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Kyoto University
APPORXIMATION OF FIXED POINTS AND PROXIMAL POINT ALGORITHMS

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ABSTRACT. In this article, we give three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space. Then we discuss weak and strong convergence theorems for nonlinear operators of accretive and monotone type in a Hilbert space or a Banach space. In particular, we state weak and strong convergence theorems for resolvents of m-accretive operators and maximal monotone operators in a Banach space. Using these results, we also consider the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

1. INTRODUCTION

We consider the following problem: Let $f_0, f_1, f_2, \ldots, f_m$ be convex continuous functions of a Hilbert space $H$ into $\mathbb{R}$. Then, the problem is to find a $z \in C$ such that

$$f_0(z) = \min \{f_0(x) : x \in C\},$$

(1)

where $C = \{x \in H : f_1(x) \leq 0, f_2(x) \leq 0, \ldots, f_m(x) \leq 0\}$. Such a problem is called the convex minimization problem. Let us define a function $g : H \to (-\infty, \infty]$ as follows:

$$g(x) = \begin{cases} f_0(x), & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, $g$ is a proper lower semicontinuous convex function and a minimizer $z \in H$ of $g$ is a solution of the convex minimization problem (1). So, let $g : H \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Consider a convex minimization problem:

$$\min \{g(x) : x \in H\}.\tag{2}$$

For such a $g$, we can define a multivalued operator $\partial g$ on $H$ by

$$\partial g(x) = \{x^* \in H : g(y) \geq g(x) + \langle x^*, y - x \rangle, y \in H\}$$

for all $x \in H$. Such a $\partial g$ is said to be the subdifferential of $g$. A monotone operator $A \subset H \times H$ is called maximal if its graph

$$G(A) = \{(x, y) : y \in Ax\}$$

is not properly contained in the graph of any other monotone operator. We know that if $A$ is a maximal monotone operator, then $R(I + \lambda A) = H$ for all $\lambda > 0$. A monotone operator $A$ is also called m-accretive if $R(I + \lambda A) = H$ for all $\lambda > 0$. 

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So, we can define, for each positive $\lambda$, the resolvent $J_{\lambda} : R(I + \lambda A) \to D(A)$ by $J_{\lambda} = (I + \lambda A)^{-1}$. We know that $J_{\lambda}$ is a nonexpansive mapping. If $g : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then $\partial g$ is a maximal monotone operator.

We know that one method for solving (2) is the proximal point algorithm first introduced by Martinet [16]. The proximal point algorithm is based on the notion of resolvent $J_{\lambda}$, i.e.,

$$J_{\lambda}x = \arg\min \left\{ g(z) + \frac{1}{2\lambda}||z - x||^2 : z \in H \right\}.$$  

The proximal point algorithm is an iterative procedure, which starts at a point $x_1 \in H$, and generates recursively a sequence $\{x_n\}$ of points $x_{n+1} = J_{\lambda_n}x_n$, where $\{\lambda_n\}$ is a sequence of positive numbers; see, for instance, Rockafellar [26].

On the other hand, Halpern [6] and Mann [15] introduced the following iterative schemes to approximate a fixed point of a nonexpansive mapping $T$ of $H$ into itself:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots$$

and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots,$$

respectively, where $x_1 = x \in H$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, Nakajo and Takahashi [18] also introduced an iterative scheme of finding a fixed point of a nonexpansive mapping in a Hilbert space by using an idea of the hybrid method in mathematical programming.

