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Unlinked Allocations in an Exchange Economy with One Good and One Bad

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Abstract

We consider an exchange economy with two commodities, of which one is a good, which generates utility to all consumers, and the other is a bad, which causes disutility to all consumers. We look into "unlinked" allocations, that is, allocations at which almost every consumer consumes either the good or the bad, but not both, and ask under what conditions there exist unlinked and individually rational allocations and also unlinked and envy-free allocations. We also examine efficient and equilibrium allocations, taking special care of the cases where the so called minimum condition is violated.

1 Introduction

1.1 Overview of the Results

In this paper, we investigate *unlinked* allocations in an exchange economy with two commodities, of which one is a good, which generates utility to all consumers, and the other is a bad, which causes disutility to all consumers. While the good can be considered just as any usual consumption good, the bad should be considered as garbage or toxic wastes. The exchange economy may consist of infinitely many consumers, and an "unlinked" allocation is, by definition, an allocation at which almost every consumer consumes either the good or the bad, but not both.

The properties we explore on unlinked allocations are the individual rationality and the envy-freeness. Propositions 4.2 shows that there exists an unlinked and individually rational allocation if and only if the initial endowment allocation is unlinked. Moreover, then, the initial endowment allocation is essentially the unique allocation that is both unlinked and individually rational. Proposition 5.2 establishes analogous results for the

*This is a by-product of the collaborative work with Akira Yamazaki.

envy-freeness. Since an equilibrium allocation is individually rational (and also envy-free in terms of net demands), these results imply that there is no unlinked equilibrium allocation unless the initial endowment allocation is unlinked.

We also look into efficient allocations in such an economy. It is well known that if utility functions were to be strictly increasing in every commodity, the strong and weak Pareto efficiency would be equivalent. On the other hand, as shown by Example 6.3, if there are more than one bad, then the strong efficiency may indeed be more stringent than the weak efficiency. We prove (Lemma 6.2) that in our setting of one good and one bad, the equivalence between the two notions of efficiency still obtains, and a careful analysis of unlinked allocations will turn out to be a critical step of the proof. Also, at an unlinked allocation, since almost every consumer consumes only one of the two commodities, the so called minimum income condition (or its generalization, the local cheaper point condition)is likely to be violated. It is therefore necessary to distinguish the notion of equilibrium from that of the so called quasi-equilibrium (or sometimes called pseudo-equilibrium). We shall also provide an equivalent condition (Proposition 8.3) for the existence of an unlinked quasi-equilibrium allocation.

1.2 Motivation

Geometrically speaking, an unlinked allocation is an allocation of which the distribution of the consumers' commodity bundles is concentrated on the axes bordering the non-negative orthant. Why should we bother to investigate unlinked allocations in details, given that their distributions appear rather unusual? There are at least three reasons why our exploration is indeed worthwhile.

First, as will be established as Proposition 7.6 below, every unlinked allocation is efficient as long as one commodity is a good and the other bad for everyone, regardless of other specifications of utility functions or the number of consumers in the economy. Our contention is that the one-good, one-bad setup is the benchmark model for the general equilibrium analysis of bads; and that the efficiency of unlinked allocations is so "robust" that they should merit special attention.

Second, as shown in Example 8 and Proposition 9 of Hara (2002), the unlinked allocations may be the only efficient allocations in an economy with an atomless space of consumers. This implies, along with Proposition 4.2, that there is no equilibrium in such an economy unless the initial endowment allocation is unlinked.[1] This pervasive non-existence of an equilibrium is an disturbing fact, because the atomless space of consumers is widely perceived as a good description of perfect competition, and should also constitute a benchmark model for the analysis of efficient allocation of bads via the price mechanism. The analysis of unlinked allocations in this paper clarifies the source of the non-existence from the viewpoint of welfare economics.

