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On the Wellposedness of the Cauchy Problem for Weakly Hyperbolic Operators

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§1. Introduction

We first consider the Cauchy problem on $[0, T] \times \mathbb{R}^n$

\[
\begin{aligned}
D_t^m u &= \sum_{j+|\alpha|=m} A_{j,\alpha}(t) D_t^j D_x^\alpha u + \sum_{j+|\alpha| \leq d} B_{j,\alpha}(t) D_t^j D_x^\alpha u + F(t, x) \\
D_t^j u(0, x) &= u_j(x) \quad (j = 0, \ldots, m - 1),
\end{aligned}
\]

where $D_t = -i \partial_t$, $D_x = -i(\partial_{x_1}, \ldots, \partial_{x_n})$ and $0 \leq d \leq m - 1$. The coefficients of the principal part depend only on $t$. We shall write for the principal part

\[ p(t, \tau, \xi) = \prod_{k=1}^{m} (\tau - \lambda_k(t, \xi)) = \tau^m - \sum_{j+|\alpha|=m} A_{j,\alpha}(t) \tau^j \xi^\alpha, \]

and for the lower order terms

\[ p_d(t, \tau, \xi) = \sum_{j+|\alpha| \leq d} B_{j,\alpha}(t) \tau^j \xi^\alpha. \]

We assume that the principal part $p$ is hyperbolic with respect to $\tau$, that is, the characteristic roots are all real-valued and named $\lambda_k(t, \xi)$ according to the rule

\[ \lambda_1(t, \xi) \geq \lambda_2(t, \xi) \geq \cdots \geq \lambda_m(t, \xi). \]

We recall that the functions $\lambda_k(t, \xi)$ are homogeneous of degree 1 in $\xi$.

In this paper we use the following notations:

\[ \Omega^k_\sigma(\xi) = \{ t \in [0, T] : |\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)| \leq \sigma \} \text{ for } 0 < \sigma < 1 \text{ and } 1 \leq k \leq m - 1. \]

\[ \Omega^*_\sigma(\xi) = \{ t \in [0, T] : |\lambda_1(t, \xi) - \lambda_m(t, \xi)| \leq \sigma \}, \quad \Omega_\sigma(\xi) = \bigcup_{k=1}^{m-1} \Omega^k_\sigma(\xi) \text{ for } 0 < \sigma < 1. \]
\( \mu(S) \) is the Lebesgue measure in \( \mathbb{R}_t \) of the set \( S \subseteq [0, T] \).

\( AC([0, T]) \) is the space of absolutely continuous functions on \([0, T] \).

\( G^s(\mathbb{R}^n) (s \geq 1) \) is the space of Gevrey functions \( g(x) \) satisfying,

\[
\sup_{x \in K} \left| D^\alpha_x g(x) \right| \leq C_K \rho_K^{|\alpha|} |\alpha|^s \quad \text{for any compact set } K \subset \mathbb{R}^n \text{ and } \alpha \in \mathbb{N}^n.
\]

Our previous result is the following:

**Theorem 0.** ([DK]). Assume that \( B_{j,\alpha} \) belong to \( C^0([0, T]) \) and \( \lambda_1, \cdots, \lambda_m \) belong to \( AC([0, T]) \) and that there exist \( C > 0, a \geq 0 \) and \( b > 0 \) such that for any \( 0 < \sigma < 1, |\xi| = 1, \)

\[
(2) \quad \mu(\Omega_\sigma(\xi)) \leq C\sigma^a,
\]

\[
(3) \quad \int_{[0,T] \setminus \Omega^k_\sigma(\xi)} \frac{|\lambda'_k(t, \xi)| + |\lambda'_{k+1}(t, \xi)|}{|\lambda_k(t, \xi) - \lambda_{k+1}(t, \xi)|} \, dt \leq C\sigma^{-b} \text{ for } 1 \leq k \leq m - 1,
\]

Then the Cauchy problem \((P_1)\) is wellposed in \( G^s \) if

\[
1 \leq s < \begin{cases} 1 + \frac{a+1}{b} & \text{when } d \leq \frac{m(a+b)}{a+b+1}, \\ \frac{m}{d+a(d-m)} & \text{when } d > \frac{m(a+b)}{a+b+1}, \end{cases}
\]

i.e., for any data \( u_j \in G^s(\mathbb{R}^n) \) and \( f \in C^0([0, T]; G^s(\mathbb{R}^n)) \) the Cauchy problem \((P_1)\) has a unique solution \( u \in C^m([0, T]; G^s(\mathbb{R}^n)) \).