In this article, we first state three convergence theorems for nonexpansive mappings in a Hilbert space. They are convergence theorems of Halpern's type, Mann's type and Nakajo-Takahashi's type. Then, we prove a strong convergence theorem of Halpern's type and a weak convergence theorem of Mann's type for inverse-strongly-monotone mappings in a Hilbert space. In Section 6, we prove weak and strong convergence theorems for resolvents of accretive operators in a Banach space. In Section 7, we consider the strong convergence of a sequence defined by resolvents of maximal monotone operators in a Banach space. Using these results, we also discuss the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

2. Preliminaries

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ denote the dual of $E$. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus of convexity of $E$ is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|^2}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If $E$ is uniformly convex, then $\delta$ satisfies that $\delta(\epsilon/\tau) > 0$ and

$$\left\| \frac{x + y}{2} \right\| \leq \tau \left( 1 - \delta \left( \frac{\epsilon}{\tau} \right) \right)$$

for every $x, y \in E$ with $\|x\| \leq \tau$, $\|y\| \leq \tau$ and $\|x - y\| \geq \epsilon$. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Then we know that
for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call $P_C$ the metric projection of $E$ onto $C$. The duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. In the case, $E$ is called smooth. The norm of $E$ is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (3) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (3) is attained uniformly for $y \in U$. It is known that if the norm of $E$ is uniformly Gâteaux differentiable, then the duality mapping $J$ is single valued and uniformly norm to weak* continuous on each bounded subset of $E$. A Banach space $E$ is said to satisfy Opial's condition [20] if for any sequence $(x_n) \subset E$, $x_n \to y$ implies

$$\liminf_{n \to \infty} \|x_n - y\| < \liminf_{n \to \infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial's condition.

Let $C$ be a closed convex subset of $E$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of $T$ by $F(T)$. A closed convex subset $C$ of $E$ is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset $D$ of $C$ into itself has a fixed point in $D$. Let $D$ be a subset of $E$. We denote the closure of the convex hull of $D$ by $\overline{D}$.

Let $I$ denote the identity operator on $E$. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup \{Az : z \in D(A)\}$ is said to be accretive if for each $x_t \in D(A)$ and $y_t \in Ax_t$, $t = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. If $A$ is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all $r > 0$. An accretive operator $A$ is said to satisfy the range condition if $D(A) \subset \bigcap_{r>0} R(I + rA)$. If $A$ is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of $A$. We also define the Yosida approximation $A_r$ by $A_r = \frac{(I - J_r)}{r}$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf \{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an accretive operator $A$ satisfying the range condition, $A^{-1} = F(J_r)$ for all $r > 0$. An accretive operator $A$ is said to be $m$-accretive if $R(I + rA) = E$ for all $r > 0$.

A multi-valued operator $A : E \to 2^{E^*}$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup \{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_1 \in D(A)$ and $y_1 \in Ax_1$, $i = 1, 2$. A monotone operator $A$ is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; see, for instance [32].

**Theorem 1.** Let $E$ be a reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a monotone operator. Then $A$ is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$. 
Theorem 2. Let $E$ be a strictly convex and smooth Banach space and let $x, y \in E$. If $(x - y, Jx - Jy) = 0$, then $x = y$.

By Theorem 1, a monotone operator $A$ in a Hilbert space $H$ is maximal if and only if $A$ is m-accretive.

3. APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS

There are three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space which are related to the problem of finding a minimizer of a convex function.

Halpern [6] introduced the following iterative scheme to approximate a fixed point of a nonexpansive mapping in a Hilbert space. For the proof, see Wittmann [36] and Takahashi [32].

Theorem 3 ([36]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies

$$\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $Px \in F(T)$.

Mann [15] also introduced the iterative scheme for finding a fixed point of a nonexpansive mapping. For the proof, see Takahashi [32].

Theorem 4 ([15]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies

$$0 \leq \alpha_n < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then, $\{x_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n \to \infty} Px_n$.

Recently, Nakajo and Takahashi [18] proved the following theorem for nonexpansive mappings in a Hilbert space by using an idea of the hybrid method in mathematical programming.

Theorem 5 ([18]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x_1 = x \in C$ and

$$\begin{align*}
(y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, \ldots,
\end{align*}$$
where \( \{\alpha_n\} \subset [0,1] \) satisfies \( \liminf_{n \to \infty} \alpha_n < 1 \) and \( P_{C_n \cap Q_n} \) is the metric projection of \( H \) onto \( C_n \cap Q_n \). Then, \( \{x_n\} \) converges strongly to \( Px_1 \in F(T) \).