Third, unlinked allocations have often been put aside when establishing some im-

[1] In the example, there is no equilibrium at all even when the initial endowment allocation is unlinked, due to its particular construction of utility functions, which violate the condition of Proposition 8.4. It is, however, possible to modify the example so that there is an equilibrium whenever the initial endowment allocation is unlinked.

portant propositions in economics. One such instance is Proposition 4.3.6 of Mas-Colell (1985), which establishes, under the assumption of monotone preference relations, the uniqueness (up to positive multiplication) of the support price vecotr of every efficient, linked allocation. Another is the irreducibility condition due to McKenzie (1959, 1981). Indeed, one can prove that the irreducibility condition is not satisfied if e is unlinked. On the other hand, Propositions 8.2 shows that in Example 8 of Hara (2002), there is an equilibrium only if e is unlinked. In our setting, therefore, there is no equilibrium whenever the irreducibility condition is met. This is a rather curious phenomenon given the vital role played by the irreducibility condition to establish the existence of an equilibrium.[2]

1.3 Organization of the Paper

The next section lays out our framework. Section 3 gives a formal definition of an unlinked allocation and an important lemma, which underlies the analysis of the paper. Sections 4 and 5 deal with individually rational and envy-free allocations. Sections 6 and 7 deal with efficiency and price supportability. Necessary and sufficient conditions for the existence of equilibria are given in Section 8.

2 Setup

The space of (names of) consumers is given by a measure space $(A, \mathscr{B}, \mu)$ with $0 < \mu(A) < \infty$. Denote by $\mathscr{V}$ the set of all real-valued functions defined on $X = R^2_+$ that are continuous, quasi-concave, strictly increasing in the first coordinate, and strictly decreasing in the second coordinate, endowed with the C^0 compact open topology. This is the space of utility functions we shall consider in this paper. The interpretation is that there are two commodities; the consumption set is the non-negative orthant; the first commodity is a good; and the second commodity is a bad. An (private ownership) economy is characterized as a pair of a Borel-measurable mapping $u : A \to \mathscr{V}$ and an integrable mapping $e : A \to X$ with $\int_A e \in R^2_{++}$. At this point and in the rest of this paper, the measurability is meant to be with respect to $\mathscr{B}$ and the Borel σ-fields of $\mathscr{V}$ and X. The interpretation is that the utility function $u(a)$, which we also write u_a, represents consumer a's preference relation and $e(a)$ is his initial endowment vector. Notice that every consumer's endowments are non-negative for both commodities; the aggregate endowments are strictly positive; but an individual consumer's endowments in either commodity may be zero.

Definition 2.1 Let $B \in \mathscr{B}$, then a measurable mapping $f : B \to X$ is an *allocation within* B if it is integrable and satisfies $\int_B f = \int_B e$. An allocation within A is also simply called an allocation.

[2]The existence of a quasi-equilibrium is even more starkly incompatible with the irreducibility condition in our setting. The necessary and sufficient condition of Proposition 8.3 for the existence of an unlinked pseudo-equilibrium in terms of the initial endowment allocation is satisfied if and only if the irreducibility condition is violated.

Note that an allocation, by definition, satisfies the resource feasibility constraint, which is met with the strict equality rather than the weak inequality $\int_B f \leq \int_B e$, to eliminate the possibility of free disposal.

3 Unlinked Allocations

The following definition is taken from Proposition 7.2.7 of Mas-Colell (1985).[3]

Definition 3.1 Let $B \in \mathscr{B}$ and f be an allocation within B, then f is *unlinked* if $f(a) \notin R^2_{++}$ for almost every $a \in B$.

The following lemma implies that no unlinked allocation within any group of consumers can be Pareto-improving for the group. It underlies the analysis of this paper.

Lemma 3.2 *Let $B \in \mathscr{B}$ and f be an allocation within B. If g is an unlinked allocation within B and $u_a(g(a)) \geq u_a(f(a))$ for almost every $a \in B$, then $g(a) = f(a)$ for almost every $a \in B$.*

Proof of Lemma 3.2 By taking B smaller if necessary, we can assume that $g(a) \notin R^2_{++}$ and $u_a(g(a)) \geq u_a(f(a))$ for *every* $a \in B$.

