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Kyoto University
APPROXIMATION OF FIXED POINTS AND PROXIMAL POINT ALGORITHMS

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ABSTRACT. In this article, we give three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space. Then we discuss weak and strong convergence theorems for nonlinear operators of accretive and monotone type in a Hilbert space or a Banach space. In particular, we state weak and strong convergence theorems for resolvents of $m$-accretive operators and maximal monotone operators in a Banach space. Using these results, we also consider the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

1. INTRODUCTION

We consider the following problem: Let $f_0, f_1, f_2, \ldots, f_m$ be convex continuous functions of a Hilbert space $H$ into $\mathbb{R}$. Then, the problem is to find a $z \in C$ such that

$$f_0(z) = \min \{f_0(x) : x \in C\},$$

where $C = \{x \in H : f_1(x) \leq 0, f_2(x) \leq 0, \ldots, f_m(x) \leq 0\}$. Such a problem is called the convex minimization problem. Let us define a function $g : H \to (-\infty, \infty]$ as follows:

$$g(x) = \begin{cases} f_0(x), & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, $g$ is a proper lower semicontinuous convex function and a minimizer $z \in H$ of $g$ is a solution of the convex minimization problem (1). So, let $g : H \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Consider a convex minimization problem:

$$\min \{g(x) : x \in H\}. \quad (2)$$

For such a $g$, we can define a multivalued operator $\partial g$ on $H$ by

$$\partial g(x) = \{x^* \in H : g(y) \geq g(x) + \langle x^*, y-x \rangle, y \in H\}$$

for all $x \in H$. Such a $\partial g$ is said to be the subdifferential of $g$. A monotone operator $A : H \times H$ is called maximal if its graph

$$G(A) = \{(x, y) : y \in Ax\}$$

is not properly contained in the graph of any other monotone operator. We know that if $A$ is a maximal monotone operator, then $R(I + \lambda A) = H$ for all $\lambda > 0$. A monotone operator $A$ is also called $m$-accretive if $R(I + \lambda A) = H$ for all $\lambda > 0$. 

So, we can define, for each positive \( \lambda \), the resolvent \( J_\lambda : R(I + \lambda A) \to D(A) \) by 
\[
J_\lambda = (I + \lambda A)^{-1}.
\]
We know that \( J_\lambda \) is a nonexpansive mapping. If \( g : H \to (-\infty, \infty] \) is a proper lower semicontinuous convex function, then \( \partial g \) is a maximal monotone operator.

We know that one method for solving (2) is the proximal point algorithm first introduced by Martinet [16]. The proximal point algorithm is based on the notion of resolvent \( J_\lambda \), i.e.,
\[
J_\lambda x = \arg \min \left\{ g(z) + \frac{1}{2\lambda} \| z - x \|^2 : z \in H \right\}.
\]
The proximal point algorithm is an iterative procedure, which starts at a point \( x_1 \in H \), and generates recursively a sequence \( \{x_n\} \) of points \( x_{n+1} = J_\lambda x_n \), where \( \{\lambda_n\} \) is a sequence of positive numbers; see, for instance, Rockafellar [26].

On the other hand, Halpern [6] and Mann [15] introduced the following iterative schemes to approximate a fixed point of a nonexpansive mapping \( T \) of \( H \) into itself:
\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \ n = 1, 2, \ldots
\]
and
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \ n = 1, 2, \ldots,
\]
respectively, where \( x_1 = x \in H \) and \( \{\alpha_n\} \) is a sequence in \([0, 1]\). Recently, Nakajo and Takahashi [18] also introduced an iterative scheme of finding a fixed point of a nonexpansive mapping in a Hilbert space by using an idea of the hybrid method in mathematical programming.

In this article, we first state three convergence theorems for nonexpansive mappings in a Hilbert space. They are convergence theorems of Halpern's type, Mann's type and Nakajo-Takahashi's type. Then, we prove a strong convergence theorem of Halpern's type and a weak convergence theorem of Mann's type for inverse-strongly-monotone mappings in a Hilbert space. In Section 6, we prove weak and strong convergence theorems for resolvents of accretive operators in a Banach space. In Section 7, we consider the strong convergence of a sequence defined by resolvents of maximal monotone operators in a Banach space. Using these results, we also discuss the convex minimization problem of finding a minimizer of a proper lower semicontinuous convex function in a Hilbert space or a Banach space.

