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actable Co-operative Game and Intractable Co-Operative ame Arising from Combinatorial Optimization Problems

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Abstract

We would like to consider about Tractability on co-operative games arising form combinatorial optimization problems. Tractability is like "Solvability" in Computational Complexity Theory. If we were able to calculate solutions of the co-operative games in polynomial time, we would say that they have good property. For example, nucleolus can be calculated in polynomial time when the game is convex game. Convexity is important property for game theory. But we consider about the original combinatorial optimization problem's structure of the games. The discrete structures induces more complicate situation. That is why I would like to consider Tractability.

1 Introduction

Suppose that a co-operative game in characteristic function form $\Gamma = (N, v)$ consists of a player set $N = \{1, \ldots, n\}$ and a characteristic function $v : 2^N \to R$. Also, $v(S)$ is equal to optimal value of combinatorial optimization problems for all $S = \{s \in 2^N : s \neq \emptyset\}$ with $v(\emptyset) = 0$. We call this type of games "Co-Operative Games Arising from Combinatorial Optimization Problem". They are Profit Games in case combinatorial optimization problems are maximizing profit problem. Otherwise, they are Cost Games in case combinatorial optimization problems are minimizing cost problems.

We want to calculate the well-known co-operative game theory solutions, i.e., core, Shapley value, nucleolus, kernel, ...etc. directly, we must solve the original combinatorial optimization problem $2^n - 1$ times for all coalitions. Also, if the original combinatorial problem is $\mathcal{NP}$-hard in Computational Complexity Theory, it would be very hard to get these solutions. That is why if n is large, it would not have possibility to calculate these solutions within useful time. Besides it may happen if the original combinatorial optimization problem is in $\mathcal{P}$.

Then we would like to consider the Tractability on co-operative games arising form combinatorial optimization problems. Tractability is like "Solvability" in Computational Complexity Theory. If we were able to calculate solutions of the co-operative games in polynomial time, we would say that they have good property. It is difficult to say what is Tractable on this type of games. But we try to say something about "Tractability".

2 What is solution of co-operative game ?

2.1 Solutions we discuss

This question is hard to answer. The plural solutions were proposed until now. Also, they are not classified. We have only some information about relationships among solutions of games. But we cannot discuss about Tractability if we suppose them. At first, core and nucleolus, we suppose. That is because these solutions are related to linear programming and deeply researched in computational complexity theory.

2.2 Linear Production Game

For preparation, we introduce "Linear Production Game" Owen [18] in short.

2.2.1 Linear Production Problem

The original optimization problem which is not combinatorial optimization problem,, is the Linear Programming Problem below.

(Notation)

p	: the number of goods
m	: the number of resources
e_j	: the profit of a unit of j th good
a_{ij}	: the amount of i th resource for a unit j th good
b_i	: the maximum amount of i th resource

(Formulation)

$$\max \quad \sum_{j=1}^{p} e_j x_j \tag{1}$$

subject to

$$\sum_{j=1}^{p} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m \tag{2}$$

$$x_j \geq 0 \quad j = 1, \ldots, p \tag{3}$$

2.2.2 Formulation of Linear Production Game

In this game, l th player supplies his own resource b_i^l. That is why the maximum amount of resource is below for a coalition $S \subseteq N = \{1, \ldots, n\}$ to gather i th resource.

$$b_i(S) = \sum_{l \in S} b_i^l \tag{4}$$

Now, we replace (2) with (4) i.e. $b_i(S)$. Then we get a "Linear Production Game" $\Lambda\Gamma = (N, v)$ with characteristic function $v(S)$ is defined its optimal value.

2.2.3 Calculation of a member of core

Suppose dual problem of this problem. We can calculate of a member of core.

$$\min \quad \sum_{i=1}^{m} b_i(S) y_i \tag{5}$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq e_j \quad j = 1, \ldots, p \tag{6}$$

$$y_i \geq 0 \quad i = 1, \ldots, m \tag{7}$$

The solution of this dual problem i.e. y_i^* is a shadow price of i th resource. That is why the l th player's resource is $b^l = (b_i^l, \ldots, b_m^l)$. The value of resources is below for the l th player.

$$z^l = \sum_{i=1}^{m} b_i^l y_i^* \quad l = 1, \ldots, n \tag{8}$$

From Duality Theorem, $z = (z^1, \ldots, z^n)$ is a member of core.