Since g is an unlinked allocation within B, for every $a \in B$, if $g_2(a) > 0$, then $g_1(a) = 0 \leq f_1(a)$. Since $u_a(g(a)) \geq u_a(f(a))$, this implies that $g_2(a) \leq f_2(a)$. This of course holds when $g_2(a) = 0$. Thus $g_2(a) \leq f_2(a)$ for every $a \in B$. Since $\int_B g_2 = \int_B e_2 = \int_B f_2$, this implies that $g_2(a) = f_2(a)$ for every $a \in B$. Since $u_a(g(a)) \geq u_a(f(a))$, this implies that $f_1(a) \geq g_1(a)$ for every $a \in B$. As before, then, $g_1(a) = f_1(a)$ for every $a \in B$. ///

4 Individual Rationality

The following definition is standard.

Definition 4.1 An allocation f is *individually rational* if $u_a(f(a)) \geq u_a(e(a))$ for almost every $a \in A$.

The first application of Lemma 3.2 is the following lemma.

Proposition 4.2 *There exists an individually rational, unlinked allocation if and only if e is unlinked. Moreover, if f is an individually rational, unlinked allocation, then $f(a) = e(a)$ for almost every $a \in A$.*

[3]There are apparently many predecessors of this definition. Also, Definition 4.3.5 of Mas-Colell (1985) is more explicit but less suited to our analysis because he assumed differentiability of utility functions while we do not.

Proof of Proposition 4.2 By Lemma 3.2, if f is an individually rational, unlinked allocation, then $f(a) = e(a)$ for almost every $a \in A$. Thus, such an allocation does indeed exists if and only if e is unlinked. ///

5 Envy-Freeness

We consider the envy-free property with respect to net demands.

Definition 5.1 An allocation f is *envy-free* if there exists a $B \in \mathscr{B}$ such that $\mu(A) = \mu(B)$ and for every $a \in B$ and every $b \in B$, if $e(a) + (f(b) - e(b)) \in X$, then $u_a(e(a) + (f(b) - e(b))) \leq u_a(f(a))$.

The definition states that at an envy-free allocation, almost no consumer can get strictly better off by receiving the net demands that another consumer receives.

Proposition 5.2 *There exists an envy-free, unlinked allocation if and only if* e *is unlinked. Moreover, if* f *is an envy-free, unlinked allocation, then* $f(a) = e(a)$ *for almost every* $a \in A$.

Proof of Proposition 5.2 Suppose that f is an envy-free, unlinked allocation and define

$$A_1 = \{a \in A \mid f_1(a) > e_1(a)\},$$
$$A_2 = \{a \in A \mid f_2(a) > e_2(a)\}.$$

Then $\mu(A_1 \cap A_2) = 0$ because f is unlinked. Suppose that $\mu(A_1) > 0$ and $\mu(A_2) > 0$, then

$$\mu(A_1 \cap (A \setminus A_2)) = \mu(A_1 \setminus (A_1 \cap A_2)) = \mu(A_1) > 0,$$
$$\mu((A \setminus A_1) \cap A_2) = \mu(A_2 \setminus (A_1 \cap A_2)) = \mu(A_2) > 0.$$

But the consumers $a \in (A \setminus A_1) \cap A_2$ would envy those $a \in A_1 \setminus (A_1 \cap A_2)$ and hence f could not be envy-free. We must thus have either $\mu(A_1) = 0$ and $\mu(A_2) = 0$. If $\mu(A_1) = 0$, then $f_1(a) \leq e_1(a)$ for almost every $a \in A$ and, since $\int_A f_1 = \int_A e_1$, $f_1(a) = e_1(a)$ for almost every $a \in A$. Then, by the envy-free property, $f_2(a) = e_2(a)$ for almost every $a \in A$. Thus $f(a) = e(a)$ for almost every $a \in A$. We can analogously show that the same conclusion is obtained also when $\mu(A_2) = 0$.