2. Preliminaries

Let \( E \) be a real Banach space with norm \( \| \cdot \| \) and let \( E^* \) denote the dual of \( E \). We denote the value of \( y^* \in E^* \) at \( x \in E \) by \( \langle x, y^* \rangle \). When \( \{x_n\} \) is a sequence in \( E \), we denote the strong convergence of \( \{x_n\} \) to \( x \in E \) by \( x_n \to x \) and the weak convergence by \( x_n \rightharpoonup x \). The modulus of convexity of \( E \) is defined by
\[
\delta(\epsilon) = \inf \left\{ 1 - \frac{\| x + y \|}{2} : \| x \| \leq 1, \| y \| \leq 1, \| x - y \| \geq \epsilon \right\}
\]
for every \( \epsilon \) with \( 0 \leq \epsilon \leq 2 \). A Banach space \( E \) is said to be uniformly convex if \( \delta(\epsilon) > 0 \) for every \( \epsilon > 0 \). If \( E \) is uniformly convex, then \( \delta \) satisfies that \( \delta(\epsilon/r) > 0 \) and
\[
\frac{\| x + y \|}{2} \leq r \left( 1 - \delta \left( \frac{\epsilon}{r} \right) \right)
\]
for every \( x, y \in E \) with \( \| x \| \leq r, \| y \| \leq r \) and \( \| x - y \| \geq \epsilon \). Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( E \). Then we know that
for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call $P_C$ the metric projection of $E$ onto $C$. The duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists. In the case, $E$ is called smooth. The norm of $E$ is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (3) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (3) is attained uniformly for $y \in U$. It is known that if the norm of $E$ is uniformly Gâteaux differentiable, then the duality mapping $J$ is single valued and uniformly norm to weak* continuous on each bounded subset of $E$. A Banach space $E$ is said to satisfy Opial’s condition [20] if for any sequence $\{x_n\} \subset E$, $x_n \to y$ implies

$$\lim_{n \to \infty} \|x_n - y\| < \lim_{n \to \infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial’s condition.

Let $C$ be a closed convex subset of $E$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of $T$ by $F(T)$. A closed convex subset $C$ of $E$ is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset $D$ of $C$ into itself has a fixed point in $D$. Let $D$ be a subset of $E$. We denote the closure of the convex hull of $D$ by $\overline{CD}$.

Let $I$ denote the identity operator on $E$. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. If $A$ is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \tau(y_1 - y_2)\|$$

for all $\tau > 0$. An accretive operator $A$ is said to satisfy the range condition if $D(A) \subset \bigcap_{\tau > 0} R(I + \tau A)$. If $A$ is accretive, then we can define, for each $\tau > 0$, a nonexpansive single valued mapping $J_\tau : R(I + \tau A) \to D(A)$ by $J_\tau = (I + \tau A)^{-1}$. It is called the resolvent of $A$. We also define the Yosida approximation $A_\tau$ by $A_\tau = (I - J_\tau)/\tau$. We know that $A_\tau x \in AJ_\tau x$ for all $x \in R(I + \tau A)$ and $\|A_\tau x\| \leq \inf\{|y| : y \in Ax\}$ for all $x \in D(A) \cap R(I + \tau A)$. We also know that for an accretive operator $A$ satisfying the range condition, $A^{-1}0 = F(J_\tau)$ for all $\tau > 0$. An accretive operator $A$ is said to be $m$-accretive if $R(I + \tau A) = E$ for all $\tau > 0$. A multi-valued operator $A : E \to 2^{E^*}$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator $A$ is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; see, for instance [32].

**Theorem 1.** Let $E$ be a reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a monotone operator. Then $A$ is maximal if and only if $R(J + \tau A) = E^*$ for all $\tau > 0$. 
Theorem 2. Let $E$ be a strictly convex and smooth Banach space and let $x, y \in E$. If $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.

