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Approximation of Expectation of Diffusion Processes based on Lie Algebra and Malliavin Calculus

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In the present paper, we refine the idea in [1] by using notions in [5]. We use the notation in [5] for free Lie algebra. Let $\Omega, \mathcal{F}, P$ be a probability space and let \{(B^i(t), \ldots, B^d(t); t \in [0, \infty])\} be a $d$-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0, \infty)$. Let $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Here $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ denotes the space of $\mathbb{R}^n$-valued smooth functions defined in $\mathbb{R}^N$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ as vector fields on $\mathbb{R}^N$.

Now let $X(t, x)$, $t \in [0, \infty)$, $x \in \mathbb{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t, x) = x + \frac{\sum_{i=0}^{d}\int_{0}^{t}V_i(X(s, x)) \circ dB^i(s)}{(1)}.$$ 

Then there is a unique solution to this equation. Moreover we may assume that with probability one $X(t, x)$ is continuous in $t$ and smooth in $x$.

Let $A = A_d = \{v_0, v_1, \ldots, v_d\}$, be an alphabet, a set of letters, and $A^*$ be the set of words consisting of $A$ including the empty word which is denoted by $1$. For $u = u^1 \cdots u^k \in A^*$, $u^j \in A$, $j = 1, \ldots, k$, $k \geq 0$, we denote by $n_i(u)$, $i = 0, \ldots, d$, the cardinal of \{\{j \in \{1, \ldots, k\}; u^j = v_i\}\}. Let \#(u) = n_0(u) + \ldots + n_d(u)$, a length of $u$, $\|u\| = |u| + n_0(u)$, and \#(u) denotes the cardinal of \{\{i \in \{0, \ldots, d\}; n_i(u) \geq 1\}\} for $u \in A^*$. Let $R(A)$ be the $\mathbb{R}$-algebra of noncommutative polynomials on $A$, $R(\langle A \rangle)$ be the $\mathbb{R}$-algebra of noncommutative formal series on $A$, $\mathcal{L}(A)$ be the free Lie algebra over $\mathbb{R}$ on the set $A$, and $\mathcal{L}(\langle A \rangle)$ be the $\mathbb{R}$ Lie algebra of free Lie series on the set $A$.

Let $\iota$ denotes the left normed bracketing operator, i.e.,

$$\iota(v_{i_1} \cdots v_{i_n}) = [\ldots[v_{i_1}, v_{i_2}], \ldots, v_{i_n}].$$

Let $p: R(\langle A \rangle) \rightarrow R[x_0, \ldots, x_d]$ denotes a natural homomorphism from the algebra of noncommutative polynomials to the algebra of commutative polynomials such that $p(u) = x_0^{n_0(u)} \cdots x_d^{n_d(u)}$, $u \in A^*$.

Vector fields $V_0, V_1, \ldots, V_d$ can be regarded as first differential operators over $\mathbb{R}^N$. Let $DO(\mathbb{R}^N)$ denotes the set of smooth differential operators over $\mathbb{R}^N$. Then $DO(\mathbb{R}^N)$ is a noncommutative algebra over $\mathbb{R}$. Let $\Phi: R(\langle A \rangle) \rightarrow DF(\mathbb{R}^N)$ be a homomorphism given by

$$\Phi(1) = Identity, \quad \Phi(v_{i_1} \cdots v_{i_n}) = V_{i_1} \cdots V_{i_n}, \quad n \geq 1, \ i_1, \ldots, i_n = 0, 1, \ldots, d.$$
Also, note that
\[
\Phi(\iota(u_1 \cdots u_n)) = [\cdots [V_{i_1}, V_{i_2}], \cdots, V_{i_n}], \quad n \geq 2, \quad i_1, \ldots, i_n = 0, 1, \ldots, d.
\]

Let \( B(t; u) \), \( t \in [0, \infty) \), \( u \in A^* \), be inductively defined by
\[
B(t; 1) = 1, \quad B(t; V_i) = B^i(t), \quad i = 0, 1, \ldots, d,
\]
and
\[
B(t; u) = \int_0^t B(s; u) \circ dB^i(s) \quad u \in A^*, \quad i = 0, \ldots, d.
\]

Also we define \( B(t; w) \) \( t \in [0, \infty) \), \( w \in \mathcal{R}(A) \) by
\[
B(t; \sum_{u \in A^*} a_u u) = \sum_{u \in A^*} a_u B(t; u),
\]
and we denote \( B(1; w) \) by \( B(w) \) for \( w \in \mathcal{B}(w) \).

Let \( A_{m}^{*} = \{ u \in A^{*} \mid ||u|| = m \} \), \( m \geq 0 \), and let
\[
\mathcal{R}(A)_{m} = \sum_{u \in A_{m}^{*}} \mathcal{R}u,
\]
and \( \mathcal{R}(A)_{\leq m} = \sum_{k=0}^{m} \mathcal{R}(A)_{k} \), \( m \geq 0 \).

Let \( j_m : \mathcal{R}(A)^{\leq m} \to \mathcal{R}(A)_{\leq m} \) be an natural sujective linear map such that \( j_m(u) = u \), \( u \in A^{*} \), \( ||u|| \leq m+1 \).

