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Viscous shock profile for $2 \times 2$ systems of hyperbolic conservation laws with an umbilic point

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1 Introduction

Let us consider a $2 \times 2$ system of conservation laws in one space dimension:

$$U_t + F(U)_x = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+ \quad (1)$$

where $U = (u, v) \in \Omega$ for a domain $\Omega \subseteq \mathbb{R}^2$ and $F = (F_1, F_2) : \Omega \to \mathbb{R}^2$ is a smooth map. We suppose that this system of equations (1) is hyperbolic, i.e. the Jacobian matrix $F'(U)$ has real eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$. If, in particular, these eigenvalues are distinct: $\lambda_1(U) < \lambda_2(U)$, the system is called strictly hyperbolic at $U$. A state $U^* \in \Omega$ is called an umbilic point, if $\lambda_1(U) = \lambda_2(U)$ and $F'(U)$ is diagonal at $U = U^*$. We suppose that the system of equations (1) is strictly hyperbolic at any $U \in \Omega \setminus \{U^*\}$ and that $U^*$ is a single umbilic point in $\Omega$. Since $U = U^*$ is an isolated umbilic point, we have the Taylor expansion of $F(U)$ near $U = U^*$:

$$F(U) = F(U^*) + \lambda^*(U - U^*) + Q(U - U^*) + O(1)|U - U^*|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbb{R}^2 \to \mathbb{R}^2$ is a homogeneous quadratic mapping. After the Galilean change of variables: $x \to x - \lambda^*t$ and $U \to U + U^*$, we observe that the system of equations (1) is reduced to

$$U_t + Q(U)_x = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+ \quad (2)$$
modulo higher order terms. Now by a change of unknown functions \( V = S^{-1}U \) with a regular constant matrix \( S \), we have a new system of equations \( V_t + P(V)_x = 0 \) where \( P(V) = S^{-1}Q(SV) \). Thus we come to

**Definition 1.1** Two quadratic mappings \( Q_1(U) \) and \( Q_2(U) \) are said to be equivalent, if there is a constant matrix \( S \in GL_2(\mathbb{R}) \) such that

\[
Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \mathbb{R}^2.
\]

A general quadratic mapping \( Q(U) \) has six coefficients and \( GL_2(\mathbb{R}) \) is a four dimensional group. Thus by the above equivalence transformations, we can eliminate four parameters. These procedures are successfully carried out by Schaeffer-Shearer [25] and they obtained the following normal forms.

Let \( Q(U) \) be a hyperbolic quadratic mapping with an isolated umbilic point \( U = 0 \), then there exist two real parameters \( a \) and \( b \) with \( a \neq 1 + b^2 \) such that \( Q(U) \) is equivalent to \( \frac{1}{2} \nabla C \) where \( \nabla = (\partial_u, \partial_v) \) and

\[
C(U) = \frac{1}{3} au^3 + bu^2v + uv^2.
\]

Moreover, if \( (a, b) \neq (a', b') \), then the corresponding quadratic mappings: \( \frac{1}{2} \nabla C \) and \( \frac{1}{2} \nabla C' \) are not equivalent.

In the following argument, we shall confine ourselves to the quadratic mapping:

\[
F(U) = Q(U) = \frac{1}{2} \nabla C(U) = \frac{1}{2} \left( \begin{array}{c} au^3 + 2buv + v^2 \\ bu^2 + 2uv \end{array} \right) (a \neq 1 + b^2).
\]

Mathematical properties of the systems of equations (1) depends on \( (a, b) \). Schaeffer-Shearer classify in [25] \( ab \)-plane into four cases: Case I is \( a < \frac{3}{4} b^2 \); Case II is \( \frac{3}{4} b^2 < a < 1 + b^2 \); for \( a > 1 + b^2 \), the boundary between Case III and Case IV is \( 4b^2 - 3(a - 2)^2 - \{16b^3 + 9(1 - 2a)b\}^2 = 0 \). We notice that these 2 x 2 system of hyperbolic conservation laws with an isolated umbilic point is a generalization of a three phase Buckley-Leverett model for oil reservoir flow where the flux functions are represented by a quotient of polynomials of degree two. In Appendix [25]: in collaboration with Marchesin and Paes-Leme, they show that the quadratic approximation of the flux functions is either Case I or Case II.