**Remark 1.** If \( \lambda_1, \cdots, \lambda_m \) and \( \psi_1, \cdots, \psi_{m-1} \) are analytic in \( t \) and vanish at \( t = 0 \) and there exist \( C > 0, c > 0 \) and \( 0 < \alpha < \beta \) such that for any \( t \in [0, T], |\xi| = 1 \)

\[
(5) \quad |\lambda_k(t, \xi)| \leq Ct^\alpha \quad \text{for } 1 \leq k \leq m,
\]

\[
(6) \quad |\lambda_{k+1}(t, \xi) - \lambda_k(t, \xi)| \geq d^\beta \quad \text{for } 1 \leq k \leq m - 1,
\]

we can take \( a = 1/\beta \) in (2) and \( b = 1 - \alpha/\beta \) in (3). Then the Cauchy problem \((P_1)\) when \( B_{j,\alpha} \equiv 0 \) is wellposed in \( G^s \) if

\[
1 \leq s < 1 + \frac{\beta+1}{\beta - \alpha},
\]

**Remark 2.** When \( \lambda_1, \cdots, \lambda_m \) vanish of infinite order, (2) can be dropped (one is forced to choose \( a = 0 \)). Then the Cauchy problem \((P_1)\) is wellposed in \( G^s \) if

\[
1 \leq s < \min \left\{ 1 + \frac{1}{b}, \frac{m}{d} \right\}.
\]
In case of constant multiplicity, we can take \( a = 0 \) in (2), but (3) does not make a sense, since \( [0, T] \backslash \Omega_{x}^{\varepsilon} = \phi \). From Remark 8 in §3, we may take \( b = 1 \). Then the Cauchy problem \((P_{1})\) is wellposed in \( G^{s} \) if \( 1 \leq s < \min\{2, m/d\} \).

**Remark 3.** When \( \lambda_{1}, \cdots, \lambda_{m} \) belong to \( C^{1}([0, T]) \), we can take \( 0 \leq a < 1 \) and \( 0 < b \leq 1 - a \) (see Lemma 3 and Remark 7). Then the Cauchy problem \((P_{1})\) is wellposed in \( G^{s} \) if

\[
1 \leq s < \min\left\{ \frac{2}{1-a}, \frac{m}{d+a(d-m)} \right\}.
\]

We next consider the Cauchy problem on \([0, T] \times \mathbb{R}_{x}^{n}\)

\[
(P_{2})\begin{cases}
D_{t}U = \sum_{|\alpha|=1} A_{\alpha}(t)D_{x}^{|\alpha|}U + B(t)U + F(t, x)
\end{cases}
\]

where \( A_{\alpha}(t) \) and \( B(t) \) are \( m \times m \) matrices.

By the Cauchy-Kowalevski Theorem any type of systems can be solvable locally in the analytic class. Weakly hyperbolic systems are solvable globally in the Gevrey classes of order \( 1 \leq s < m/(m-1) \). In this paper the main result is the following:

**Theorem 1.** Assume that \( A_{\alpha} \) and \( B \) belong to \( C^{m-1}([0, T]) \) and \( \lambda_{1}, \cdots, \lambda_{m} \) belong to \( AC([0, T]) \) and satisfy (2) and (3). Then the Cauchy problem \((P_{2})\) is wellposed in \( G^{s} \) if

\[
1 \leq s < \min\left\{ 1 + \frac{a+1}{b}, \frac{m}{m-1-a} \right\},
\]

i.e., for any data \( U_{0} \in G^{s}(\mathbb{R}^{n}) \) and \( F \in C^{m-1}([0, T]; G^{s}(\mathbb{R}^{n})) \) the Cauchy problem \((P_{2})\) has a unique solution \( U \in C^{1}([0, T]; G^{s}(\mathbb{R}^{n})) \).

**Remark 4.** When \( \lambda_{1}, \cdots, \lambda_{m} \) belong to \( C^{1}([0, T]) \), we can take \( 0 \leq a < 1 \) and \( 0 < b \leq 1 - a \) (see Lemma 3 and Remark 7). Then the Cauchy problem \((P_{2})\) is wellposed in \( G^{s} \) if

\[
1 \leq s < \min\left\{ \frac{2}{1-a}, \frac{m}{m-1-a} \right\} = \frac{m}{m-1-a}.
\]

We also consider the special case when

\[
\sum_{|\alpha|=1} A_{\alpha}(t)D_{x}^{|\alpha|} = \left( \begin{array}{cccc}
\lambda_{1}(t, D_{x}) & \psi_{1}(t, D_{x}) & 0 & \cdots & 0 \\
0 & \lambda_{2}(t, D_{x}) & \psi_{2}(t, D_{x}) & \cdots & \\
\vdots & \vdots & \ddots & \ddots & 0 \\
\vdots & & & \lambda_{m-1}(t, D_{x}) & \psi_{m-1}(t, D_{x}) \\
0 & \cdots & & 0 & \lambda_{m}(t, D_{x})
\end{array} \right),
\]
where $\lambda_k(t, \xi) \ (1 \leq k \leq m)$ and $\psi_k(t, \xi) \ (1 \leq k \leq m-1)$ are homogeneous of degree 1 in $\xi$.