By Theorem 1, a monotone operator $A$ in a Hilbert space $H$ is maximal if and only if $A$ is m-accretive.

3. APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS

There are three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space which are related to the problem of finding a minimizer of a convex function.

Halpern [6] introduced the following iterative scheme to approximate a fixed point of a nonexpansive mapping in a Hilbert space. For the proof, see Wittmann [36] and Takahashi [32].

Theorem 3 ([36]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n$, $n = 1, 2, \ldots$, where $\{\alpha_n\} \subset [0, 1]$ satisfies

$$\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$ 

Then, $\{x_n\}$ converges strongly to $Px \in F(T)$.

Mann [15] also introduced the iterative scheme for finding a fixed point of a nonexpansive mapping. For the proof, see Takahashi [32].

Theorem 4 ([15]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$, $n = 1, 2, \ldots$, where $\{\alpha_n\} \subset [0, 1]$ satisfies

$$0 \leq \alpha_n < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty.$$ 

Then, $\{x_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n \to \infty} Px_n$.

Recently, Nakajo and Takahashi [18] proved the following theorem for nonexpansive mappings in a Hilbert space by using an idea of the hybrid method in mathematical programming.

Theorem 5 ([18]). Let $C$ be a closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(T)$. Let $x_1 = x \in C$ and

$$\begin{align*}
y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, \ldots,
\end{align*}$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies

$$\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$
where \( \{\alpha_n\} \subset [0,1] \) satisfies \( \lim \inf_{n \to \infty} \alpha_n < 1 \) and \( P_{C_n \cap Q_n} \) is the metric projection of \( H \) onto \( C_n \cap Q_n \). Then, \( \{x_n\} \) converges strongly to \( Px \in F(T) \).

Shioji and Takahashi [27] extended Theorem 3 to that of a Banach space whose norm is uniformly Gâteaux differentiable. Let \( C \) and \( D \) be closed convex subsets of a Banach space \( E \) and let \( D \) be a subset of \( C \). Then, a mapping \( P \) of \( C \) onto \( D \) is called sunny if

\[
P(Px + t(x - Px)) = Px
\]

whenever \( Px + t(x - Px) \in C \) for \( x \in C \) and \( t \geq 0 \).

**Theorem 6 ([27]).** Let \( E \) be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let \( C \) be a nonempty closed convex subset of \( E \) and let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \) is nonempty. Let \( \{\alpha_n\} \) be a sequence of real numbers such that

\[
0 \leq \alpha_n \leq 1, \quad \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.
\]

Suppose \( x_1 = x \in C \) and \( \{x_n\} \) is given by

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots.
\]

Then, \( \{x_n\} \) converges strongly to \( Px \in F(T) \), where \( P \) is a unique sunny nonexpansive retraction of \( C \) onto \( F(T) \).

Reich [22] extended also Mann's result to that of a Banach space whose norm is Fréchet differentiable.

**Theorem 7 ([22]).** Let \( C \) be a closed convex subset of a uniformly convex Banach space \( E \) with a Fréchet differentiable norm, let \( T : C \to C \) be a nonexpansive mapping such that \( F(T) \) is nonempty, and let \( \{\alpha_n\} \) be a real sequence such that \( 0 \leq \alpha_n \leq 1 \) and \( \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty \). If \( x_1 = x \in C \) and

\[
x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n)x_n, \quad n = 1, 2, \ldots,
\]

then \( \{x_n\} \) converges weakly to a fixed point of \( T \).