The Riemann problem for (1) is the Cauchy problem with initial data of the form

\[
U(x, 0) = \begin{cases} 
U_L & \text{for } x < 0, \\
U_R & \text{for } x > 0
\end{cases}
\]
where \( U_L, U_R \) are constant states in \( \Omega \). A jump discontinuity defined by

\[
U(x, t) = \begin{cases} 
U_L & \text{for } x < st, \\
U_R & \text{for } x > st
\end{cases}
\]  

(7)
is a piecewise constant weak solution to the Riemann problem, provided these quantities satisfy the Rankine-Hugoniot condition:

\[
s(U_R - U_L) = F(U_R) - F(U_L).
\]  

(8)

We say that the above discontinuity is a \( j \)-compressive shock wave \((j = 1, 2)\) if it satisfies the Lax entropy conditions:

\[
\lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R)
\]

(9) (Lax [16], [17]). Here we adopt the convention \( \lambda_0 = -\infty \) and \( \lambda_3 = \infty \). The presence of an umbilic point bring us to face with non-classical: overcompressive shocks and crossing shocks. We say that a piecewise constant weak solution (7) is an overcompressive shock if it satisfies

\[
\lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L).
\]

(10)

We say also that a piecewise constant weak solution (7) is a crossing shock if it satisfies

\[
\lambda_1(U_R) < s < \lambda_2(U_R), \quad \lambda_1(U_L) < s < \lambda_2(U_L).
\]

(11)

In this note, we shall confine ourselves to Case II of the representative quadratic mapping \( F(U) = Q(U) \) defined by (5). Our aim is to show that there is no crossing shock with viscous profile on the complement of medians \( M_1 \cup M_3 \) hence the associated vector field \( X_s(U_L, U) \) is structurally stable on the complement of \( M_1 \cup M_3 \) in Case II. In Section 2, we introduce the vector field \( X_s(U_L, U) \) which allows us to determine the existence of a viscous profile to the shock wave solutions. Then we classify the character of critical points for the vector field \( X_s(U_L, U) \). In Section 3, we show that there is no crossing shock with viscous profile on the complement of \( M_1 \cup M_3 \). In Section 4, as conclusion, we show that the vector field \( X_s(U_L, U) \) is structurally stable on the complement of \( M_1 \cup M_3 \) in Case II.

## 2 Viscous Shock Profiles

One admissibility condition for shock wave solutions (7) to the Riemann problem (6) for a hyperbolic system of conservation laws (1) is to obtain these
solutions as limits of travelling wave solutions to an associated parabolic equation:

\[ U_t + F(U)_x = \epsilon (B(U)U_x)_x, \epsilon > 0 \tag{12} \]

with an admissible matrix \(B(U)\) in [4, 8, 9, 21, 28, 31]. More precisely, let \(U_L\) and \(U_R\) be two constant states to Riemann problem (1), (6). If there exists a shock \(U(x, t)\) (7) with speed \(s\) to this Riemann problem and the two constant states \(U_L\) and \(U_R\) are connected through a travelling wave solution \(U_{\epsilon}(x, t) = U\left(\frac{x-st}{\epsilon}\right)\) to (12) with shock speed \(s\) which converges to the shock wave \(U(x, t)\) (7) as \(\epsilon\) tends to 0, then we say that this shock (7) satisfies the \textit{viscosity admissibility criterion} and that it has a \textit{viscous shock profile} \(U_{\epsilon}(x, t) = U\left(\frac{x-st}{\epsilon}\right)\). The travelling wave \(U_{\epsilon}(x, t)\) should satisfy, by integrating (12), the \(2 \times 2\) system of nonlinear ordinary equations:

\[ B(U)U_{\xi} = -s(U - U_L) + f(U) - f(U_L) \tag{13} \]

with \(\xi = \frac{x-st}{\epsilon}\) and the boundary conditions at the infinity

\[ \lim_{\xi \to -\infty} U(\xi) = U_L, \lim_{\xi \to \infty} U(\xi) = U_R. \tag{14} \]

The conditions (13), (14) required for the travelling wave solution imply automatically the Rankine-Hugoniot condition (8) for the Riemann problem. The existence of shock with a viscous profile is equivalent to the system of (13) with the boundary condition (14).

