<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>ある数列を付随した偶数点のエルリック共形形式 (自動形形式と表現群上の局部体における表現群表現)</td>
</tr>
<tr>
<td>著者</td>
<td>水松 聖一郎</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 (2003), 1338: 25-29</td>
</tr>
<tr>
<td>発行日</td>
<td>2003-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/43402">http://hdl.handle.net/2433/43402</a></td>
</tr>
<tr>
<td>タイプ</td>
<td>学術雑誌論文</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>出版社</td>
</tr>
</tbody>
</table>

京都大学
Certain series attached to an even number of elliptic modular forms

Shin-ichiro Mizumoto
Department of Mathematics,
Tokyo Institute of Technology

1 Results

Let $n \in \mathbb{Z}_{>0}$, $k := (k_1, \ldots, k_n) \in (\mathbb{Z}_{>0})^n$, $m = (m_1, \ldots, m_n) \in (\mathbb{Z}_{>0})^n$ and $s \in \mathbb{C}$. We put

$$Q_k^{(n)}(m, s) := \int_{0}^{\infty} e^{s+|k|-n-1} dt \cdot \prod_{j=1}^{n} \int_{0}^{\infty} u_j^{k_j-2} e^{-4\pi m_j u_j} e^{t(u_j\theta(iu_j) - 1)} du_j,$$  \hspace{1cm} (1)

here $|k| := \sum_{j=1}^{n} k_j$ and

$$\theta(z) := \sum_{l=-\infty}^{\infty} e^{\pi i l^2 z}$$

is the Jacobi theta function. The right-hand side of (1) converges absolutely and locally uniformly for $\text{Re}(s) > \frac{n}{2}$. It is easy to see

$$Q_k^{(n)}(m, \sigma) > 0 \quad \text{for} \quad \frac{n}{2} < \sigma \in \mathbb{R}.$$

For $w \in \mathbb{Z}$ let $M_w$ be the space of holomorphic modular forms of weight $w$ for $SL_2(\mathbb{Z})$ and $S_w$ be the space of cusp forms in $M_w$. Let $f_j$ and $g_j$ be elements of $M_{k_j}$ such that $f_j(z)g_j(z)$ is a cusp form for each $j = 1, \ldots, n$. Let

$$f_j(z) = \sum_{l=0}^{\infty} a_j(l) e^{2\pi i l z} \quad \text{and} \quad g_j(z) = \sum_{l=0}^{\infty} b_j(l) e^{2\pi i l z}$$  \hspace{1cm} (2)

be the Fourier expansions. The series we treat here is the following:
\[ D(s; f_1, \ldots, f_n; g_1, \ldots, g_n) := \sum_{m=(m_1, \ldots, m_n) \in (\mathbb{Z}_{>0})^n} \left( \prod_{j=1}^{n} a_j(m_j)\overline{b_j(m_j)} \right) Q_k^{(n)}(m, s). \] (3)

The right-hand side of (3) converges absolutely and locally uniformly for
\[ \text{Re}(s) > \frac{n}{2} (\max_{1 \leq j \leq n} (k_j) + 1). \]

**Theorem 1.**
(i) The series (3) has a meromorphic continuation to the whole s-plane.

(ii) Let \((, )\) be the Petersson inner product. Then the function
\[
\sum_{\nu=1}^{n} \sum_{1 \leq i_1 < \ldots < i_\nu \leq n} \left( \prod_{j \neq i_1, \ldots, i_\nu}^{n} (f_j, g_j) \right) \cdot D(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu})
\]
is invariant under the substitution \(s \mapsto n - s\); it has possible simple poles at \(s = 0\) and \(s = n\) with residues \(-\prod_{j=1}^{n} (f_j, g_j)\) and \(\prod_{j=1}^{n} (f_j, g_j)\) respectively, and is holomorphic elsewhere.