**Problem.** Is a Hilbert space in Theorem 5 replaced by a uniformly convex and smooth Banach space?

4. **Approximating solutions of variational inequalities**

Let \( C \) be a closed convex subset of a Hilbert space \( H \). Then, a mapping \( A \) of \( C \) into \( H \) is called inverse-strongly-monotone if there exists a positive real number \( \alpha \) such that

\[
\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2
\]

for all \( x, y \in C \); see [4] and [14]. For such a case, \( A \) is called \( \alpha \)-inverse-strongly-monotone. If a mapping \( T \) of \( C \) into itself is nonexpansive, then \( A = I - T \) is \( \frac{1}{2} \)-inverse-strongly-monotone and \( F(T) = \text{VI}(C,A) \); for example, see [8]. A mapping \( A \) of \( C \) into \( H \) is called strongly monotone if there exists a positive number \( \eta \) such that

\[
\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2
\]

for all \( x, y \in C \). In such a case, we say that \( A \) is \( \eta \)-strongly monotone. If \( A \) is \( \eta \)-strongly monotone and \( k \)-Lipschitz continuous, i.e., \( \|Ax - Ay\| \leq k\|x - y\| \) for all \( x, y \in C \), then \( A \) is \( \frac{\eta}{k} \)-inverse-strongly-monotone; see [14]. Let \( f \) be a continuously Fréchet differentiable convex function \( H \) and let \( \nabla f \) be the gradient of \( f \). If \( \nabla f \) is
\[\frac{1}{\alpha}\text{-Lipschitz continuous, then } \nabla f \text{ is an } \alpha\text{-inverse-strongly-monotone mapping of } C \text{ into } H; \text{ see [1]. We also have that for all } x, y \in C \text{ and } \lambda > 0,\]
\[
\|(I - \lambda A)x - (I - \lambda A)y\|^2 = \|(x - y) - \lambda(Ax - Ay)\|^2
\]
\[
= \|x - y\|^2 - 2\langle x - y, Ax - Ay \rangle + \lambda^2\|Ax - Ay\|^2
\]
\[
\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.
\]

So, if \(\lambda \leq 2\alpha\), then \(I - \lambda A\) is a nonexpansive mapping of \(C\) into \(H\).

**Theorem 8 ([7]).** Let \(C\) be a closed convex subset of a Hilbert space \(H\). Let \(A\) be an \(\alpha\)-inverse-strongly-monotone mapping of \(C\) into \(H\) and let \(S\) be a nonexpansive mapping of \(C\) into itself such that \(F(S) \cap VI(C, A) \neq \phi\). Let \(x_1 = x \in C\) and let \(\{x_n\}\) be a sequence defined by
\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \ldots,
\]
where \(\{\alpha_n\} \subset [0, 1)\) and \(\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)\) satisfy
\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.
\]
Then, \(\{x_n\}\) converges strongly to \(z = P_{F(S) \cap VI(C, A)}x\).

**Theorem 9 ([34]).** Let \(C\) be a closed convex subset of a Hilbert space \(H\). Let \(A\) be an \(\alpha\)-inverse-strongly-monotone mapping of \(C\) into \(H\) and let \(S\) be a nonexpansive mapping of \(C\) into itself such that \(F(S) \cap VI(C, A) \neq \phi\). Let \(x_1 = x \in C\) and let \(\{x_n\}\) be a sequence defined by
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n = 1, 2, \ldots,
\]
where \(\{\alpha_n\}\) and \(\{\lambda_n\}\) satisfy
\[
0 < c \leq \alpha_n \leq d < 1 \quad \text{and} \quad 0 < a \leq \lambda_n \leq b < 2\alpha.
\]
Then, \(\{x_n\}\) converges weakly to \(z \in F(S) \cap VI(C, A)\).

5. Proximal point algorithms in Hilbert spaces

We consider two proximal point algorithms for solving (2) in Section 1, with parameters \(\{r_n\}\), starting at an initial point \(x_1\) in a Hilbert space \(H\).

**Theorem 10 ([9]).** Let \(H\) be a Hilbert space and let \(A \subset H \times H\) be a maximal monotone operator. Let \(x_1 = x \in H\) and let \(\{x_n\}\) be a sequence defined by
\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{r_n}x_n, \quad n = 1, 2, \ldots,
\]
where \(\{\alpha_n\} \subset [0, 1]\) and \(\{r_n\} \subset (0, \infty)\) satisfy
\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \to \infty} r_n = \infty.
\]
If \(A^{-1}0 \neq \phi\), then \(\{x_n\}\) converges strongly to \(Px \in A^{-1}0\), where \(P\) is the metric projection of \(H\) onto \(A^{-1}0\).