Let \(X_{s}(U, U_{L})\) be the vector field

\[ X_{s}(U, U_{L}) = -s(U - U_L) + F(U) - F(U_L). \tag{15} \]

The shock wave solution (7) has a viscous shock profile if and only if there exists an orbit along the vector-field \(X_{s}(U, U_L)\) from the critical point \(U_L\) to the critical point \(U_R\) of this vector-field.

Let \(p\) be a critical point of a vector field \(X\). We say that \(p\) is hyperbolic if \(dX\) has two eigenvalues with non-zero real part at \(p\). Clearly the eigenvalues of \(dX_{s}(U, U_L)\) are \(-s + \lambda_j(U)\). In particular, \(dX_{s}(U, U_L)\) has real eigenvalues. The critical point \(U\) of \(X_{s}\) is not hyperbolic if and only if \(s = \lambda_j(U)\) (\(j = 1\) or \(2\)).

\textbf{Proposition 2.1} \hspace{1em} \textit{The shock wave (7) is}
- 1-compressive shock if and only if \( U_L \) is repeller and \( U_R \) is saddle.
- 2-compressive shock if and only if \( U_L \) is saddle and \( U_R \) is attractor.
- overcompressive shock if and only if \( U_L \) is repeller and \( U_R \) is attractor.
- crossing shock if and only if \( U_L \) and \( U_R \) are saddles.

For all above shocks, both critical point \( U_L \) and \( U_R \) are hyperbolic. Moreover there exists a shock wave (7) with a viscous profile if and only if there exists an orbit connecting two critical points of the vector field \( X_s \).

We say, for example, repeller-saddle connection or simply R-S connection an orbit from a repeller point to a saddle point.

In Case II, we investigate the critical points of the vector-field \( X_s(U, U_L) \) in the finite part of the \( U \)-plane and at the infinity. The Poincaré transformation [2, 9] enables us to make a one-to-one correspondence from \( U \)-plane including the infinity to the sphere \( S^2 \) by identifying two antipodal points. The line joining two antipodal points of \( S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\} \) intercepts the plane \( P_1 = \{(u, v, -1); (u, v) \in \mathbb{R}^2\} \simeq U \) -plane at one point. This mapping induces the vector field \( X_s(U, U_L) \) on \( U \)-plane to the vector field \( X_s^{S^2}(U, U_L) \) on the sphere \( S^2 \) minus the equator \( \{x_3 = 0\} \). The equator \( \{x_3 = 0\} \) corresponds to \( \infty \times S^1 \) of \( U \)-plane. Similarly the line joining the origin and a point on \( P_2 = \{(1, w, -z); (w, z) \in \mathbb{R}^2\} \) intersects \( S^2 \) at two antipodal points. By this mapping, a vector field on \( P_2 \) is induced to a vector field on the sphere \( S^2 \) minus the equator \( \{x_1 = 0\} \). Therefore the composition of two mappings above transforms a point \( (1, w, -z) \in P_2 \) to a point \( (u, v, 1) \in P_1 \):

\[
u = 1/z, \ v = w/z \text{ if } z \neq 0, \]

or equivalently

\[
w = v/u, \ z = 1/u \text{ if } u \neq 0. \]

For \( u = 0 \), we take instead of the plane \( P_2 \) the plane \( P'_2 = \{(w, 1, -z); (w, z) \in \mathbb{R}^2\} \). Similarly a point \( (w, 1, -z) \in P'_2 \) corresponds to a point \( (u, v, 1) \in P_1 \):

\[
w = u/v, \ z = 1/v \text{ if } v \neq 0. \]

By the mapping from \( P_2 \) to \( P_1 \), the differential equation

\[
\frac{dv}{du} = \frac{-sv + F_2(U)}{-su + F_1(U)}
\]

of the vector field \( X_s(U, U_L) \) is induced to the differential equation

\[
\frac{dz}{dw} = \frac{\Psi}{\Xi} \tag{16}
\]
where
\[
\Psi = -z\{-sz(1-zu_L) + F_1(1, w) - z^2F_1(U_L)\},
\]
\[
\Xi = -w\{-sz(1-zu_L) + F_1(1, w) - z^2F_1(U_L)\} + F_2(1, w) - z^2F_2(U_L) - sz(w-zv_L).
\]

The right-hand side of the differential equation (16) is well-defined also for \(\{z = 0\}\) which corresponds to the equator \(\{x_3 = 0\}\) of \(S^2\) then to the infinity of \(U\)-plane.

