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Certain series attached to an even number of elliptic modular forms

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1 Results

Let \( n \in \mathbb{Z}_{>0} \), \( k := (k_1, \ldots, k_n) \in (\mathbb{Z}_{>0})^n \), \( m = (m_1, \ldots, m_n) \in (\mathbb{Z}_{>0})^n \) and \( s \in \mathbb{C} \). We put

\[
Q_k^{(n)}(m,s) := \int_0^\infty e^{\Sigma_{j=1}^{n} k_j t - n - 1} dt \prod_{j=1}^{n} \int_0^\infty u_j^{k_j - 2} e^{-4\pi m_j u_j} (\sqrt{u_j} \theta(iu_j) - 1) du_j ;
\] (1)

here \( |k| := \Sigma_{j=1}^{n} k_j \) and

\[
\theta(z) := \sum_{l=-\infty}^{\infty} e^{\pi i l^2 z}
\]
is the Jacobi theta function. The right-hand side of (1) converges absolutely and locally uniformly for \( \text{Re}(s) > \frac{n}{2} \). It is easy to see

\[
Q_k^{(n)}(m, \sigma) > 0 \quad \text{for} \quad \frac{n}{2} < \sigma \in \mathbb{R}.
\]

For \( w \in \mathbb{Z} \) let \( M_w \) be the space of holomorphic modular forms of weight \( w \) for \( SL_2(\mathbb{Z}) \) and \( S_w \) be the space of cusp forms in \( M_w \). Let \( f_j \) and \( g_j \) be elements of \( M_{k_j} \) such that \( f_j(z)g_j(z) \) is a cusp form for each \( j = 1, \ldots, n \). Let

\[
f_j(z) = \sum_{l=0}^{\infty} a_j(l) e^{2\pi ilz} \quad \text{and} \quad g_j(z) = \sum_{l=0}^{\infty} b_j(l) e^{2\pi ilz}
\] (2)

be the Fourier expansions. The series we treat here is the following:
\[ D(s; f_1, \ldots, f_n; g_1, \ldots, g_n) := \sum_{m=(m_1, \ldots, m_n) \in (\mathbb{Z}_{>0})^n} \left( \prod_{j=1}^{n} a_j(m_j)b_j(m_j) \right) Q_k^{(n)}(m, s). \tag{3} \]

The right-hand side of (3) converges absolutely and locally uniformly for
\[ \text{Re}(s) > \frac{n}{2} \left( \max_{1 \leq j \leq n} (k_j) + 1 \right). \]

**Theorem 1.**
(i) The series (3) has a meromorphic continuation to the whole \( s \)-plane.
(ii) Let \((, )\) be the Petersson inner product. Then the function
\[ \sum_{\nu=1}^{n} \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \left( \prod_{\substack{1 \leq j \leq n \leq n \\text{otherwise}}} (f_j, g_j) \right) \cdot D(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu}) \]
is invariant under the substitution \( s \mapsto n-s \); it has possible simple poles at \( s = 0 \) and \( s = n \) with residues \( - \prod_{j=1}^{n} (f_j, g_j) \) and \( \prod_{j=1}^{n} (f_j, g_j) \) respectively, and is holomorphic elsewhere.

In case where every \( g_j \) is the Eisenstein series we have

**Corollary.** Suppose \( f_j \in S_{k_j} \) \((j = 1, \ldots, n)\) with Fourier expansions as in (2). For \( l \in \mathbb{Z}_{>0} \) put
\[ \sigma_{\nu}(l) := \sum_{d|l} d^\nu \quad \text{for} \quad \nu \in \mathbb{C}. \]

Then the series
\[ S(s; f_1, \ldots, f_n) := \sum_{m=(m_1, \ldots, m_n) \in (\mathbb{Z}_{>0})^n} \left( \prod_{j=1}^{n} a_j(m_j)\sigma_{k_j-1}(m_j) \right) Q_k^{(n)}(m, s) \]
has a holomorphic continuation to the whole \( s \)-plane and satisfies the functional equation
\[ S(s; f_1, \ldots, f_n) = S(n-s; f_1, \ldots, f_n). \]
2 A key to the proof: an integral of Rankin-Selberg type

We use the following type of Eisenstein series for the Siegel modular group $\Gamma_n := Sp_{2n}(\mathbb{Z})$ whose properties were studied by Kohnen-Skoruppa [2], Yamazaki [5], and Deitmar-Krieg [1]:

$$E_s^{(n)}(Z) := \sum_{M \in \Delta_{n,n-1} \setminus \Gamma_n} \left( \frac{\det(\text{Im}(M(Z)))}{\det(\text{Im}(M(Z)^*))} \right)^s.$$  \hspace{1cm} (4)

Here $s \in \mathbb{C}$, $Z$ is a variable on $H_n$, the Siegel upper half space of degree $n$,

$$\Delta_{n,n-1} := \left\{ \begin{pmatrix} * & \ast \\ 0(1,2n-1) & * \end{pmatrix} \in \Gamma_n \right\},$$

$M$ runs over a complete set of representatives of $\Delta_{n,n-1} \setminus \Gamma_n$; for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D$ being $n \times n$ blocks,

$$M(Z) := (AZ + D)(CZ + D)^{-1}$$

and $M(Z)^*$ is the upper left $(n-1) \times (n-1)$ block of $M(Z)$. We understand that

$$\det(\text{Im}(M(Z)^*)) = 1$$

if $n = 1$. The right-hand side of (4) converges absolutely and locally uniformly for $\text{Re}(s) > n$. Put

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s).$$

By [1][5], the Eisenstein series (4) has meromorphic continuation in $s$ to the whole $s$-plane; the function $\xi(2s)E_s^{(n)}(Z)$ is invariant under the substitution $s \mapsto n - s$ and is holomorphic except for the simple poles at $s = 0$ and $s = n$ with residues $-1/2$ and $1/2$, respectively.