We have thus shown that if f is an envy-free, unlinked allocation, then $f(a) = e(a)$ for almost every $a \in A$. This proves that if e is unlinked, then it is the only envy-free and unlinked allocation; otherwise, there is no envy-free, unlinked allocation. ///

6 Efficiency

In the presence of a bad, it is helpful to distinguish two notions of efficient allocations.

Definition 6.1 An allocation is *(strongly) efficient* if there is no allocation g such that $u_a(g(a)) \geq u_a(f(a))$ for almost every $a \in A$ and there exists a $B \in \mathscr{B}$ such that $\mu(B) > 0$ and $u_a(g(a)) > u_a(f(a))$ for every $a \in B$. It is *weakly efficient* if there is no allocation g such that $u_a(g(a)) > u_a(f(a))$ for almost every $a \in A$.

It is well known that the two notions of efficiency are equivalent with each other if utility functions are strictly increasing in every commodity. This equivalence is still valid in our present setting of one good and one bad.

Lemma 6.2 *An allocation is strongly efficient if and only if it is weakly efficient.*

Proof of Lemma 6.2 It is sufficient to prove that if there exists a weak improvement g on an allocation f, then there also exists a strong improvement on f. Define

$$B = \{a \in A \mid u_a(g(a)) > u_a(f(a))\},$$
$$C = \{a \in A \mid g(a) \in R^2_{++}\}$$

If $\mu(B \cap C) > 0$, then it is possible to re-distribute small amounts of the good from $B \cap C$ to $A \setminus B$ to obtain a strong improvement on f. Since $\mu(C) > 0$ by Lemma 3.2, if $\mu(B \cap C) = 0$, then $\mu((A \setminus B) \cap C) > 0$. We shall construct another weak improvement h on f in the following manner. For each $a \in (A \setminus B) \cap C$, let $k(a) \in R^2_{++}$ be a sufficiently short vector such that consumer a is indifferent between $f(a)$ and $f(a) - k(a)$. We can then transfer $\int_C k$ to B so that, even after the transfers, every consumer $a \in B$ is strictly better off than at f. Denote the resulting allocation by h, then h is still a weak improvement on f and, if B and C are defined for h in place of g as above, then h satisfies $\mu(B \cap C) > 0$. Then, just as before, we can construct a strong improvement on f by transferring small amounts of the good from B to $A \setminus B$. ///

As there is no need to distinguish the two notions of efficiency, we refer to them simply as "efficiency" in the rest of this paper.

We show by means of an example that the equivalence would no longer hold if there were more than two types of bads.

Example 6.3 Let $A = \{1, 2, 3\}$, $\mathscr{B}$ be the power set of A, and μ be the counting measure. In this example, assume that there are three commodities, of which the first one is a good and the others are bads. More specifically, define $u_a(x) = x_1 - x_2^2 - x_3^2$ and $e(a) = (1, 1, 1)$ for each $a \in A$. Also define an allocation f by

$$f(a) = \begin{cases} (0, 2, 1) & \text{for } a = 1, \\ (0, 1, 2) & \text{for } a = 2, \\ (3, 0, 0) & \text{for } a = 3. \end{cases}$$

Then f is weakly efficient because the consumer $a = 3$ cannot be made strictly better off. But f is not strongly efficient because the marginal rates of substitution between the two bads are not equated between consumers $a = 1$ and $a = 2$.

7 Supportability

Two versions of price supportability are given below. By a *price vector*, we shall always mean a non-zero vector in R^2.

Definition 7.1 An allocation f is *strongly supportable* by a price vector p if for almost every $a \in A$ and every $x \in X$, $p \cdot x > p \cdot f(a)$ whenever $u_a(x) > u_a(f(a))$. It is *weakly supportable* by p if for almost every $a \in A$ and every $x \in X$, $p \cdot x \geq p \cdot f(a)$ whenever $u_a(x) > u_a(f(a))$.

