<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>アプロクシメーション・オブ・エクスペクテーション・オブ・ディフュージョン・プロセス・ベース・オン・リ氏アレーブシーとマラヴィン・カジュレックス（マシテムチック・エコノミックス）</td>
</tr>
<tr>
<td>著者(s)</td>
<td>Kusuoka, Shigeo</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 2003(2003), 1337: 205-209</td>
</tr>
<tr>
<td>発行年</td>
<td>2003-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/43394">http://hdl.handle.net/2433/43394</a></td>
</tr>
<tr>
<td>形式</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>publisher</td>
</tr>
</tbody>
</table>

京都大学
Approximation of Expectation of Diffusion Processes
based on Lie Algebra and Malliavin Calculus

Shigeo KUSUOKA
Graduate School of Mathematical Sciences
The University of Tokyo

In the present paper, we refine the idea in [1] by using notions in [5]. We use the notation in [5] for free Lie algebra. Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(\{(B^1(t), \ldots, B^d(t); t \in [0, \infty))\}\) be a d-dimensional Brownian motion. Let \(B^0(t) = t, t \in [0, \infty)\). Let \(V_0, V_1, \ldots, V_d \in C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)\). Here \(C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)\) denotes the space of \(\mathbb{R}^n\)-valued smooth functions defined in \(\mathbb{R}^N\) whose derivates of any order are bounded. We regard elements in \(C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)\) as vector fields on \(\mathbb{R}^N\).

Now let \(X(t, x), t \in [0, \infty), x \in \mathbb{R}^N\), be the solution to the Stratonovich stochastic integral equation

\[
X(t, x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s, x)) \circ dB^i(s). \tag{1}
\]

Then there is a unique solution to this equation. Moreover we may assume that with probability one \(X(t, x)\) is continuous in \(t\) and smooth in \(x\).

Let \(A = A_d = \{v_0, v_1, \ldots, v_d\}\), be an alphabet, a set of letters, and \(A^*\) be the set of words consisting of \(A\) including the empty word which is denoted by \(1\). For \(u = u^1 \cdots u^k \in A^*, u^j \in A, j = 1, \ldots, k, k \geq 0\), we denote by \(n_{i}(u)\), \(i = 0, \ldots, d\), the cardinal of \(\{j \in \{1, \ldots, k\}; u^j = v_i\}\). Let \(|u| = n_0(u) + \ldots + n_d(u)\), a length of \(u\), \(|u| = |u| + n_0(u)\), and \(|u| = \text{cardinal of } \{i \in \{0, \ldots, d\}; n_i(u) \geq 1\} \text{ for } u \in A^*\). Let \(R(A)\) be the \(\mathbb{R}\)-algebra of noncommutative polynomials on \(A\), \(R(\langle A \rangle)\) be the \(\mathbb{R}\)-algebra of noncommutative formal series on \(A\), \(L(A)\) be the free Lie algebra over \(\mathbb{R}\) on the set \(A\), and \(L(\langle A \rangle)\) be the \(\mathbb{R}\)-Lie algebra of free Lie series on the set \(A\).

Let \(\iota\) denotes the left normed bracketing operator, i.e.,

\[
\iota(u_1, \ldots, u_n) = [\ldots [v_{i_1}, v_{i_2}], \ldots, v_{i_n}]\]

Let \(p : R(A) \rightarrow R[x_0, \ldots, x_d]\) denotes a natural homomorphism from the algebra of noncommutative polynomials to the the algebra of commutative polynomials such that \(p(u) = x_0^{n_0(u)} \cdots x_d^{n_d(u)}\), \(u \in A^*\).