It is important for us in above discussion that we can calculate a member of core in polynomial time because we can solve Linear Programming in polynomial time.

3 Combinatorial Optimization Structure

Some readers guess that if the original optimization problem is solved in polynomial time, we would be able to calculate a member of core or nucleolus in polynomial time. It is not true. Granot and Huberman [12] and Kalai and Zemel [16] discussed the existence of core for Linear Programming Games in which Linear Production Game is. On such games, if the original problem had feasible solution, a solution would be found in polynomial time using the ellipsoid algorithm or the interior point algorithm which are polynomial time algorithms for Linear Programming. Also, they guess that if the original combinatorial optimization problem has **submodular property i.e. a kind of discrete convexity**, the solution of the game would be calculated in polynomial time. But combinatorial optimization structure makes situation much harder. But we found the following proposition is true.

Proposition 1
If the original combinatorial optimization problem is $\mathcal{NP}$-hard, we would be unable to get the solution of the game arising from it in polynomial time.

3.1 Minimal Cost Spanning Tree (MCST) Game

MCST Game is deeply researched in this type of games. Also, MCST Problem is in $\mathcal{P}$. Then we consider the typical game whose original combinatorial optimization problem is in $\mathcal{P}$ We introduce some results of research. Then, useful information is gotten.

3.1.1 MCST Problem

The MCST Problem is posed on a connected graph $G = (V, E)$ with positive cost $\dot{w}_e \in E$. We are looking for the shortest spanning tree of G. We can find its solution by Kruskal algorithm or Prim algorithm (see text books of Discrete Mathematics or Graph Theory) in polynomial time.

3.1.2 MCST Game Definition

The MCST Game is usually defined on complete graph. Suppose water, electricity and cable television networks. 0 is defined as a special node as one common supplier. Users are Players as a set of other nodes which is denoted by $N = \{1, \ldots, n\}$. And its characteristic function is defined by

$$v(S) = \min\{\sum_{e \in T_s} w_e : T_s \text{ is a spanning tree of } G(S \cup \{0\}, E(S)) \\ \text{which is a subgraph of } G(N \cup \{0\}, E)\}.$$

3.1.3 Tractability of MCST Game

Megiddo [17] and Granot and Huberman [12] show that nucleolus is calculated in polynomial time for subclasses of MCST Game. However, Faigle, Kern and Kuipers [8] shows that it is $\mathcal{NP}$-hard to get nucleolus for MCST Game in general. Then if the original combinatorial optimization problem is in $\mathcal{P}$, it would be not to calculate the nucleolus of its game in polynomial time. We call this types of games "Intractable Game". That is why it is very interesting to know if we can calculate in polynomial time for what kind of this game.

3.2 Facility Location Game and Traveling Salesman Game

These original combinatorial optimization problems are famous problems which are $\mathcal{NP}$-hard. That is why they are "Intractable Game". But Goemans and Skutella [11] show that a member of core is calculated in polynomial time for subclass of the game. Its original Facility Location Problem has the special property which is the no integer gap. Then we can calculate a member of core as the Linear Production Game. In this case, the original combinatorial optimization problem is $\mathcal{NP}$-hard in general. But its subclass is in $\mathcal{P}$. This is "Tractable" case. But as we wrote, if the original combinatorial optimization problem is in $\mathcal{P}$, we would not say that a member of core or nucleolus is calculated in polynomial time.

As for Traveling Salesman Game, its original problem i.e. Traveling Salesman Problem is $\mathcal{NP}$-hard in general. But in Burkard et al. [2] and Gilmore [5] shows that subclasses of this problem are solved in polynomial time. If we know that a member of core or nucleolus is calculated in polynomial time for what kind of subclasses of Traveling Salesman Game, it would be for good information on defining "Tractability".

4 Remarks

Now, we have not defined "Tractability" specifically yet. But some information is gotten to try to define it. On our next research step, we will gather "Tractable" Game cases and seek something similarity from structures of thier original combinatorial problems via matriod theoretical approach and discrete convexity analysis theory.

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