**Theorem 2.** Assume that $B$ belongs to $C^{m-1}([0, T])$ and $\lambda_1, \cdots, \lambda_m$ and $\psi_1, \cdots, \psi_{m-1}$ belong to $C^{m-1}([0, T])$ and satisfy (2) and that there exists $C > 0$ and $b > 0$ such that for any $0 < \sigma < 1$, $|\xi| = 1$

\[ \int_{[0, T] \setminus \Omega^*_\sigma(\xi)} \frac{|\lambda'_k(t, \xi)|}{|\lambda_1(t, \xi) - \lambda_m(t, \xi)|} dt \leq C\sigma^{-b} \text{ for } 1 \leq k \leq m, \]

\[ \int_{[0, T] \setminus \Omega^*_\sigma(\xi)} \frac{|\psi'_k(t, \xi)|}{|\lambda_1(t, \xi) - \lambda_m(t, \xi)|} dt \leq C\sigma^{-b} \text{ for } 1 \leq k \leq m-1. \]

Then the Cauchy problem $(P_2)$ is wellposed in $G^s$ if

\[ 1 \leq s < 1 + \frac{a+1}{b+m-2}. \]

**Remark 5.** (9) is a more relaxed condition than (3) (when $m = 2$, (9) is the same as (3)). Since $\lambda_1, \cdots, \lambda_m$ belong to $C^{m-1}([0, T]) \subset C^1([0, T])$, we find that $b \leq 1 - a$ and by Theorem 1 we get the wellposedness in $G^s$ if

\[ 1 \leq s < \frac{m}{m-1-a} \left(1 + \frac{a+1}{1-a+m-2}\right) \leq 1 + \frac{a+1}{b+m-2} \]

(see Remark 4). The Gevrey index (11) for this special case is larger than (8).

**Example A.** Theorem 2 can be also applied when the principal part has the form of the Jordan matrix:

\[ \sum_{|\alpha| = 1} A_{\alpha}(t)D_x^\alpha \equiv \sum_{|\alpha| = 1} \begin{pmatrix} \lambda_1(t) & 1 & 0 & \cdots & 0 \\ 0 & \lambda_2(t) & 1 & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \lambda_{m-1}(t) & 1 \\ 0 & \cdots & 0 & \lambda_m(t) \end{pmatrix} D_x^\alpha. \]

Since (10) is valid for any $0 < b(\leq 1 - a)$, with only (9) we obtain (11).

**Example B.** Yamahara [Y] studied 4 by 4 systems with the principal part

\[ \sum_{|\alpha| = 1} A_{\alpha}(t)D_x^\alpha \equiv \begin{pmatrix} \lambda t^\beta & 1 & 0 & 0 \\ 0 & \lambda t^\beta & \psi t^\alpha & 0 \\ 0 & 0 & \mu t^\beta & 1 \\ 0 & 0 & 0 & \mu t^\beta \end{pmatrix} D_x, \text{ where } \alpha, \beta > 0, \lambda \neq \mu \text{ and } \psi \neq 0. \]
and proved that the Cauchy problem \((P_2)\) is wellposed in \(G^s\) if

\[
1 \leq s < \begin{cases} 
2 & \text{for } \alpha \geq 2\beta, \\
1 + \frac{\beta}{3\beta - \alpha} & \text{for } 0 < \alpha < 2\beta, 
\end{cases}
\]

and the Cauchy problem \((P_2)\) is not wellposed in \(G^s\) otherwise. In particular, let us consider the case when \(0 < \alpha < \beta\). Taking \(m = 4\), \(a = 0\) and \(b = 1 - \alpha/\beta\) which is determined by (10), by Theorem 2 we can also get the optimal Gevrey index (12) if \(0 < \alpha < \beta\).

\[\S 2.\] Sketch of Proof of Theorem 0

When \(s = 1\), the Cauchy problem \((P_1)\) is wellposed in the class of real analytic functions. Therefore we can suppose that \(s > 1\) for the proof. By Fourier transform with respect to \(x\), the Cauchy problem \((P_1)\) turns into

\[
\begin{cases}
p(t, D_t, \xi)\hat{u} = \hat{f}(t, \xi) + p_d(t, D_t, \xi)\hat{u} \\
D_t^j\hat{u}(0, \xi) = \hat{u}_j(\xi) \quad (j = 0, \cdots, m-1) 
\end{cases}
\]