**Theorem 11 ([9]).** Let \(H\) be a Hilbert space and let \(A \subset H \times H\) be a maximal monotone operator. Let \(x_1 = x \in H\) and let \(\{x_n\}\) be a sequence defined by
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)J_{r_n}x_n, \quad n = 1, 2, \ldots,
\]
where \( \{ \alpha_n \} \subset [0, 1] \) and \( \{ r_n \} \subset (0, \infty) \) satisfy \( \alpha_n \in [0, k] \) for some \( k \) with \( 0 < k < 1 \) and \( \lim_{n \to \infty} r_n = \infty \). If \( A^{-1}0 \neq \phi \), then \( \{ x_n \} \) converges weakly to \( v \in A^{-1}0 \), where \( v = \lim_{n \to \infty} Px_n \) and \( P \) is the metric projection of \( H \) onto \( A^{-1}0 \).

Using Theorems 10 and 11, we obtain the following theorems.

**Theorem 12** ([9]). Let \( H \) be a Hilbert space and let \( f : H \to (-\infty, \infty] \) be a lower semicontinuous proper convex function. Let \( x_1 = x \in H \) and let \( \{ x_n \} \) be a sequence defined by

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,
\]

\[
J_{r_n} x_n = \arg \min \left\{ f(z) + \frac{1}{2r_n} \| z - x_n \|^2 : z \in H \right\},
\]

where \( \{ \alpha_n \} \subset [0, 1] \) and \( \{ r_n \} \subset (0, \infty) \) satisfy

\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \to \infty} r_n = \infty.
\]

If \( (\partial f)^{-1}0 \neq \phi \), then \( \{ x_n \} \) converges weakly to \( v \in H \), which is the minimizer of \( f \) nearest to \( x \). Further

\[
f(x_{n+1}) - f(v) \leq \alpha_n (f(x_n) - f(v)) + \frac{1 - \alpha_n}{r_n} \| J_{r_n} x_n - v \| \| J_{r_n} x_n - x_n \|.
\]

**Theorem 13** ([9]). Let \( H \) be a Hilbert space and let \( f : H \to (-\infty, \infty] \) be a lower semicontinuous proper convex function. Let \( x_1 = x \in H \) and let \( \{ x_n \} \) be a sequence defined by

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,
\]

\[
J_{r_n} x_n = \arg \min \left\{ f(z) + \frac{1}{2r_n} \| z - x_n \|^2 : z \in H \right\},
\]

where \( \{ \alpha_n \} \subset [0, 1] \) and \( \{ r_n \} \subset (0, \infty) \) satisfy \( \alpha_n \in [0, k] \) for some \( k \) with \( 0 < k < 1 \) and \( \lim_{n \to \infty} r_n = \infty \). If \( (\partial f)^{-1}0 \neq \phi \), then \( \{ x_n \} \) converges weakly to \( v \in H \), which is a minimizer of \( f \). Further

\[
f(x_{n+1}) - f(v) \leq \alpha_n (f(x_n) - f(v)) + \frac{1 - \alpha_n}{r_n} \| J_{r_n} x_n - v \| \| J_{r_n} x_n - x_n \|.
\]

Solodov and Svaiter [29] also proved the following strong convergence theorem.

**Theorem 14** ([29]). Let \( H \) be a Hilbert space and let \( A \subset H \times H \) be a maximal monotone operator. Let \( x \in H \) and let \( \{ x_n \} \) be a sequence defined by

\[
\begin{cases}
x_1 = x \in H, \\
0 = v_n + \frac{1}{r_n} (y_n - x_n), \quad v_n \in Ay_n, \\
H_n = \{ z \in H : \langle z - y_n, v_n \rangle \leq 0 \}, \\
W_n = \{ z \in H : \langle z - x_n, x_1 - x_n \rangle \leq 0 \}, \\
x_{n+1} = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots,
\end{cases}
\]

where \( \{ r_n \} \) is a sequence of positive numbers. If \( A^{-1}0 \neq \phi \) and \( \lim \inf_{n \to \infty} r_n > 0 \), then \( \{ x_n \} \) converges strongly to \( P_{A^{-1}0} x_1 \).
6. Convergence Theorems for Accretive Operators

In this section, we study a strong convergence theorem of Halpern's type for accretive operators in a Banach space. We need the following lemma for the proof of our theorem.