Vector fields \(V_0, V_1, \ldots, V_d\) can be regarded as first differential operators over \(\mathbb{R}^N\). Let \(\mathcal{DO}(\mathbb{R}^N)\) denotes the set of smooth differential operators over \(\mathbb{R}^N\). Then \(\mathcal{DO}(\mathbb{R}^N)\) is a noncommutative algebra over \(\mathbb{R}\). Let \(\Phi : R(A) \rightarrow \mathcal{DF}(\mathbb{R}^N)\) be a homomorphism given by

\[
\Phi(1) = \text{Identity}, \quad \Phi(u_1 \cdots u_n) = V_{i_1} \cdots V_{i_n}, \quad n \geq 1, i_1, \ldots, i_n = 0, 1, \ldots, d.
\]
Also, note that
\[ \Phi(\iota(u_{1} \cdots u_{n})) = [\cdots [V_{i_{1}}, V_{i_{2}}], \cdots, V_{i_{n}}], \quad n \geq 2, \ i_{1}, \ldots, i_{n} = 0, 1, \ldots, d. \]

Let \( B(t; u), \ t \in [0, \infty), \ u \in A^{*}, \) be inductively defined by
\[ B(t; 1) = 1, \ B(t; V_{i}) = B^{i}(t), \ i = 0, 1, \ldots, d, \]
and
\[ B(t; uv) = \int_{0}^{t} B(s; u) \circ dB^{i}(s) \quad u \in A^{*}, \ i = 0, \ldots, d. \]

Also we define \( B(t; w), \ t \in [0, \infty), \ w \in \mathbb{R}(A) \) by
\[ B(t; \sum_{u \in A^{*}} a_{u} u) = \sum_{u \in A^{*}} a_{u} B(t; u), \]
and we denote \( B(1; w) \) by \( B(w) \) for \( w \in \mathbb{B}(w) \).

Let \( A_{m}^{*} = \{ u \in A^{*} | \| u \| = m \} \), \( m \geq 0 \), and let \( \mathbb{R}(A)_{m} = \sum_{u \in A_{m}^{*}} \mathbb{R}u \)
and \( \mathbb{R}(A)_{\leq m} = \sum_{k=0}^{m} \mathbb{R}(A)_{k} \), \( m \geq 0 \).

Let \( j_{m} : \mathbb{R}(A)_{\leq m} \rightarrow \mathbb{R}(A)_{\leq m} \) be a natural sujective linear map such that \( j_{m}(u) = u, \ u \in A^{*}, \| u \| = m, \) and \( j_{m}(u) = 0, \ u \in A^{*}, \| u \| \geq m + 1 \).

Let \( \mathcal{L}(A)_{m} = \mathcal{L}(A) \cap \mathbb{R}(A)_{m} \)
and \( \mathcal{L}(A)_{\leq m} = \mathcal{L}(A) \cap \mathbb{R}(A)_{\leq m}, \ m \geq 1 \).

Let \( A^{**} = \{ u \in A^{*} ; u \neq 1, v_{0} \} \)
and \( A_{\leq m}^{**} = \{ u \in A^{**} ; \| u \| \leq m \} \), \( m \geq 1 \).

Let \( \Psi_{s} : \mathbb{R}(A)_{\leq m} \rightarrow \mathbb{R}(A)_{\leq m}, \ s > 0, \) be given by
\[ \Psi_{s}(\sum_{m=0}^{\infty} x_{m}) = \sum_{m=0}^{\infty} s^{m/2} x_{m}, \quad x_{m} \in \mathbb{R}(A)_{m}, \ m \geq 0. \]

Now we introduce a condition (UFG) on the family of vector field \( \{ V_{0}, V_{1}, \ldots, V_{d} \} \) as follows.
(UFG) There are an integer \( \ell \) and \( \varphi_{u,u'} \in C_{b}^{\infty}(\mathbb{R}^{N}) \), \( u \in A^{**}, \ u' \in A_{\leq \ell}^{*} \), satisfying the following.
\[ \Phi(\iota(u)) = \sum_{u' \in A_{\leq \ell}^{*}} \varphi_{u,u'} \Phi(\iota(u')) \quad u \in A^{**}. \]

Let us define a semi-norm \( \| \cdot \|_{V,n}, \ n \geq 1, \) on \( C_{0}^{\infty}(\mathbb{R}^{N}, \mathbb{R}) \) by
\[ \| f \|_{V,n} = \sum_{k=1}^{n} \sum_{u_{1},\ldots,u_{k} \in A^{**}, \| u_{1}\cdots u_{k} \| = n} \| \Phi(\iota(u_{1})\cdots\iota(u_{k}))f \|_{\infty}. \]

Now let us define a semigroup of linear operators \( \{ P_{t} \}_{t \in [0, \infty)} \) by
\[ (P_{t}f)(x) = E[f(X(t, x))], \quad t \in [0, \infty), \ f \in C_{b}^{\infty}(\mathbb{R}^{N}). \]

Then we can prove the following ([2]).