Shioji and Takahashi [27] extended Theorem 3 to that of a Banach space whose norm is uniformly Gâteaux differentiable. Let \( C \) and \( D \) be closed convex subsets of a Banach space \( E \) and let \( D \) be a subset of \( C \). Then, a mapping \( P \) of \( C \) onto \( D \) is called sunny if

\[
P(Px + t(x - Px)) = Px
\]

whenever \( Px + t(x - Px) \in C \) for \( x \in C \) and \( t \geq 0 \).

**Theorem 6** ([27]). Let \( E \) be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let \( C \) be a nonempty closed convex subset of \( E \) and let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \) is nonempty. Let \( \{\alpha_n\} \) be a sequence of real numbers such that

\[
0 \leq \alpha_n \leq 1, \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.
\]

Suppose \( x_1 = x \in C \) and \( \{x_n\} \) is given by

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n) Tx_n, \quad n = 1, 2, \ldots.
\]

Then, \( \{x_n\} \) converges strongly to \( Px \in F(T) \), where \( P \) is a unique sunny nonexpansive retraction of \( C \) onto \( F(T) \).

Reich [22] extended also Mann's result to that of a Banach space whose norm is Fréchet differentiable.

**Theorem 7** ([22]). Let \( C \) be a closed convex subset of a uniformly convex Banach space \( E \) with a Fréchet differentiable norm, let \( T : C \to C \) be a nonexpansive mapping such that \( F(T) \) is nonempty, and let \( \{\alpha_n\} \) be a real sequence such that \( 0 \leq \alpha_n \leq 1 \) and \( \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty \). If \( x_1 = x \in C \) and

\[
x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n)x_n, \quad n = 1, 2, \ldots,
\]

then \( \{x_n\} \) converges weakly to a fixed point of \( T \).

**Problem.** Is a Hilbert space in Theorem 5 replaced by a uniformly convex and smooth Banach space?

### 4. Approximating Solutions of Variational Inequalities

Let \( C \) be a closed convex subset of a Hilbert space \( H \). Then, a mapping \( A \) of \( C \) into \( H \) is called inverse-strongly-monotone if there exists a positive real number \( \alpha \) such that

\[
\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2
\]

for all \( x, y \in C \); see [4] and [14]. For such a case, \( A \) is called \( \alpha \)-inverse-strongly-monotone. If a mapping \( T \) of \( C \) into itself is nonexpansive, then \( A = I - T \) is \( \frac{1}{2} \)-inverse-strongly-monotone and \( F(T) = \text{VI}(C, A) \); for example, see [8]. A mapping \( A \) of \( C \) into \( H \) is called strongly monotone if there exists a positive number \( \eta \) such that

\[
\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2
\]

for all \( x, y \in C \). In such a case, we say that \( A \) is \( \eta \)-strongly monotone. If \( A \) is \( \eta \)-strongly monotone and \( k \)-Lipschitz continuous, i.e., \( \|Ax - Ay\| \leq k\|x - y\| \) for all \( x, y \in C \), then \( A \) is \( \frac{\eta}{k^2} \)-inverse-strongly-monotone; see [14]. Let \( f \) be a continuously Fréchet differentiable convex function \( H \) and let \( \nabla f \) be the gradient of \( f \). If \( \nabla f \) is
\[ \alpha \text{-Lipschitz continuous, then } \nabla f \text{ is an } \alpha \text{-inverse-strongly-monotone mapping of } C \text{ into } H; \text{ see [1]. We also have that for all } x, y \in C \text{ and } \lambda > 0, \]
\[ \|(I - \lambda A)x - (I - \lambda A)y\|^2 = \|(x - y) - \lambda (Ax - Ay)\|^2 \]
\[ = \|x - y\|^2 - 2\langle x - y, Ax - Ay \rangle + \lambda^2 \|Ax - Ay\|^2 \]
\[ \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \]

So, if \( \lambda \leq 2\alpha \), then \( I - \lambda A \) is a nonexpansive mapping of \( C \) into \( H \).