Theorem 1 follows from the following integral representation:

**Theorem 2.** For

$$F_j(z) := \overline{f_j(z)}g_j(z) \text{Im}(z)^{k_j}$$

we have

$$\left( \begin{pmatrix} \cdots & E_s^{(n)}(Z_1) & \cdots \\ 0 & \cdots & 0 \end{pmatrix}, F_1(z_1), \cdots, F_n(z_n) \right)$$
\[
\frac{1}{2\xi(2s)} \sum_{\nu=1}^{n} \sum_{1 \leq i_{1} < \ldots < i_{\nu} \leq n} \left( \prod_{1 \leq j \leq \nu} (f_{j}, g_{j}) \right) \\
\cdot D(s; f_{i_{1}}, \ldots, f_{i_{\nu}}; g_{1}, \ldots, g_{\nu}).
\]

**Remark.** Define a symmetric positive definite matrix

\[
P_{Z} := \begin{pmatrix} 1_{n} & X & 0 \\ 0 & 1_{n} \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} 1_{n} & 0 \end{pmatrix}.
\]

Then

\[
E_{s}^{(n)}(Z) = \frac{1}{2\xi(2s)} \sum_{h \in \mathbb{Z}(2n,1)_{-\{0\}}} (hP_{Z}h)^{\cdot} \quad \text{for Re}(s) > n.
\]

### 3 Supplementary remarks

(i) Let

\[
\varphi_{j}(z) = \sum_{l=1}^{\infty} c_{j}(l)e^{2\pi ilz}
\]

be holomorphic primitive cusp forms of weight 1 for \( \Gamma_{0}(N_{j}) \) with odd characters \( \chi_{j} \) where \( N_{j} \in \mathbb{Z}_{>0} \) and \( j = 1, \ldots, n \). Suppose \( n \geq 3 \). Then by Kurokawa [3, Theorem 5], the Dirichlet series

\[
\sum_{l=1}^{\infty} c_{1}(l) \cdots c_{n}(l)l^{-s}
\]

has meromorphic continuation in the region Re\((s) > 0\) but has the line Re\((s) = 0\) as a natural boundary. (Cf. also [4, Theorem 8].) Thus it is a nontrivial problem to find a series associated with more than two elliptic modular forms which has analytic continuation to the whole \( s \)-plane.

(ii) In case \( n = 1 \) we have

\[
D(s; f_{1}; g_{1}) = 2\xi(2s)(4\pi)^{1-k_{1}-s}\Gamma(s + k_{1} - 1)D(s + k_{1} - 1, f_{1}, g_{1})
\]

for Re\((s) > (k_{1} + 1)/2\), where

\[
D(s, f_{1}, g_{1}) := \sum_{m=1}^{\infty} a_{1}(m)b_{1}(m)m^{-s}.
\]

Thus in this case Theorem 1 states nothing but the well-known properties of the Rankin series \( D(s, f_{1}, g_{1}) \).
(iii) In case $n = 2$ we have

$$D(s; f_1, f_2; g_1, g_2) = 2^{6-2|k|} \pi^{2-|k|} (2\pi)^{-2e} \frac{\Gamma(s)\Gamma(s+|k|-2)\Gamma(s+k_1-1)\Gamma(s+k_2-1)}{\Gamma(2s+|k|-2)}$$

$$\sum_{m_1, m_2 \in \mathbb{Z}_{>0}} a_1(m_1)a_2(m_2)\overline{b_1(m_1)b_2(m_2)} m_1^{1-k_1-\epsilon} m_2^{1-k_2} \lambda_1^{-2\epsilon} F(s, s+k_1-1; 2s+|k|-2; 1 - \frac{m_2\lambda_2^2}{m_1\lambda_1^2})$$

for $\text{Re}(s) > \max(k_1, k_2) + 1$, where $F = _2F_1$ is the hypergeometric function.

(iv) The function $Q_k^{(n)}(m, s)$ has another representation:

$$Q_k^{(n)}(m, s) = 2^{3n-|k|+1} \pi^{n-|h|} \left( \prod_{j=1}^{n} m_j^{\frac{1}{j}} \right) \sum_{\lambda_1, \ldots, \lambda_n \in \mathbb{Z}_{>0}} \left( \prod_{j=1}^{n} \lambda_j^{k_j-1} \right)$$

$$\int_0^{\infty} t^{2s-1+|k|-n} \prod_{j=1}^{n} K_{k_j-1}(4\sqrt{\pi m_j\lambda_j} t) dt$$

for $\text{Re}(s) > n/2$, where $K_{\nu}$ is the modified Bessel function of order $\nu$.

References