**Theorem 1** Assume that the family of vector fields satisfies the condition (UFG). Then for any \( n \geq 1 \) there is a constant \( C > 0 \) such that
\[ \| P_{t}f \|_{V,n} \leq C \frac{1}{t^{n/2}} \| f \|_{\infty}, \quad f \in C_{b}^{\infty}(\mathbb{R}^{N}), \ t \in (0,1]. \]
Let us think of a family \( \{Q(s); s \in (0,1]\} \) of linear operators in \( C_b(\mathbb{R}^N) \).

**Definition 2** We say that \( Q(s), s \in (0,1], \) is \( m \)-similar, \( m \geq 1, \) if there are a constant \( C > 0 \) and \( n \geq m + 1 \) such that

\[
\| Pf - Q(s)f(x) \|_\infty \leq C( \sum_{k=m+1}^{n} s^{k/2} \| \nabla f \|_{k} + s^{(m+1)/2} \| \nabla f \|_\infty),
\]

and

\[
\| Q(s)f - f \|_\infty \leq C s^{1/2} \| \nabla f \|_\infty
\]

for any \( s \in (0,1], \) and \( f \in C^\infty(\mathbb{R}^N; \mathbb{R}) \).

Let \( T > 0 \) and \( \gamma > 0 \). Let \( t_k = t_k^{(n)} = \frac{k^\gamma T}{n^\gamma}, n \geq 1, k = 0, 1, \ldots, n, \) and let \( s_k = s_k^{(n)} = t_k - t_{k-1}, k = 1, \ldots, n. \) Then we have the following.

**Theorem 3** Let \( m \geq 1 \) and \( Q(s), s > 0 \) be an \( m \)-similar family of linear operators in \( C_b(\mathbb{R}^N) \). Then we have the following.

For \( \gamma \in (0, m-1) \), there is a constant \( C > 0 \) such that

\[
\| Pf - Q(s_{h})Q(s_{n-1}) \cdots Q(s_{1})f \|_\infty \leq C n^{-\gamma/2} \| \nabla f \|_\infty, \quad f \in C^\infty(\mathbb{R}^N), n \geq 1.
\]

For \( \gamma = m - 1 \), there is a constant \( C > 0 \) such that

\[
\| Pf - Q(s_{n})Q(s_{n-1}) \cdots Q(s_{1})f \|_\infty \leq C n^{-m+1} \log(n+1) \| \nabla f \|_\infty,
\]

\( f \in C^\infty(\mathbb{R}^N), n \geq 1. \)

For \( \gamma > m - 1 \), there is a constant \( C > 0 \) such that

\[
\| Pf - Q(s_{n})Q(s_{n-1}) \cdots Q(s_{1})f \|_\infty \leq C n^{-m+1}, \| \nabla f \|_\infty, \quad f \in C^\infty(\mathbb{R}^N), n \geq 1.
\]

**Definition 4** We say that a \( \mathcal{L}((A)) \)-valued random variable \( Z \) is \( m \)-\( \mathcal{L} \)-moment similar, \( m \geq 1, \) if

\[
E[(j_m(Z),j_m(Z))^n] < \infty \quad \text{for any } n \geq 1,
\]

and if

\[
E[j_m(\exp(Z))] = E[j_m(X(1))].
\]

**Theorem 5** Let \( m \geq 1 \) and \( Z \) be a \( \mathcal{L}((A)) \)-valued \( m \)-\( \mathcal{L} \)-moment similar random variable. Also, let \( Y : (0,1] \times \Omega \to C(\mathbb{R}^N; \mathbb{R}^N) \) be a measurable map such that

\[
\sup_{s \in (0,1], x \in \mathbb{R}^N} s^{-(m+1)/2} E[|Y(s)(x)|] < \infty
\]

and

\[
E[\sup_{|x| \leq n} |Y(s)(x)|] < \infty, \quad s \in (0,1], n \geq 1.
\]

Let us define a linear map \( Q(s), s > 0, \) in \( C_b(\mathbb{R}^N) \) by

\[
(Q(s)f)(x) = E[f(\exp(\Phi(j_m(\Psi_s(Z)))) + Y(s)(x)), \quad f \in C_b(\mathbb{R}^N).
\]

Then \( \{Q(s); s \in (0,1]\} \) is \( m \)-similar.
We can show the following characterization theorem for 5-L-moment similar random variables.