Let \(0 < \sigma < 1\) and \(\varphi(r)\) be a non-negative function such that \(\varphi \in C_0^\infty(\mathbb{R})\), \(\varphi(r) \equiv 0\) for \(|r| \geq 2\) and \(\varphi(r) \equiv 1\) for \(|r| \leq 1\). We define

\[
\omega(t, \xi) = \sigma|\xi| \sum_{l=1}^{m-1} \varphi(\sigma^{-1}\{\lambda_l(t, \frac{\xi}{|\xi|}) - \lambda_{l+1}(t, \frac{\xi}{|\xi|})\}), \\
\mu_k(t, \xi) = \lambda_k(t, \xi) + ik\omega(t, \xi) \quad \text{for } k = 1, \cdots, m.
\]

Moreover we denote by \(q(t, \tau, \xi)\) the polynomial of degree \(m\) in \(\tau\)

\[
q(t, \tau, \xi) = \prod_{k=1}^{m} (\tau - \mu_k(t, \xi)).
\]

Now we set the energy density

\[
E(t, \xi) = \frac{1}{2} \sum_{l=1}^{m} |q_l(t, D_t, \xi)\hat{u}|^2,
\]

where \(q_l(t, \tau, \xi)\) is the polynomial of degree \(m-1\) in \(\tau\) defined by

\[
q_l(t, \tau, \xi) = \frac{q(t, \tau, \xi)}{\tau - \mu_l(t, \xi)} (= \prod_{k=1, k \neq l}^{m} (\tau - \mu_k(t, \xi))).
\]
Hence we can derive the energy estimate

\[
\sqrt{E(t, \xi)} \leq \exp \left\{ C \int_{0}^{T} \left( \max_{1 \leq k \leq m-1} \frac{|\lambda'_k| + |\lambda'_{k+1}| + |\omega'|}{|\lambda_k - \lambda_{k+1}| + \omega} \right) \right. \\
\left. + \omega + \frac{|\xi|^d}{\prod_{k=1}^{m-1} |\lambda_k - \lambda_{k+1}| + \omega^{m-1}} \right\} \times \left\{ \sqrt{E(0, \xi)} + \int_{0}^{T} |\hat{f}(t, \xi)| dt \right\}.
\]

By (2)-(4), there exist \( C > 0 \) and \( \rho > 0 \) such that for any \((t, \xi) \in [0, T] \times \mathbb{R}^n \setminus 0 \)

\[
\sqrt{E(t, \xi)} \leq C \exp \left\{ \rho |\xi|^\frac{1}{s} \right\} \left\{ \sqrt{E(0, \xi)} + \int_{0}^{T} |\hat{f}(t, \xi)| dt \right\}.
\]

### §3. Sketch of Proof of Theorem 1

In Theorem 1 the coefficients \( A_{\alpha}, B \) and \( F \) belong to \( C^{m-1}([0, T]) \) and the assumptions for the characteristic roots of the principal part are the same as Theorem 0. So, we shall use the result of Theorem 0. Let \( C(t, \tau, \xi) \) be the cofactor matrix of \( \tau I - \sum_{|\alpha|=1} A_{\alpha}(t)\xi^\alpha \), i.e.,

\[
\begin{cases} \\
C(t, \tau, \xi) \cdot \left\{ \tau I - \sum_{|\alpha|=1} A_{\alpha}(t)\xi^\alpha \right\} = p(t, \tau, \xi)I, \\
\left\{ \tau I - \sum_{|\alpha|=1} A_{\alpha}(t)\xi^\alpha \right\} \cdot C(t, \tau, \xi) = p(t, \tau, \xi)I,
\end{cases}
\]

where \( I \) is the \( m \times m \) identity matrix and the polynomial \( p \) has degree \( m \):

\[
p(t, \tau, \xi) = \det \left\{ \tau I - \sum_{|\alpha|=1} A_{\alpha}(t)\xi^\alpha \right\}.
\]

Multiplying \( \left\{ D_t - \sum_{|\alpha|=1} A_{\alpha}(t)D_x^\alpha \right\} \) by \( C(t, D_t, D_x) \), we have the following operator:

\[
p(t, D_t, D_x)I + G(t, D_t, D_x),
\]

where \( G \) contains all the coefficients \( A_{\alpha}, B \) and their derivatives up to order \( m-1 \) in each row. Applying Theorem 0 into (13), we get the existence and the uniqueness.
REFERENCES