**Theorem 8 ([7]).** Let \( C \) be a closed convex subset of a Hilbert space \( H \). Let \( A \) be an \( \alpha \)-inverse-strongly-monotone mapping of \( C \) into \( H \) and let \( S \) be a nonexpansive mapping of \( C \) into itself such that \( F(S) \cap VI(C, A) \neq \phi \). Let \( x_1 = x \in C \) and let \( \{x_n\} \) be a sequence defined by
\[ x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \ldots, \]
where \( \{\alpha_n\} \subset [0, 1) \) and \( \{\lambda_n\} \subset [a, b] \subset (0, 2\alpha) \) satisfy
\[ \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \]

Then, \( \{x_n\} \) converges strongly to \( z = P_{F(S) \cap VI(C, A)} x \).

**Theorem 9 ([34]).** Let \( C \) be a closed convex subset of a Hilbert space \( H \). Let \( A \) be an \( \alpha \)-inverse-strongly-monotone mapping of \( C \) into \( H \) and let \( S \) be a nonexpansive mapping of \( C \) into itself such that \( F(S) \cap VI(C, A) \neq \phi \). Let \( x_1 = x \in C \) and let \( \{x_n\} \) be a sequence defined by
\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \ldots, \]
where \( \{\alpha_n\} \) and \( \{\lambda_n\} \) satisfy
\[ 0 < c \leq \alpha_n \leq d < 1 \text{ and } 0 < a \leq \lambda_n \leq b < 2\alpha. \]

Then, \( \{x_n\} \) converges weakly to \( z \in F(S) \cap VI(C, A) \).

5. **Proximal point algorithms in Hilbert spaces**

We consider two proximal point algorithms for solving (2) in Section 1, with parameters \( \{r_n\} \), starting at an initial point \( x_1 \) in a Hilbert space \( H \).

**Theorem 10 ([9]).** Let \( H \) be a Hilbert space and let \( A \subset H \times H \) be a maximal monotone operator. Let \( x_1 = x \in H \) and let \( \{x_n\} \) be a sequence defined by
\[ x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{r_n}x_n, \quad n = 1, 2, \ldots, \]
where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy
\[ \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n \to \infty} r_n = \infty. \]

If \( A^{-1}0 \neq \phi \), then \( \{x_n\} \) converges strongly to \( Px \in A^{-1}0 \), where \( P \) is the metric projection of \( H \) onto \( A^{-1}0 \).

**Theorem 11 ([9]).** Let \( H \) be a Hilbert space and let \( A \subset H \times H \) be a maximal monotone operator. Let \( x_1 = x \in H \) and let \( \{x_n\} \) be a sequence defined by
\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)J_{r_n}x_n, \quad n = 1, 2, \ldots, \]
where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \alpha_n \in [0, k] \) for some \( k \) with \( 0 < k < 1 \) and \( \lim_{n \to \infty} r_n = \infty \). If \( A^{-1}0 \neq \phi \), then \( \{x_n\} \) converges weakly to \( v \in A^{-1}0 \), where \( v = \lim_{n \to \infty} Px_n \) and \( P \) is the metric projection of \( H \) onto \( A^{-1}0 \).

Using Theorems 10 and 11, we obtain the following theorems.