Lemma 15 ([35]). Let $E$ be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator which satisfies the range condition. Suppose that every weakly compact convex subset of $E$ has the fixed point property for nonexpansive mappings. Let $C$ be a nonempty closed convex subset of $E$ such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. If $A^{-1}0 \neq \emptyset$, then the strong limit $t \to \infty J_{t}x$ exists and belongs to $A^{-1}0$ for all $x \in C$.

See also Reich [23]. Using this result, we prove the following theorem. The proof is mainly due to Wittmann [36] and Shioji and Takahashi [27].

Theorem 16 ([10]). Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let $A \subset E \times E$ be an accretive operator which satisfies the range condition, and let $C$ be a nonempty closed convex subset of $E$ such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $x_{1} = x \in C$ and let $\{x_{n}\}$ be a sequence generated by

$$x_{n+1} = \alpha_{n}x + (1 - \alpha_{n})J_{r_{n}}x_{n}, \quad n = 1, 2, \ldots,$$

where $\{\alpha_{n}\} \subset [0, 1]$ and $\{r_{n}\} \subset (0, \infty)$ satisfy

$$\lim_{n \to \infty} \alpha_{n} = 0, \quad \sum_{n=0}^{\infty} \alpha_{n} = \infty \quad \text{and} \quad \lim_{n \to \infty} r_{n} = \infty.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_{n}\}$ converges strongly to an element of $A^{-1}0$.

As a direct consequence of Theorem 16, we have the following:

Theorem 17. Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $A \subset E \times E$ be an $m$-accretive operator. Let $x_{1} = x \in E$ and let $\{x_{n}\}$ be a sequence generated by

$$x_{n+1} = \alpha_{n}x + (1 - \alpha_{n})J_{r_{n}}x_{n}, \quad n = 1, 2, \ldots,$$

where $\{\alpha_{n}\} \subset [0, 1]$ and $\{r_{n}\} \subset (0, \infty)$ satisfy

$$\lim_{n \to \infty} \alpha_{n} = 0, \quad \sum_{n=0}^{\infty} \alpha_{n} = \infty \quad \text{and} \quad \lim_{n \to \infty} r_{n} = \infty.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_{n}\}$ converges strongly to an element of $A^{-1}0$.

Next, we prove a weak convergence theorem for Mann's type for accretive operators in a Banach space. Before proving the theorem, we need the following two lemmas.

Lemma 18 ([3]). Let $C$ be a closed bounded convex subset of a uniformly convex Banach space $E$ and let $T$ be a nonexpansive mapping of $C$ into itself. If $\{x_{n}\}$ converges weakly to $z \in C$ and $\{x_{n} - Tx_{n}\}$ converges strongly to 0, then $Tz = z$.

Lemma 19 ([22]). Let $E$ be a uniformly convex Banach space whose norm is Fréchet differentiable, let $C$ be a nonempty closed convex subset of $E$ and let $\{T_{0}, T_{1}, T_{2}, \ldots\}$ be a sequence of nonexpansive mappings of $C$ into itself such that $\bigcap_{n=0}^{\infty} F(T_{n})$ is nonempty. Let $x \in C$ and $S_{n} = T_{n}T_{n-1} \cdots T_{0}$ for all $n = 1, 2, \ldots$. Then the set $\bigcap_{n=0}^{\infty} \{S_{m}x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=0}^{\infty} F(T_{n})$. 
For the proof of Lemma 19, see Takahashi and Kim [33]. Now we can prove the following weak convergence theorem.