We consider the critical points of \(X_s(U, U_L)\) at the infinity. They satisfy \(z = 0\) then
\[
-wF_1(1, w) + F_2(1, w) = -\Phi(w) = -(w^3 + 2bw^2 + (a-2)w - b) = 0
\]
which has three distinct real roots \(\mu_1, \mu_2, \mu_3\) for \(a < 1+b^2\). The corresponding vector field of (16) is \(\dot{w} = \Xi, \dot{z} = \Psi\) and its Jacobian matrix at \(z = 0\) is
\[
\begin{pmatrix}
-F_1(1,w) - wF_1'(1,w) + F_2'(1,w) & 0 \\
0 & -F_1(1,w)
\end{pmatrix}.
\]

We have already known [3] the configuration of the roots \(\mu_i\) of \(\Phi(w) = 0\). For \(b > 0\),
\[
in \text{Case II, } \mu_1 < -b < \mu_2 < -b/2 < 0 < \mu_3.
\]

Then we have
\[
-F_1(1,w) - wF_1'(1,w) + F_2'(1,w) = -\Phi'(w) \begin{cases} < 0 & \text{for } w = \mu_1, \mu_3, \\ > 0 & \text{for } w = \mu_2 \end{cases}
\]
and
\[
-F_1(1,w) = -\frac{1}{w}(\Phi(w) + 2w + b) \begin{cases} < 0 & \text{for } \mu_1, \mu_2, \\ > 0 & \text{for } \mu_3. \end{cases}
\]

Therefore in Case II, \(\mu_1\) is a attractor, \(\mu_2\) is a saddle and \(\mu_3\) is a repeller. On account of the fact that, at the antipodal point, the character of a critical point is the inverse, we have

**Theorem 2.1** The vector field \(X_s(U, U_L)\) has six singularities at infinity. In Case II, two are repellers, two are attractors and two are saddles.

We investigate critical points of \(X_s(U, U_L)\) in the bounded region of \(U\)-plane. Owing to the Poincaré-Hopf theorem, we can show
Theorem 2.2  The vector field $X_s(U,U_L)$ has two, three or four critical points in the bounded region of $U$–plane. In Case II,

(i) if the vector field $X_s(U,U_L)$ has four critical points in the bounded region of $U$–plane, then the critical points are two nodes and two saddles.

(ii) if the vector field $X_s(U,U_L)$ has three critical points in the bounded region of $U$–plane, then the critical points are one node, one saddle and one saddle-node.

(iii) if the vector field $X_s(U,U_L)$ has two critical points in the bounded region of $U$–plane, then the critical points are one node and one saddle or two saddle-nodes.

Let us recall the notion of structurally stable vector fields. Let $\chi(M^2)$ be the space of all vector fields of $C^1$ class on a 2-dimensional compact manifold $M^2$ with the $C^1$-topology.

Definition 2.1  A vector field $X \in \chi(M^2)$ is said to be structurally stable if there exists a neighborhood $N$ of $X$ in $\chi(M^2)$ such that for any $Y \in N$, there exists a homeomorphism $\rho : M^2 \to M^2$ which maps any orbit of $X$ to an orbit $Y$.

The following theorem due to Peixoto [24] gives a characterization of structurally stable vector fields.

Theorem 2.3  A vector field $X \in \chi(M^2)$ is structurally stable if and only if it satisfies the following conditions:

- there are only a finite number of critical points and all are hyperbolic,
- there are only a finite number of closed orbits and all are hyperbolic,
- the $\omega$-limit sets and $\alpha$-limit sets of any orbit consist only of critical points or closed orbits,
- there are no saddle-saddle connections.

Since both eigenvalues of $X_s(U_L,U)$ are real, we have

Proposition 2.2  The vector field $X_s(U_L,U)$ has no closed orbits, nor singular closed orbit, nor $\omega$-limit sets, nor $\alpha$-limit sets.

The most unstable connection is clearly saddle-saddle connection. We will show in the next section that there are no saddle-saddle connections on the complement of $M_1 \cup M_3$ in Case II.
3 Saddle-Saddle Connections

The aim of this section is to show that there is no crossing shock on the complement of $M_1 \cup M_3$ in the Case II.