For each consumer a, we shall refer to the utility maximization condition for strong supportability as the *strong utility maximization condition* and the utility maximization condition for weak supportability as the *weak utility maximization condition.* The above notion of weak supportability has been considered as part of definitions of a pseudo-equilibrium or quasi-equilibrium in Hildenbrand (1968), Mas-Colell, Whinston, and Green (1995, Definition 16.D.1), Hurwicz and Richter (2001), and probably many others. As they have pointed out, with the locally non-satiation assumption on utility functions, then the weak utility maximization condition is equivalent to the cost minimization condition, that is, for every $x \in X$, if $u_a(x) \geq u_a(f(a))$, then $p \cdot x \geq p \cdot f(a)$. Given this, we see that, for each consumer $a \in A$, the weak utility maximization condition is equivalent to the strong utility maximization condition if $p \cdot f(a) > \inf\{p \cdot x \mid x \in X\}$, that is, the so called minimum income condition is met. We thus obtain the following lemma.

Lemma 7.2 *If an allocation f is weakly supportable by a price vector p and if $p \cdot f(a) > \inf\{p \cdot x \mid x \in X\}$ for almost every $a \in A$, then f is strongly supportable by the same price vector p.*

The following lemma is a consequence of the one-good, one-bad assumption.

Lemma 7.3 *If an allocation is weakly supportable by a price vector p, then $p_1 > 0$ and $p_2 \leq 0$. If it is strongly supportable by p, then $p_2 < 0$.*

Proof of Lemma 7.3 Suppose that an allocation f is weakly supportable by p. Then there exist a $B \in \mathscr{B}$ such that $\mu(B) > 0$ and $f_2(a) > 0$ for every $a \in B$. If $p_2 > 0$, then the weak utility maximization condition for $a \in B$ would be the strong utility maximization condition, but it could then not be met because the second commodity is a bad. We must thus have $p_2 \leq 0$. Of course, if $p_1 < 0$, then the weak utility maximization condition would be the strong maximization condition for every consumer, which could then not be met because the first commodity is a good. We thus have $p_1 \geq 0$. Given $p_2 \leq 0$ and $p \neq 0$, if $p_1 = 0$, then $p_2 < 0$, and again the weak utility maximization condition would be the strong maximization condition for every consumer, which could not be met because the first commodity is a good. Thus $p_1 > 0$. The first part of this lemma is thus established. If $p_2 = 0$, then the strong utility maximization condition would not be met by any $a \in B$. The second part thus follows. ///

The following lemma is, in a sense, converse to Lemma 7.3.

Lemma 7.4 *If an allocation is weakly supported by a price vector p with $p_2 < 0$, then it is strongly supported by p.*

This follows from Lemma 7.2 and $\inf\{p \cdot x \mid x \in X\} = -\infty$ if $p_2 < 0$. The two lemmas together imply that a weakly supportable allocation is strongly supportable if and only if there exists a weakly supporting price vector for which the bad has a negative price.

Since strong and weak efficiency are equivalent by Lemma 6.2, by combining the first and second welfare theorems, we obtain the following equivalence.

Theorem 7.5 *An allocation is efficient if and only if it is weakly supportable.*

Proof of Theorem 7.5 The proof is so standard that we only give a sketch. The only point worth making is that the following standard argument is valid even when the utility functions are decreasing in the bad.

If f is an allocation weakly supportable by a price vector p, then, since $\int_A f \in R^2_{++}$, there exists a $B \in \mathscr{B}$ such that $\mu(B) > 0$ and $p \cdot f(a) > \inf\{\{p \cdot x \mid x \in X\}$ for every $a \in B$. Thus the weak utility maximization condition is indeed the strong utility maximization condition for every $a \in B$. Now, if there were to exist a strong improvement g on f, then $p \cdot g(a) \geq p \cdot f(a)$ for almost every $a \in A$ and $p \cdot g(a) > p \cdot f(a)$ for almost every $a \in B$. This is a contradiction. Hence f is efficient, by Lemma 6.2.