Theorem 12 ([9]). Let \( H \) be a Hilbert space and let \( f : H \to (-\infty, \infty] \) be a lower semicontinuous proper convex function. Let \( x_1 = x \in H \) and let \( \{x_n\} \) be a sequence defined by

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n}, x_n, \quad n = 1, 2, \ldots,
\]

\[
J_{r_n} x_n = \arg\min_{z \in H} \left\{ f(z) + \frac{1}{2r_n} \|z - x_n\|^2 : z \in H \right\},
\]

where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy

\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \to \infty} r_n = \infty.
\]

If \( (\partial f)^{-1}0 \neq \phi \), then \( \{x_n\} \) converges strongly to \( v \in H \), which is the minimizer of \( f \) nearest to \( x \). Further

\[
f(x_{n+1}) - f(v) \leq \alpha_n (f(x) - f(v)) + \frac{1 - \alpha_n}{r_n} \|J_{r_n} x_n - v\| \|J_{r_n} x_n - x_n\|.
\]

Theorem 13 ([9]). Let \( H \) be a Hilbert space and let \( f : H \to (-\infty, \infty] \) be a lower semicontinuous proper convex function. Let \( x_1 = x \in H \) and let \( \{x_n\} \) be a sequence defined by

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,
\]

\[
J_{r_n} x_n = \arg\min_{z \in H} \left\{ f(z) + \frac{1}{2r_n} \|z - x_n\|^2 : z \in H \right\},
\]

where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \alpha_n \in [0, k] \) for some \( k \) with \( 0 < k < 1 \) and \( \lim_{n \to \infty} r_n = \infty \). If \( (\partial f)^{-1}0 \neq \phi \), then \( \{x_n\} \) converges weakly to \( v \in H \), which is a minimizer of \( f \). Further

\[
f(x_{n+1}) - f(v) \leq \alpha_n (f(x) - f(v)) + \frac{1 - \alpha_n}{r_n} \|J_{r_n} x_n - v\| \|J_{r_n} x_n - x_n\|.
\]

Solodov and Svaiter [29] also proved the following strong convergence theorem.

Theorem 14 ([29]). Let \( H \) be a Hilbert space and let \( A \subset H \times H \) be a maximal monotone operator. Let \( x_1 \in H \) and let \( \{x_n\} \) be a sequence defined by

\[
\begin{align*}
x_1 &= x \in H, \\
0 &= v_n + \frac{1}{r_n} (y_n - x_n), \quad v_n \in Ay_n, \\
H_n &= \{z \in H : \langle z - y_n, v_n \rangle \leq 0\}, \\
W_n &= \{z \in H : \langle z - x_n, x_1 - x_n \rangle \leq 0\}, \\
x_{n+1} &= P_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots,
\end{align*}
\]

where \( \{r_n\} \) is a sequence of positive numbers. If \( A^{-1}0 \neq \phi \) and \( \lim \inf_{n \to \infty} r_n > 0 \), then \( \{x_n\} \) converges strongly to \( P_{A^{-1}0} x_1 \).
6. CONVERGENCE THEOREMS FOR ACCRETEIV OPERATORS

In this section, we study a strong convergence theorem of Halpern's type for accretive operators in a Banach space. We need the following lemma for the proof of our theorem.

**Lemma 15 ([35]).** Let $E$ be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator which satisfies the range condition. Suppose that every weakly compact convex subset of $E$ has the fixed point property for nonexpansive mappings. Let $C$ be a nonempty closed convex subset of $E$ such that $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$. If $A^{-1}0 \neq \emptyset$, then the strong limit $\lim_{t \to \infty} J_{t}x$ exists and belongs to $A^{-1}0$ for all $x \in C$.

See also Reich [23]. Using this result, we prove the following theorem. The proof is mainly due to Wittmann [36] and Shioji and Takahashi [27].

**Theorem 16 ([10]).** Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let $A \subset E \times E$ be an accretive operator which satisfies the range condition, and let $C$ be a nonempty closed convex subset of $E$ such that $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $x_{1} = x \in C$ and let $\{x_{n}\}$ be a sequence generated by

$$x_{n+1} = \alpha_{n}x + (1 - \alpha_{n}) J_{r_{n}}x_{n}, \quad n = 1, 2, \ldots,$$

where $\{\alpha_{n}\} \subset [0, 1]$ and $\{r_{n}\} \subset (0, \infty)$ satisfy

$$\lim_{n \to \infty} \alpha_{n} = 0, \quad \sum_{n=0}^{\infty} \alpha_{n} = \infty \quad \text{and} \quad \lim_{n \to \infty} r_{n} = \infty.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_{n}\}$ converges strongly to an element of $A^{-1}0$.