**Theorem 20** ([10]). Let $E$ be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, let $A \subset E \times E$ be an accretive operator which satisfies the range condition, and let $C$ be a nonempty closed convex subset of $E$ such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

As a direct consequence of Theorem 20, we have the following:

**Theorem 21.** Let $E$ be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let $A \subset E \times E$ be an m-accretive operator. Let $x_1 = x \in E$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

**7. Convergence Theorems for Maximal Monotone Operators**

In this section, we study strong convergence theorems for resolvents of maximal monotone operators in a Banach space. Let $E$ be a uniformly convex and smooth Banach space and let $A$ be a maximal monotone operator from $E$ into $E^*$ such that $A^{-1}0 \neq \emptyset$. For $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

By Theorems 1 and 2, this equation has a unique solution $x_r$. We denote $J_r$ by $x_r = J_r x$ and such $J_r, \ r > 0$ are called resolvents of $A$. Now, we extend Solodov and Svaiter's result [29].

**Theorem 22** ([19]). Let $E$ be a uniformly convex and smooth Banach space and let $A$ be a maximal monotone operator from $E$ into $E^*$ such that $A^{-1}0 \neq \emptyset$. Suppose $\{x_n\}$ is the sequence generated by

$$\begin{cases}
  x_1 \in E, \\
  y_n = J_{r_n} x_n, \\
  H_n = \{z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\
  W_n = \{z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geq 0\}, \\
  x_{n+1} = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots,
\end{cases}$$

where $\{r_n\}$ is a sequence of positive numbers. If $A^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0} x_1$. 


Next, we establish another extension of Solodov and Svaiter's result [29]. Before establishing it, we give a definition. Let $E$ be a reflexive, strictly convex and smooth Banach space. The function \( \phi : E \times E \to (-\infty, \infty) \) is defined by
\[
\phi(x, y) = \|x\|^2 - 2(x, Jy) + \|y\|^2
\]
for \( x, y \in E \). Let \( C \) be a nonempty closed convex subset of \( E \) and let \( x \in E \). Then there exists a unique element \( x_0 \in C \) such that
\[
\phi(x_0, x) = \inf \{ \phi(z, x) : z \in C \}.
\] (4)

So, if \( C \) is a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space \( E \) and \( x \in E \), we define the mapping \( Q_C \) of \( E \) onto \( C \) by \( Q_C x = x_0 \), where \( x_0 \) is defined by (4). It is easy to see that in a Hilbert space, the mapping \( Q_C \) is coincident with the metric projection.

**Theorem 23** ([11]). Let \( E \) be a uniformly convex and uniformly smooth Banach space and let \( A \) be a maximal monotone operator from \( E \) into \( E^* \) such that \( A^{-1} 0 \neq \phi \). Let \( Q_r = (J + rA)^{-1}J \) for all \( r > 0 \) and let \( \{x_n\} \) be the sequence generated by
\[
\begin{aligned}
x_1 &\in E, \\
y_n &= Q_{r_n}x_n, \\
H_n &= \{z \in E : (z - y_n, Jx_n - Jy_n) \leq 0\}, \\
W_n &= \{z \in E : (z - x_n, Jx_1 - Jx_n) \leq 0\}, \\
x_{n+1} &= Q_{H_n \cap W_n}x_1, \quad n = 1, 2, \ldots,
\end{aligned}
\]
where \( \{r_n\} \) is a sequence of positive numbers such that \( \liminf_{n \to \infty} r_n > 0 \). Then, \( \{x_n\} \) converges strongly to \( Q_{A^{-1}0}x_1 \).

Recently, Kohsaka and Takahashi [12] proved a strong convergence theorem of Halpen's type for maximal monotone operators in a Banach space.

**Theorem 24** ([12]). Let \( E \) be a smooth and uniformly convex Banach space and let \( A \subset E \times E^* \) be a maximal monotone operator. Let \( Q_r = (J + rA)^{-1}J \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows:
\[
x_1 = x \in E, \\
x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n)JQ_{r_n}x_n), \quad n = 1, 2, \ldots,
\]
where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy
\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n \to \infty} r_n = \infty.
\]
If \( A^{-1} 0 \neq \phi \), then \( \{x_n\} \) converges strongly to \( Q_{A^{-1}0}x \).

**Problem.** If \( E \) and \( E^* \) are uniformly convex Banach spaces, does Theorem 11 hold for maximal monotone operators \( A \subset E \times E^* \)?