Suppose conversely that f is efficient. Define a correspondence Φ of A into X by $\Phi(a) = \{x \in X \mid u_a(x) > u_a(f(a))\}$. Then the integral $\int_A \Phi \subseteq X$ is non-empty (as, for example, $a \mapsto f(a) + (1,0)$ is an integrable selection of Φ) and convex because $\Phi(a)$ is convex for every $a \in A$.[4] Moreover, by efficiency, $\int_A e \notin \int_A \Phi$. Thus, there is a price vector p that supports $\int_A \Phi$ at $\int_A e$. Then f is weakly supportable by p. ///

It is of course well known (as exemplified by Figure 16.D.2 of Mas-Colell, Whinston, and Green (1995)) that strong supportability does not necessarily follow from efficiency.

The last lemma of this subsection is the weak supportability of unlinked allocations.

Proposition 7.6 *Every unlinked allocation is weakly supportable by a price vector p with $p_1 > 0$ and $p_2 = 0$, and thus efficient.*

The first part is straightforward. Efficiency follows from Theorem 7.5. As mentioned in the introduction, in an economy with an atomless space of consumers, the converse may be the case as well.

[4]Consult Hildenbrand (1974, D.II. 2) for details.

8 Equilibrium

Definition 8.1 An allocation f is a *strong equilibrium allocation* if there exists a price vector p such that f is strongly supportable by p and

$$p \cdot e(a) = p \cdot f(a) \tag{1}$$

for almost every $a \in A$. It is a *weak equilibrium allocation* if there exists a price vector p such that f is weakly supportable by p and (1) holds.

The strong equilibrium allocation is commonly known simply as an equilibrium allocation, but we opt for adding the adjective "strong" to distinguish it from a weak equilibrium allocation. The weak equilibrium concept coincides with the pseudo-equilibrium and quasi-equilibrium referred to in Section 7, but we call it an "weak" equilibrium for simplicity. Even if the equality is replaced by a weak inequality $\geq$, the condition would still be equivalent in our setup, because the utility functions are strictly increasing in the good.

Based on the previous results, we first give a necessary condition for the existence of an unlinked strong equilibrium allocations. It implies that a strong equilibrium must necessarily be the essentially no-trade equilibrium.

Proposition 8.2 *If there exists an unlinked strong equilibrium allocation, then e is unlinked. Moreover, if f is a strong equilibrium unlinked allocation, then $f(a) = e(a)$ for almost every $a \in A$.*

This follows from Proposition 4.2 or 5.2, because every strong equilibrium allocation is efficient, individually rational, and envy-free.

Next, we give an equivalent condition for the existence of an unlinked weak equilibrium allocation.

Proposition 8.3 *There exists an unlinked weak equilibrium allocation if and only if $\mu(\{a \in A \mid e_1(a) = 0\}) > 0$. Moreover, then, an unlinked allocation f is a weak equilibrium allocation if and only if $f_1(a) = e_1(a)$ for almost every $a \in A$.*

In words, there exists an unlinked weak equilibrium allocation if and only if there is a positive proportion of consumers not endowed with the good. Since f is unlinked, the above proposition implies that $f_2(a) = 0$ for almost every $a \in A$ with $e_1(a) > 0$. That is, the burden of consuming the bad is borne exclusively by the poor consumers who are not endowed with the good at all.

Proof of Proposition 8.3 Define $B = \{a \in A \mid e_1(a) = 0\}$. Suppose that f is an unlinked weak equilibrium allocation. If it also a strong equilibrium allocation, then, by Proposition 8.2, e is unlinked and $f(a) = e(a)$ for almost every $a \in A$. Thus, of course, $f_1(a) = e_1(a)$ for almost every $a \in A$, and $\mu(B) \geq \mu(\{a \in A \mid e_2(a) > 0\}) > 0$. Suppose next that f is not a strong equilibrium allocation. By Lemma 7.4, if p is a corresponding support price vector, then $p_1 > 0$ and $p_2 = 0$. For almost every $a \in B$, $p_1 f_1(a) = p \cdot f(a) = p \cdot e(a) = p_1 e_1(a) = 0$. Thus $f_1(a) = 0 = e_1(a)$. For every $a \notin B$ with

$e_1(a) > 0$, the weak utility maximization condition is equivalent to the strong one; and the latter implies that $f_1(a) = e_1(a)$. Thus $f_1(a) = e_1(a)$ for almost every $a \in A$. Since f is linked, this implies that $\mu(B) \geq \mu(\{a \in A \mid f_2(a) > 0\}) > 0$.