**Theorem 3.1** A crossing shock has a viscous profile if and only if this profile comes from a saddle-saddle connection which is a straight line on the median $M_j = \{(u, v); v = \mu_j u\} (j = 1, 2, 3)$.

**Proof.** Suppose that there is a crossing shock. It is obvious, from Proposition 2.1 and its following remark, that the existence of a crossing shock is equivalent to the existence of a S-S connection. The next lemma is due to Chicone [6].

**Lemma 3.1** Let $X = {}^t(\Psi, \Xi)$ be a quadratic vector field on the plane where $\Psi$ and $\Xi$ are relatively prime polynomials. Then every saddle-saddle connection lies on a straight line.

To accomplish the proof of the theorem, we make of a use of a strategy of Gomes [9]. Let $U_L$ and $U_R$ be two saddle points connected by a straight orbit $L: U = {}^t(1, k)t + U_L$. Owing to the fact that the segment $\tilde{L}$ from $U_L$ to $U_R$ is invariant under the vector field $X_s$, we have $(X_s|_{\overline{L}}, {}^t(-k, 1)) = 0$.

Denoting $U = {}^t(u, v)$ and $U_L = {}^t(u_L, v_L)$, we have, from the above equation,

$$F_2(U) - F_2(U_L) = k(F_1(U) - F_1(U_L)),$$

i.e. $(kF_1(1, k) - F_2(1, k))u^2 = 0$ modulo polynomial of $u$ of degree $\leq 1$. It implies that

$$kF_1(1, k) - F_2(1, k)(= \Phi(k)) = 0,$$

(22)

then $k = \mu_j$ ($j = 1, 2$ or $3$). Substituting $k = \mu_j$ into (21), we obtain

$$k^2(bu_L + v_L) + k((a - 1)u_L + bv_L) - (bu_L + v_L) = 0,$$

(23)

(22) $\times u_L - (23)$ gives us $(k^2 + bk - 1)(ku_L - v_L) = 0$. Because clearly $k^2 + bk - 1 \neq 0$, we have $ku_L = v_L$. Then $L$ is on a median. Therefore the straight orbit lies on the medians and every median is invariant of the vector field $X_s$, which proves the assertion. The converse is quite clear.

In the context of the above proof, we showed
Corollary 3.1  i) Every median $M_j$ is invariant under the vector field $X_s$ and every straight line orbit lies on a median. ii) The orbit of any saddle-saddle connection lies on a median.

Let us investigate the structure of orbits on the medians. Let $U_L = {}^t(u_L, v_L)$ be a point on a median $M = \{ U = {}^t(u, v); v = \mu u \}$ where $\mu = \mu_j (1 \leq j \leq 3)$. Owing to Corollary 3.1, the orbit through $U_L$ lies on the median $M$. Then we have

$$
X_s(U, U_L) = \{(a + 2b\mu + \mu^2)(u^2 - u_L^2) - s(u - u_L)\} \left( \frac{1}{\mu} \right). \tag{24}
$$

Let $U_1 = {}^t(u_1, v_1)$ be a point $X_s(U_1, U_L) = 0 (U_1 \neq U_L)$. Then we have $v_1 = \mu u_1$ and

$$
u_1 = -u_L + \frac{\mu}{b + 2\mu} s. \tag{25}$$

If $u_1 < u_L$ i.e. $u_L > \frac{\mu}{2(b + 2\mu)} s$, then both components of $X_s(U, U_L)$ are negative for $u_1 < u < u_L$ and positive for $u < u_1$ and for $u > u_L$. Hence there is an orbit from $U_L$ to $U_1$.

If $u_1 > u_L$ i.e. $u_L < \frac{\mu}{2(b + 2\mu)} s$, then both components of $X_s(U, U_L)$ are negative for $u_L < u < u_1$ and positive for $u < u_L$ and for $u > u_1$. Hence there is an orbit from $U_1$ to $U_L$.

In any case, there is an orbit between $U_L$ and $U_1$. Therefore we have

Theorem 3.2 Any point $U_L$ on a median $M_j (1 \leq j \leq 3)$ can be connected via one shock to a point $U_1$ on the common median $M_j$ and this shock has a viscous profile.