Let \( \Psi_{s} : \mathcal{R}(\langle A \rangle)^{\leq m} \to \mathcal{R}(\langle A \rangle)^{\leq m} \), \( s > 0 \), be given by
\[
\Psi_{s}(\sum_{m=0}^{\infty} x_{m}) = \sum_{m=0}^{\infty} s^{m/2} x_{m}, \quad x_{m} \in \mathcal{R}(A)_{m}, \quad m \geq 0.
\]

Now we introduce a condition (UFG) on the family of vector field \( \{V_0, V_1, \ldots, V_d\} \) as follows.
(UFG) There are an integer \( \ell \) and \( \varphi_{u,u'} \in C_{b}^{\infty}(\mathbb{R}^N) \), \( u \in A^{**}, u' \in A_{\leq \ell}^{**} \), satisfying the following.
\[
\Phi(\iota(u)) = \sum_{u' \in A_{\leq \ell}^{**}} \varphi_{u,u'}(\iota(u')) \quad u \in A^{**}.
\]

Let us define a semi-norm \( \| \cdot \|_{V_n} \), \( n \geq 1 \), on \( \mathcal{C}_b^\infty(\mathbb{R}^N; \mathbb{R}) \) by
\[
\| f \|_{V_n} = \sum_{k=1}^{n} \sum_{u_1, \ldots, u_k \in A^{**}, ||u_1 \cdots u_k|| = m} \| \Phi(\iota(u_1) \cdots \iota(u_k))f \|_{\infty}.
\]

Now let us define a semigroup of linear operators \( \{P_t\}_{t \in [0, \infty)} \) by
\[
(P_t f)(x) = E[f(X(t, x))], \quad t \in [0, \infty), \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^N).
\]

Then we can prove the following ([2]).

**Theorem 1** Assume that the family of vector fields satisfies the condition (UFG). Then for any \( n \geq 1 \) there is a constant \( C > 0 \) such that
\[
\| P_t f \|_{V_n} \leq \frac{C}{t^{n/2}} \| f \|_{\infty}, \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^N), \quad t \in (0, 1].
\]
Let us think of a family \( \{Q_s; s \in (0, 1]\} \) of linear operators in \( C_b(\mathbb{R}^N) \).

**Definition 2** We say that \( Q_s \), \( s \in (0, 1] \), is \( m \)-similar, \( m \geq 1 \), if there are a constant \( C > 0 \) and \( n \geq m + 1 \) such that

\[
\| P_s f - Q_s f(x) \|_{\infty} \leq C \left( \sum_{k=m+1}^{n} s^{k/2} \| f \|_{V,k} + s^{(m+1)/2} \| \nabla f \|_{\infty} \right),
\]

and

\[
\| Q_s f - f \|_{\infty} \leq Cs^{1/2} \| \nabla f \|_{\infty}
\]

for any \( s \in (0, 1] \), and \( f \in C_0^\infty(\mathbb{R}^N; \mathbb{R}) \).

Let \( T > 0 \) and \( \gamma > 0 \). Let \( t_k = t_k^{(n)} = \frac{k^\gamma T}{n^\gamma}, n \geq 1, k = 0, 1, \ldots, n \), and let \( s_k = s_k^{(n)} = t_k - t_{k-1}, k = 1, \ldots, n \). Then we have the following.

**Theorem 3** Let \( m \geq 1 \) and \( Q_s, s > 0 \) be an \( m \)-similar family of linear operators in \( C_b(\mathbb{R}^N) \). Then we have the following.

For \( \gamma \in (0, m-1) \), there is a constant \( C > 0 \) such that

\[
\| P_T f - Q_{s_h} Q_{s_{n-1}} \cdots Q_{s_{1}} f \|_{\infty} \leq Cn^{-\gamma/2} \| \nabla f \|_{\infty}, \quad f \in C_0^\infty(\mathbb{R}^N), \quad n \geq 1.
\]

For \( \gamma = m - 1 \), there is a constant \( C > 0 \) such that

\[
\| P_T f - Q_{s_n} Q_{s_{n-1}} \cdots Q_{s_{1}} f \|_{\infty} \leq Cn^{-(m-1)/2} \log(n+1) \| \nabla f \|_{\infty}, \quad f \in C_0^\infty(\mathbb{R}^N), \quad n \geq 1.
\]

For \( \gamma > m - 1 \), there is a constant \( C > 0 \) such that

\[
\| P_T f - Q_{s_n} Q_{s_{n-1}} \cdots Q_{s_{1}} f \|_{\infty} \leq Cn^{-(m-1)/2}, \quad f \in C_0^\infty(\mathbb{R}^N), \quad n \geq 1.
\]

**Definition 4** We say that a \( \mathcal{L}((A)) \)-valued random variable \( Z \) is \( m \)-\( \mathcal{L} \)-moment similar, \( m \geq 1 \), if

\[
E[(j_m(Z), j_m(Z))^n] < \infty \quad \text{for any } n \geq 1,
\]

and if

\[
E[j_m(\exp(Z))] = E[j_m(X(1))].
\]