Conversely, suppose that $\mu(B) > 0$. Define an allocation f by

$$f(a) = \begin{cases} (e_1(a), 0) & \text{if } a \in A \setminus B, \\ \left(0, \dfrac{1}{\mu(B)} \displaystyle\int_A e_2\right) & \text{if } a \in B. \end{cases}$$

Then f is an unlinked weak equilibrium allocation under $p = (1, 0)$. Finally, if $\mu(B) > 0$ and f is an unlinked allocation such that $f_1(a) = e_1(a)$ for almost every $a \in A$, then it is a weak equilibrium allocation under the price vector $p = (1, 0)$, where almost every consumer $a \notin B$ satisfy the strong utility maximization condition and almost every consumer $a \in B$ satisfy the weak utility maximization condition merely because $p \cdot e(a) = 0$. ///

While Proposition 8.3 gives a necessary and sufficient condition for a weak equilibrium allocation to exist, Proposition 8.2 gives only a necessary condition for a strong equilibrium allocation to exist. We now offer a sufficient condition, which is in terms of the support prices of the initial endowment allocation e.

Since every $v \in \mathscr{V}$ is continuous, strictly increasing in the first coordinate, and quasi-concave, for every $x \in X$, there exists a price vector p that supports the at-least-as-good-as set $\{y \in X \mid v(y) \geq v(x)\}$, and $p_1 > 0$ for every such price vector p. Denote by $\Xi(v, x)$ the set of all price vectors p that support $\{y \in X \mid v(y) \geq v(x)\}$ and let $\Gamma(v, x) = \{-p_2/p_1 \mid p \in \Xi(v, x)\}$. Then $\Gamma(v, x)$ is an interval that is non-empty and bounded from above. So let $\gamma(v, x) = \sup \Gamma(v, x) \geq 0$. Then $\gamma(v, x)$ is the amount of the good necessary to be given to the consumer having utility function v in order to (marginally) keep him equally well off when he is given an additional unit of the bad. In particular, for every $v \in \mathscr{V}$, every $x \in X \setminus R^2_{++}$, and price vector p, we have $p \cdot y \geq p \cdot x$ for every $y \in X$ with $v(y) > v(x)$ if and only if $-p_2/p_1 \leq \gamma(v, x)$. We can now state the condition for the existence of an unlinked strong equilibrium allocation.

Proposition 8.4 *There is an unlinked strong equilibrium allocation if and only if e is an unlinked allocation and the essential infimum of the function $a \mapsto \gamma(u(a), e(a))$ of A into R_+ is strictly positive.*

In other words, there exists an unlinked strong equilibrium allocation if and only if e is unlinked and there is a strictly positive lower bound on the marginal disutility, as measured in terms of the good, evaluated at e, across almost all consumers.

Proof of Proposition 8.4 If e is unlinked and the the essential infimum of the function $a \mapsto \gamma(u(a), e(a))$ is strictly positive, let δ be this positive number. We show that the unlinked allocation e is a strong equilibrium allocation under $p = (1, -\delta)$. As pointed out above, the weak utility maximization condition is then met at $e(a)$ under p for almost every $a \in A$. But since $\inf\{p \cdot x \mid x \in X\} = -\infty$, the strong utility maximization condition is also met. Hence e is a strong equilibrium allocation.

Suppose conversely that f is an unlinked strong equilibrium allocation, with a corresponding strong equilibrium price vector p. Then, by Proposition 8.2, $f = e$ almost everywhere, and, by Lemma 7.3, $p_1 > 0$ and $p_2 < 0$. By the strong utility maximization condition, therefore, $-p_2/p_1 \leq \gamma(u(a), e(a))$ for almost every $a \in A$ and the essential infimum of the function $a \mapsto \gamma(u(a), e(a))$ is greater than or equal to $-p_2/p_1 > 0$. ///

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