**Theorem 6** Let $Z$ be an $\mathcal{L}_{\leq 5}(A)$-valued random variable. Then $Z$ is 5-L moment similar, if and only if there are random variables $\xi_i$, $i = 1, \ldots, d$, $\eta_{ij}$, $1 \leq i < j \leq d$, $\zeta_{ij}^{(3)}$, $i, j = 1, \ldots, d$, $i \neq j$, $\zeta_{ij}^{(4)}$, $i = 1, \ldots, d$, and $\mathcal{L}_m(A)$-valued random variables $\rho^{(m)}$, $m = 3, 4, 5$, satisfying the following. 

1. \[ Z = \sum_{i=1}^{d} \xi_i v_i + (v_0 + \sum_{1 \leq i < j \leq d} \eta_{ij}^{(2)}[v_i, v_j]) + (\sum_{1 \leq i \neq j \leq d} \zeta_{ij}^{(3)}[[v_i, v_j], v_j] + \rho^{(3)}) + (\sum_{i=1}^{d} \zeta_{ij}^{(4)}[[v_0, v_i], v_i] + \rho^{(4)}) + \rho^{(5)} \]

2. \[ E[\xi_i] = E[\xi_i^3] = E[\xi_i^5] = 0, \quad i = 1, \ldots, d, \]
   \[ E[\eta_{ij}] = 0, \quad E[\eta_{ij}^2] = 1, \quad 1 \leq i < j \leq d, \]
   \[ E[\prod_{i=1}^{d} \xi_i^\alpha : \prod_{1 \leq j < d} \eta_{ij}^\beta j^j] = \prod_{i=1}^{d} E[\xi_i^\alpha : ] \prod_{1 \leq i < j \leq d} E[\eta_{ij}^\beta j^j] \]
   for any non-negative integers $\alpha_i$, $i = 1, \ldots, d$, and $\beta_{ij}$, $1 \leq i < j \leq d$ with $\sum_{i=1}^{d} \alpha_i + \sum_{1 \leq i < j \leq d} 2\beta_{ij} \leq 5$.

3. \[ E[\zeta_{ij}^{(3)}] = 0, \quad E[\xi_i \zeta_{ij}^{(3)}] = \frac{1}{12} \delta_{ik}, \text{ for any } 1 \leq i, j, k \leq d, i \neq j, \text{ and} \]
   \[ E[(\xi_i \xi_j \zeta_{k\ell}^{(3)})] = 0, \quad 1 \leq i, j, k, \ell \leq d, \ k \neq \ell, \]
   \[ E[(\eta_{ij} \zeta_{k\ell}^{(3)})] = 0, \quad 1 \leq i, j, k, \ell \leq d, \ i < j, \ k \neq \ell, \]
   and
   \[ E[(\prod_{i=1}^{d} \xi_i^\alpha : \prod_{1 \leq i < j \leq d} \eta_{ij}^\beta j^j)\rho^{(3)}] = 0 \]
   for any non-negative integers $\alpha_i$, $i = 1, \ldots, d$, and $\beta_{ij}$, $1 \leq i < j \leq d$ with $\sum_{i=1}^{d} \alpha_i + \sum_{1 \leq i < j \leq d} 2\beta_{ij} \leq 2$.

4. \[ E[\zeta_{ij}^{(4)}] = \frac{1}{12}, \quad E[\xi_i \zeta_{ij}^{(4)}] = 0, \quad 1 \leq i, j \leq d, \text{ and} \]
   \[ E[\rho^{(4)}] = E[\xi_i \rho^{(4)}] = 0, \quad i = 1, \ldots, d. \]

5. \[ E[\rho^{(5)}] = 0. \]
References


[3] Lyons, T., and N. Victoir, Cubature on Wiener Space, Preprint
