\
&= (v - \mu_j u) \{ \mu_j v - \frac{1}{\mu_j} (a\mu_j - b)u + s \} \\
&= (v - \mu_j u) \{ \mu_j v + (\mu_j^3 + 2b\mu_j - 2)u + s \}.
\end{align*}
\]
Therefore we have $v = \mu_j u$ and

$$v = \frac{1}{\mu_j^2}(a\mu_j - b)u - \frac{s}{\mu_j}$$  \hspace{1cm} (30)

or equivalently

$$v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right)u - \frac{s}{\mu_j}.$$  \hspace{1cm} (31)

Substituting $v = \mu_j u$ into $X_s(U, U_L) = 0$, we obtain as above $U = U_L, U_1$.

Similarly substituting $v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right)u - \frac{s}{\mu_j}$ into $X_s(U, U_L)$, we obtain

$$X_s(U, U_L) = x_s^1(U, U_L) \left(\begin{array}{l}1 \\ \mu_j\end{array}\right)$$  \hspace{1cm} (32)

where

$$x_s^1(U, U_L) = \left(-3b - 2\mu_j + \frac{4}{\mu_j}\right)u^2 + s \left(2b + \mu_j - \frac{4}{\mu_j}\right)u$$

$$+ \frac{s^2}{\mu_j} - (b + 2\mu_j)u_L^2 + s\mu_j u_L.$$  \hspace{1cm} (33)

Therefore on the line $v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right)u - \frac{s}{\mu_j}$, the vector field $X_s(U, U_L)$ has the constant direction $\pm^t(1, \mu_j)$ and passing through the critical point, $X_s(U, U_L)$ changes the sign. It occurs only in the case of saddle points, which proves the proposition.

### 4 Structural Stability

Applying Theorem 3.3 and Proposition 2.2 to Theorem 2.3, a vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ if and only if there are only a finite number of singularities and all are hyperbolic. Even if there are many variations of critical points as stated in Theorem 2.2, in any case, a vector field $X_s(U_L, U)$ has at most four critical points in bounded region and six critical points at infinity of $U$-plane and all of these are hyperbolic. Therefore we have

**Theorem 4.1** A vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II.
References