As a direct consequence of Theorem 16, we have the following:

**Theorem 17.** Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $A \subset E \times E$ be an $m$-accretive operator. Let $x_{1} = x \in E$ and let $\{x_{n}\}$ be a sequence generated by

$$x_{n+1} = \alpha_{n}x + (1 - \alpha_{n}) J_{r_{n}}x_{n}, \quad n = 1, 2, \ldots,$$

where $\{\alpha_{n}\} \subset [0, 1]$ and $\{r_{n}\} \subset (0, \infty)$ satisfy

$$\lim_{n \to \infty} \alpha_{n} = 0, \quad \sum_{n=0}^{\infty} \alpha_{n} = \infty \quad \text{and} \quad \lim_{n \to \infty} r_{n} = \infty.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_{n}\}$ converges strongly to an element of $A^{-1}0$.

Next, we prove a weak convergence theorem for Mann's type for accretive operators in a Banach space. Before proving the theorem, we need the following two lemmas.

**Lemma 18 ([3]).** Let $C$ be a closed bounded convex subset of a uniformly convex Banach space $E$ and let $T$ be a nonexpansive mapping of $C$ into itself. If $\{x_{n}\}$ converges weakly to $z \in C$ and $\{x_{n} - Tx_{n}\}$ converges strongly to $0$, then $Tz = z$.

**Lemma 19 ([22]).** Let $E$ be a uniformly convex Banach space whose norm is Fréchet differentiable, let $C$ be a nonempty closed convex subset of $E$ and let $\{T_{0}, T_{1}, T_{2}, \ldots\}$ be a sequence of nonexpansive mappings of $C$ into itself such that $\bigcap_{n=0}^{\infty} F(T_{n})$ is nonempty. Let $x \in C$ and $S_{n} = T_{n}T_{n-1}\cdots T_{0}$ for all $n = 1, 2, \ldots$. Then the set $\bigcap_{n=0}^{\infty} \overline{\mathrm{co}}\{S_{m}x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=0}^{\infty} F(T_{n})$. 
Theorem 20 ([10]). Let $E$ be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial’s condition, let $A \subset E \times E$ be an accretive operator which satisfies the range condition, and let $C$ be a nonempty closed convex subset of $E$ such that $\overline{D(A)} \supset C \subset \bigcap_{r>0} R(I+rA)$. Let $x_{1}=x \in C$ and let $\{x_{n}\}$ be a sequence generated by

$$x_{n+1} = \alpha_{n}x_{n} + (1-\alpha_{n})J_{r_{n}}x_{n}, \quad n = 1, 2, \ldots,$$

where $\{\alpha_{n}\} \subset [0, 1]$ and $\{r_{n}\} \subset (0, \infty)$ satisfy

$$\limsup_{n \to \infty} \alpha_{n} < 1 \quad \text{and} \quad \liminf_{n \to \infty} r_{n} > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_{n}\}$ converges weakly to an element of $A^{-1}0$.

As a direct consequence of Theorem 20, we have the following:

Theorem 21. Let $E$ be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial’s condition and let $A \subset E \times E$ be an m-accretive operator. Let $x_{1}=x \in E$ and let $\{x_{n}\}$ be a sequence generated by

$$x_{n+1} = \alpha_{n}x_{n} + (1-\alpha_{n})J_{r_{n}}x_{n}, \quad n = 1, 2, \ldots,$$

where $\{\alpha_{n}\} \subset [0, 1]$ and $\{r_{n}\} \subset (0, \infty)$ satisfy

$$\limsup_{n \to \infty} \alpha_{n} < 1 \quad \text{and} \quad \liminf_{n \to \infty} r_{n} > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_{n}\}$ converges weakly to an element of $A^{-1}0$.