**Theorem 5** Let \( m \geq 1 \) and \( Z \) be a \( \mathcal{L}((A)) \)-valued \( m \)-\( \mathcal{L} \)-moment similar random variable. Also, let \( Y : (0,1] \times \Omega \rightarrow C(\mathbb{R}^N; \mathbb{R}^N) \) be a measurable map such that

\[
\sup_{s \in (0,1], \epsilon \in \mathbb{R}^N} s^{-(m+1)/2} E[|Y(s)(x)|] < \infty
\]

and

\[
E[\sup_{|s| \leq n} |Y(s)(x)|] < \infty, \quad s \in (0, 1], \quad n \geq 1.
\]

Let us define a linear map \( Q_s; s > 0 \), in \( C_b(\mathbb{R}^N) \) by

\[
(Q_s f)(x) = E[f(\exp(\Phi(j_m(\Psi_s(Z))))(x) + Y(s)(x))], \quad f \in C_b(\mathbb{R}^N).
\]

Then \( \{Q_s; s \in (0, 1]\} \) is \( m \)-similar.
We can show the following characterization theorem for 5-\(\mathcal{L}\)-moment similar random variables.

**Theorem 6** Let \(Z\) be an \(\mathcal{L}_{\leq 5}(A)\)-valued random variable. Then \(Z\) is 5-\(\mathcal{L}\) moment similar, if and only if there are random variables \(\xi_{i}, i = 1, \ldots, d\), \(\eta_{ij}, 1 \leq i < j \leq d\), \(\zeta^{(3)}_{ij}\), \(i, j = 1, \ldots, d\), \(i \neq j\), \(\zeta^{(4)}_{ij}\), \(i = 1, \ldots, d\), and \(\mathcal{L}_{m}(A)\)-valued random variables \(\rho^{(m)}\), \(m = 3, 4, 5\), satisfying the following.

1. \(Z = \sum_{i=1}^{d} \xi_{i}v_{i} + (v_{0} + \sum_{1 \leq i < j \leq d} \eta_{ij}^{(2)}[v_{i}, v_{j}]) + \sum_{1 \leq i \neq j \leq d} \zeta^{(3)}_{ij}[[v_{i}, v_{j}], v_{j}] + \rho^{(3)} + \sum_{i=1}^{d} \zeta^{(4)}_{ii}[[v_{0}, v_{i}], v_{i}] + \rho^{(4)} + \rho^{(5)}\).

2. \(E[\xi_{i}] = E[\xi_{i}^{3}] = E[\xi_{i}^{5}] = 0, \quad i = 1, \ldots, d\),
   \(E[\eta_{ij}] = 0, \quad E[\eta_{ij}^{2}] = 1, \quad 1 \leq i < j \leq d\),
   \(E[\prod_{i=1}^{d} \xi_{i}^{\alpha_{i}} \prod_{1 \leq j < \ell \leq d} \eta_{ij}^{\beta_{ij}}] = \prod_{i=1}^{d} E[\xi_{i}^{\alpha_{i}}] \prod_{1 \leq j < \ell \leq d} E[\eta_{ij}^{\beta_{ij}}]\)
   for any non-negative integers \(\alpha_{i}, i = 1, \ldots, d\), and \(\beta_{ij}, 1 \leq i < j \leq d\) with \(\sum_{i=1}^{d} \alpha_{i} + \sum_{1 \leq i < j \leq d} 2\beta_{ij} \leq 5\).

3. \(E[\zeta^{(3)}_{ij}] = 0, \quad E[\xi_{i} \zeta^{(3)}_{ij}] = \frac{1}{12} \delta_{ik}\) for any \(1 \leq i, j, k \leq d, i \neq j\), and
   \(E[(\xi_{i} \zeta_{j\ell}^{(3)}] = 0, \quad 1 \leq i, j, \ell \leq d, \quad k \neq \ell,\)
   \(E[(\eta_{ij} \zeta_{k\ell}^{(3)}] = 0, \quad 1 \leq i, j, \ell \leq d, \quad i < j, \quad k \neq \ell,\)
   and
   \(E[(\prod_{i=1}^{d} \xi_{i}^{\alpha_{i}} \prod_{1 \leq j < \ell \leq d} \eta_{ij}^{\beta_{ij}}) \rho^{(3)}] = 0\)
   for any non-negative integers \(\alpha_{i}, i = 1, \ldots, d\), and \(\beta_{ij}, 1 \leq i < j \leq d\) with \(\sum_{i=1}^{d} \alpha_{i} + \sum_{1 \leq i < j \leq d} 2\beta_{ij} \leq 2\).

4. \(E[\zeta^{(4)}_{ij}] = \frac{1}{12}, \quad E[\xi_{i} \zeta^{(4)}_{ij}] = 0, \quad 1 \leq i, j \leq d\), and
   \(E[\rho^{(4)}] = E[\xi_{i} \rho^{(4)}] = 0, \quad i = 1, \ldots, d.\)

5. \(E[\rho^{(5)}] = 0.\)
References


[3] Lyons, T., and N. Victoir, Cubature on Wiener Space, Preprint