Furthermore the character of shock waves on the median $M_j (1 \leq j \leq 3)$ can be determined in Case II by the following two propositions

Proposition 3.1 Let $b \geq 0$. On the median $M_2$, there is no crossing shock in Case II.

Proof. On the median $M_2 = \{ {}^t(u, v); v = \mu_2 u \}$, the system (1) is reduced to the equation

$$
v_t + \left( \frac{b}{\mu_2^2} + \frac{2}{\mu_2} \right) \left( \frac{v^2}{2} \right)_x = 0. \tag{26}$$
Then the speed of shock wave joining \( U_+ = (u_+, v_+) \) and \( U_- = (u_-, v_-) \) is
\[
s(U_+, U_-) = \frac{b + 2\mu_j}{2\mu_j^2}(v_+ + v_-).
\]
The Jacobian matrix \( F'(U) \) on the median \( M_2 \) is
\[
F'(U) = \begin{pmatrix} au + bv & bu + v \\ bu + v & u \end{pmatrix} = \frac{1}{\mu_j} \begin{pmatrix} a + b\mu_j & b + \mu_j \\ b + \mu_j & 1 \end{pmatrix} v.
\]
As we have already seen in Proposition 5.1 [3], the eigenvalues of \( F'(U) \) are
\[
\lambda(U) = \left( \frac{a}{\mu_j} + 2b + \mu_j \right) v = \frac{b + 2\mu_j}{\mu_j^2} v \quad \text{and} \quad \lambda^\perp(U) = \left( \frac{1}{\mu_j} - b - \mu_j \right) v
\]
and its eigenvectors are \( (v, \mu_j v) \) and \( (\mu_j v, v) \) respectively. We can determine \( \lambda_1(U) \) and \( \lambda_2(U) \) according to the sign of \( v \) (or \( u \)). In fact, we have
\[
\lambda(U) - \lambda^\perp(U) = \frac{v}{\mu_j^2} (1 + \mu_j^2) (\mu_j + b).
\] (27)
On the median \( M_2 \), taking into account of (18), for \( v > 0 \), \( \lambda_1(U) = \lambda^\perp(U) \), \( \lambda_2(U) = \lambda(U) \) and, for \( v < 0 \), \( \lambda_1(U) = \lambda(U) \), \( \lambda_2(U) = \lambda^\perp(U) \).
Suppose that there is a crossing shock on the median \( M_2 \). We have four cases: \( i) v_+ \geq 0, v_- > 0 \), \( ii) v_+ > 0, v_- \leq 0 \), \( iii) v_+ < 0, v_- \geq 0 \), \( iv) v_+ \leq 0, v_- < 0 \).
In case \( i \), we would have
\[
s(U_+, U_-) - \lambda_2(U_+) = \frac{2\mu_j + b}{2\mu_j^2} (v_- - v_+) < 0,
\]
which is not possible to realize. In case \( ii \), we would have
\[
s(U_+, U_-) - \lambda_1(U_-) = \frac{2\mu_j + b}{2\mu_j^2} (v_+ - v_-) > 0 \quad \text{then} \quad v_+ < v_-
\]
which is not possible to realize. In case \( iii \), we would have
\[
s(U_+, U_-) - \lambda_1(U_+) = \frac{2\mu_j + b}{2\mu_j^2} (v_- - v_+) > 0 \quad \text{then} \quad v_- < v_+
\]
which is not possible to realize. In case \( iv \), we would have
\[
s(U_+, U_-) - \lambda_1(U_+) = \frac{2\mu_j + b}{2\mu_j^2} (v_- - v_+) < 0,
\]
\[
s(U_+, U_-) - \lambda_1(U_-) = \frac{2\mu_j + b}{2\mu_j^2} (v_+ - v_-) < 0
\]
which is not possible to realize.

Therefore there is no crossing shock on the median $M_2$.

**Proposition 3.2** Let $b \geq 0$. Suppose that $(a, b)$ belongs to Case II. On the median $M_1$, there is a saddle-saddle connection from $U_-$ to $U_+$ if and only if $v_- < 0 < v_+$. On the median $M_3$, there is a saddle-saddle connection from $U_-$ to $U_+$ if and only if $v_+ < 0 < v_-$. We can prove this proposition using a similar strategy as Proposition 3.1. Combining Corollary 3.1, Proposition 3.1 and Proposition 3.2, we have

**Theorem 3.3** There is no saddle-saddle connection nor crossing shock with viscous profile on the complement of $M_1 \cup M_3$ in Case II.