7. Convergence Theorems for Maximal Monotone Operators

In this section, we study strong convergence theorems for resolvents of maximal monotone operators in a Banach space. Let $E$ be a uniformly convex and smooth Banach space and let $A$ be a maximal monotone operator from $E$ into $E^{*}$ such that $A^{-1}0 \neq \phi$. For $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_{r} - x) + rAx_{r}.$$

By Theorems 1 and 2, this equation has a unique solution $x_{r}$. We denote $J_{r}$ by $x_{r} = J_{r}x$ and such $J_{r}$, $r > 0$ are called resolvents of $A$. Now, we extend Solodov and Svaiter’s result [29].

Theorem 22 ([19]). Let $E$ be a uniformly convex and smooth Banach space and let $A$ be a maximal monotone operator from $E$ into $E^{*}$ such that $A^{-1}0 \neq \phi$. Suppose $\{x_{n}\}$ is the sequence generated by

$$\begin{align*}
x_{1} &\in E, \\
y_{n} &\in J_{r_{n}}x_{n}, \\
H_{n} &\subseteq \{z \in E : \langle y_{n} - z, J(x_{n} - y_{n}) \rangle \geq 0\}, \\
W_{n} &\subseteq \{z \in E : \langle x_{n} - z, J(x_{1} - x_{n}) \rangle \geq 0\}, \\
x_{n+1} &\in P_{H_{n} \cap W_{n}}x_{1}, \quad n = 1, 2, \ldots,
\end{align*}$$

where $\{r_{n}\}$ is a sequence of positive numbers. If $A^{-1}0 \neq \phi$ and $\liminf_{n \to \infty} r_{n} > 0$, then $\{x_{n}\}$ converges strongly to $P_{A^{-1}0}x_{1}$. 


Next, we establish another extension of Solodov and Svaiter’s result [29]. Before establishing it, we give a definition. Let $E$ be a reflexive, strictly convex and smooth Banach space. The function $\phi: E \times E \to (-\infty, \infty)$ is defined by
$$\phi(x, y) = \|x\|^2 - 2(x, Jy) + \|y\|^2$$
for $x, y \in E$. Let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that
$$\phi(x_0, x) = \inf\{\phi(z, x) : z \in C\}.$$  
(4)

So, if $C$ is a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space $E$ and $x \in E$, we define the mapping $Q_C$ of $E$ onto $C$ by $Q_Cx = x_0$, where $x_0$ is defined by (4). It is easy to see that in a Hilbert space, the mapping $Q_C$ is coincident with the metric projection.

**Theorem 23** ([11]). Let $E$ be a uniformly convex and uniformly smooth Banach space and let $A$ be a maximal monotone operator from $E$ into $E^*$ such that $A^{-1}0 \neq \phi$. Let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be the sequence generated by

$$x_1 \in E,$$
$$y_n = Q_{r_n}x_n,$$
$$H_n = \{z \in E : (z - y_n, Jx_n - Jy_n) \leq 0\},$$
$$W_n = \{z \in E : (z - x_n, Jx_n - Jx_n) \leq 0\},$$
$$x_{n+1} = Q_{H_n \cap W_n}x_1, \quad n = 1, 2, \ldots,$$

where $\{r_n\}$ is a sequence of positive numbers such that $\lim \inf_{n \to \infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $Q_{A^{-1}0}x_1$.

Recently, Kohsaka and Takahashi [12] proved a strong convergence theorem of Halpen’s type for maximal monotone operators in a Banach space.

**Theorem 24** ([12]). Let $E$ be a smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. Let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be a sequence defined as follows:

$$x_1 = x \in E,$$
$$x_{n+1} = J^{-1}(\alpha_nJx + (1 - \alpha_n)JQ_{r_n}x_n), \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \to \infty} r_n = \infty.$$

If $A^{-1}0 \neq \phi$, then $\{x_n\}$ converges strongly to $Q_{A^{-1}0}x$.