We can apply Theorems 22, 23 and 24 to find a minimizer of a convex function \( f \). Let \( E \) be a real Banach space and let \( f : E \to (-\infty, \infty] \) be a proper convex lower semicontinuous function. Then the subdifferential \( \partial f \) of \( f \) is as follows:
\[
\partial f(z) = \{v \in E^* : f(y) \geq f(z) + (y - z, v), \forall y \in E\}, \quad \forall z \in E.
\]
Theorem 25 ([19]). Let $E$ be a uniformly convex and smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that \( \{r_n\} \subset (0, \infty) \) satisfies \( \lim \inf_{n \to \infty} r_n > 0 \) and let \( \{x_n\} \) be the sequence generated by
\[
\begin{align*}
x_1 & \in E \\
y_n & = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \| z - x_n \|^2 \right\}, \\
H_n & = \{ z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0 \}, \\
W_n & = \{ z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geq 0 \}, \\
x_{n+1} & = P_{H_n \cap W_n}x_1, \quad n = 1, 2, \ldots,
\end{align*}
\]
If \( (\partial f)^{-1}0 \neq \phi \), then \( \{x_n\} \) converges strongly to the minimizer of \( f \) nearest to \( x_1 \).

Proof. Since \( f : E \to (-\infty, \infty] \) is a proper convex lower semicontinuous function, by Rockafellar [24], the subdifferential \( \partial f \) of \( f \) is a maximal monotone operator. We also know that
\[
y_n = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \| z - x_n \|^2 \right\}
\]
is equivalent to
\[
0 \in \partial f(y_n) + \frac{1}{r_n} J(y_n - x_n).
\]
So, we have
\[
0 \in J(y_n - x_n) + r_n \partial f(y_n).
\]
Using Theorem 22, we get the conclusion. \( \square \)

Theorem 26 ([11]). Let $E$ be a uniformly convex and uniformly smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that \( \{r_n\} \subset (0, \infty) \) satisfies \( \lim \inf_{n \to \infty} r_n > 0 \) and let \( \{x_n\} \) be the sequence generated by
\[
\begin{align*}
x_1 & \in E \\
y_n & = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \| z \|^2 - \frac{1}{r_n} \langle z, Jx_n \rangle \right\}, \\
0 & = v_n + \frac{1}{r_n} (Jy_n - Jx_n), \quad v_n \in \partial f(y_n), \\
H_n & = \{ z \in E : \langle z - y_n, v_n \rangle \leq 0 \}, \\
W_n & = \{ z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0 \}, \\
x_{n+1} & = P_{H_n \cap W_n}x_1, \quad n = 1, 2, \ldots.
\end{align*}
\]
If \( (\partial f)^{-1}0 \neq \phi \), then \( \{x_n\} \) converges strongly to the minimizer of \( f \) nearest to \( x_1 \).

Proof. We also know that
\[
y_n = \arg \min_{z \in E} \left\{ f(z) + \frac{1}{2r_n} \| z \|^2 - \frac{1}{r_n} \langle z, Jx_n \rangle \right\}
\]
is equivalent to
\[
0 \in \partial f(y_n) + \frac{1}{r_n} Jy_n - \frac{1}{r_n} Jx_n.
\]
So, we have \( v_n \in \partial f(y_n) \) such that \( 0 = v_n + \frac{1}{r_n} (Jy_n - Jx_n) \). Using Theorem 23, we get the conclusion. \( \square \)

Using Theorem 24, we get the following theorem.
Theorem 27 ([12]). Let $E$ be a smooth and uniformly convex Banach space and let $f : E \rightarrow (-\infty, \infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}0$ is nonempty. Let $\{x_n\}$ be a sequence defined as follows:

\[
x_1 = x \in E,
\]

\[
y_n = \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r_n} \|y\|^2 - \frac{1}{r_n} \langle y, Jx_n \rangle \right\},
\]

\[
x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n), \quad n = 1, 2, \ldots,
\]

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n \to \infty} r_n = \infty.
\]

Then, $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0}x$.

REFERENCES