The relation $X_s(U, U_L) = 0$ is the intersection of two quadratic equations

$F_1(U) - F_1(U_L) - s(u - u_L) = 0$ and $F_2(U) - F_2(U_L) - s(v - v_L) = 0$.

Then it consists of at most four points including $U_L$ and $U_1$. In fact, the others are two saddle points. More precisely

**Proposition 3.3** Let $U_L$ be a point on a median $M_j (1 \leq j \leq 3)$. The set $X_s(U, U_L) = 0$ consists of at most four points. The others critical points than $U_L$ and $U_1$ consist only of saddle points.

**Proof.** Let $U_L$ be a point on a median $M_j : v_L = \mu_j u_L$. The equation

$X_s(U, U_L) = 0$ implies that

\begin{align*}
F_1(U) - F_1(U_L) - s(u - u_L) &= 0, \quad (28) \\
F_2(U) - F_2(U_L) - s(v - v_L) &= 0. \quad (29)
\end{align*}

$(29) - (28) \times \mu_j$ implies that

$(a \mu_j - b)u^2 + 2(b \mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv + \{F_2(U_L) - \mu_j F_1(U_L)\} = 0.$

Here

\[
F_2(U_L) - \mu_j F_1(U_L) = (b - a \mu_j)u_L^2 + 2(1 - b \mu_j)u_L v_L - \mu_j v_L^2
\]

\[
= u_L^2 \{(b - a \mu_j) + 2\mu_j (1 - b \mu_j) - \mu_j^3\}
\]

\[
= -u_L^2 \{\mu_j^3 + 2b \mu_j^2 + (a - 2) \mu_j - b\}
\]

\[
= 0.
\]

Hence we have

\[
0 = (a \mu_j - b)u^2 + 2(b \mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv
\]

\[
= (v - \mu_j u)\{\mu_j v - \frac{1}{\mu_j}(a \mu_j - b)u + s\}
\]

\[
= (v - \mu_j u)\{\mu_j v + (\mu_j^2 + 2b \mu_j - 2)u + s\}.
\]
Therefore we have \( v = \mu_j u \) and

\[
v = \frac{1}{\mu_j^2} (\alpha \mu_j - b) u - \frac{s}{\mu_j} \tag{30}
\]

or equivalently

\[
v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right) u - \frac{s}{\mu_j}. \tag{31}
\]

Substituting \( v = \mu_j u \) into \( X_s(U, U_L) = 0 \), we obtain as above \( U = U_L, U_1 \).

Similarly substituting \( v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right) u - \frac{s}{\mu_j} \) into \( X_s(U, U_L) \), we obtain

\[
X_s(U, U_L) = x_s^1(U, U_L) \left( \frac{1}{\mu_j} \right) \tag{32}
\]

where

\[
x_s^1(U, U_L) = \left(-3b - 2\mu_j + \frac{4}{\mu_j}\right) u^2 + s \left(2b + \mu_j - \frac{4}{\mu_j}\right) u \tag{33}
\]

\[+ \frac{s^2}{\mu_j} - (b + 2\mu_j) u_L^2 + s \mu_j u_L. \tag{34}\]

Therefore on the line \( v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right) u - \frac{s}{\mu_j} \), the vector field \( X_s(U, U_L) \) has the constant direction \( \pm^t (1, \mu_j) \) and passing through the critical point, \( X_s(U, U_L) \) changes the sign. It occurs only in the case of saddle points, which proves the proposition.

### 4 Structural Stability

Applying Theorem 3.3 and Proposition 2.2 to Theorem 2.3, a vector field \( X_s(U_L, U) \) is structurally stable on the complement of \( M_1 \cup M_3 \) if and only if there are only a finite number of singularities and all are hyperbolic. Even if there are many variations of critical points as stated in Theorem 2.2, in any case, a vector field \( X_s(U_L, U) \) has at most four critical points in bounded region and six critical points at infinity of \( U \)-plane and all of these are hyperbolic. Therefore we have

**Theorem 4.1** A vector field \( X_s(U_L, U) \) is structurally stable on the complement of \( M_1 \cup M_3 \) in Case II.
References