**Problem.** If $E$ and $E^*$ are uniformly convex Banach spaces, does Theorem 11 hold for maximal monotone operators $A \subset E \times E^*$?

We can apply Theorems 22, 23 and 24 to find a minimizer of a convex function $f$. Let $E$ be a real Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then the subdifferential $\partial f$ of $f$ is as follows:

$$\partial f(z) = \{v \in E^* : f(y) \geq f(z) + (y - z, v), \forall y \in E\}, \quad \forall z \in E.$$
Theorem 25 ([19]). Let $E$ be a uniformly convex and smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that $\{r_n\} \subset (0, \infty)$ satisfies $\lim \inf_{n \to \infty} r_n > 0$ and let $\{x_n\}$ be the sequence generated by

$$
\begin{align*}
x_1 & \in E \\
y_n & = \arg \min_{z \in E} \{ f(z) + \frac{1}{2r_n} \| z - x_n \|^2 \}, \\
H_n & = \{ z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0 \}, \\
W_n & = \{ z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geq 0 \}, \\
x_{n+1} & = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots
\end{align*}
$$

If $(\partial f)^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to the minimizer of $f$ nearest to $x_1$.

Proof. Since $f : E \to (-\infty, \infty]$ is a proper convex lower semicontinuous function, by Rockafellar [24], the subdifferential $\partial f$ of $f$ is a maximal monotone operator. We also know that

$$
y_n = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \| z - x_n \|^2 \right\}
$$

is equivalent to

$$
0 \in \partial f(y_n) + \frac{1}{r_n} J(y_n - x_n).
$$

So, we have

$$
0 \in J(y_n - x_n) + r_n \partial f(y_n).
$$

Using Theorem 22, we get the conclusion. \(\square\)

Theorem 26 ([11]). Let $E$ be a uniformly convex and uniformly smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that $\{r_n\} \subset (0, \infty)$ satisfies $\lim \inf_{n \to \infty} r_n > 0$ and let $\{x_n\}$ be the sequence generated by

$$
\begin{align*}
x_1 & \in E \\
y_n & = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \| z \|^2 - \frac{1}{r_n} \langle z, Jx_n \rangle \right\}, \\
0 & = v_n + \frac{1}{r_n} (Jy_n - Jx_n), \quad v_n \in \partial f(y_n), \\
H_n & = \{ z \in E : \langle z - y_n, v_n \rangle \leq 0 \}, \\
W_n & = \{ z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0 \}, \\
x_{n+1} & = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots
\end{align*}
$$

If $(\partial f)^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to the minimizer of $f$ nearest to $x_1$.

Proof. We also know that

$$
y_n = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \| z \|^2 - \frac{1}{r_n} \langle z, Jx_n \rangle \right\}
$$

is equivalent to

$$
0 \in \partial f(y_n) + \frac{1}{r_n} Jy_n - \frac{1}{r_n} Jx_n.
$$

So, we have $v_n \in \partial f(y_n)$ such that $0 = v_n + \frac{1}{r_n} (Jy_n - Jx_n)$. Using Theorem 23, we get the conclusion. \(\square\)

Using Theorem 24, we get the following theorem.
Theorem 27 ([12]). Let $E$ be a smooth and uniformly convex Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}0$ is nonempty. Let $\{x_n\}$ be a sequence defined as follows:
\[
x_1 = x \in E,
\]
\[
y_n = \arg \min_{y \in E} \left\{ f(y) + \frac{1}{2r_n} \|y\|^2 - \frac{1}{r_n} \langle y, Jx_n \rangle \right\},
\]
\[
x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n), \quad n = 1, 2, \ldots,
\]
where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy
\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \to \infty} r_n = \infty.
\]
Then, $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0}x$.

REFERENCES