In case where every \(g_j\) is the Eisenstein series we have

**Corollary.** Suppose \(f_j \in S_{k_j} (j = 1, \ldots, n)\) with Fourier expansions as in (2). For \(l \in \mathbb{Z}_{>0}\) put
\[ \sigma_\nu(l) := \sum_{d \mid l} d^\nu \text{ for } \nu \in \mathbb{C}. \]
Then the series
\[ S(s; f_1, \ldots, f_n) := \sum_{m=(m_1, \ldots, m_n) \in (\mathbb{Z}_{>0})^n} \left( \prod_{j=1}^{n} a_j(m_j)\sigma_{k_j-1}(m_j) \right) Q_k^{(n)}(m, s) \]
has a holomorphic continuation to the whole s-plane and satisfies the functional equation
\[ S(s; f_1, \ldots, f_n) = S(n - s; f_1, \ldots, f_n). \]
2 A key to the proof: an integral of Rankin-Selberg type

We use the following type of Eisenstein series for the Siegel modular group $\Gamma_n := Sp_{2n}(\mathbb{Z})$ whose properties were studied by Kohnen-Skoruppa [2], Yamazaki [5], and Deitmar-Krieg [1]:

$$E_{\epsilon}^{(n)}(Z) := \sum_{M \in \Delta_{n,n-1} \backslash \Gamma_n} \left( \frac{\det(\text{Im}(M(Z)))}{\det(\text{Im}(M(Z)^*))} \right)^{\epsilon}.$$  \hspace{1cm} (4)

Here $s \in \mathbb{C}$, $Z$ is a variable on $H_n$, the Siegel upper half space of degree $n$,

$$\Delta_{n,n-1} := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma_n \right\},$$

$M$ runs over a complete set of representatives of $\Delta_{n,n-1} \backslash \Gamma_n$; for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D$ being $n \times n$ blocks,

$$M(Z) := (AZ + D)(CZ + D)^{-1}$$

and $M(Z)^*$ is the upper left $(n-1) \times (n-1)$ block of $M(Z)$. We understand that

$$\det(\text{Im}(M(Z)^*)) = 1$$

if $n = 1$. The right-hand side of (4) converges absolutely and locally uniformly for $\text{Re}(s) > n$. Put

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left( \frac{s}{2} \right) \zeta(s).$$

By [1][5], the Eisenstein series (4) has meromorphic continuation in $s$ to the whole $s$-plane; the function $\xi(2s)E_{\epsilon}^{(n)}(Z)$ is invariant under the substitution $s \mapsto n - s$ and is holomorphic except for the simple poles at $s = 0$ and $s = n$ with residues $-1/2$ and $1/2$, respectively.

Theorem 1 follows from the following integral representation:

**Theorem 2.** For

$$F_{j}(z) := \overline{f_{j}(z)}g_{j}(z)\text{Im}(z)^{k_{j}}$$

we have

$$\left( \begin{array}{l} \ldots \\ \left( \begin{array}{c} \left( \begin{array}{c} z_{1} \\ \cdots \\ 0 \end{array} \right), F_{1}(z_{1}) \end{array} \right), \cdots \end{array} \right), F_{n}(z_{n}) \right)$$
\[
= \frac{1}{2\xi(2s)} \sum_{\nu=1}^{n} \sum_{1 \leq i_{1} < \ldots < i_{\nu} \leq n} \left( \prod_{1 \leq j \leq \nu \atop j \neq i_{1}, \ldots, i_{\nu}} (f_{j}, g_{j}) \right) \cdot D(s; f_{i_{1}}, \ldots, f_{i_{\nu}}; g_{i_{1}}, \ldots, g_{i_{\nu}})
\]

**Remark.** Define a symmetric positive definite matrix

\[
P_{Z} := \begin{pmatrix} 1_{n} & iX & 0 \\ 0 & 1_{n} & iY \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} 1_{n} & 0 \\ 0 & 1_{n} \end{pmatrix}
\]

Then

\[
E_{s}^{(n)}(Z) = \frac{1}{2\xi(2s)} \sum_{h \in \mathbb{Z}^{(2n,1)}_{-\{0\}}} (hP_{Z}h)^{-s} \text{ for } \text{Re}(s) > n.
\]

### 3 Supplementary remarks

(i) Let

\[
\varphi_{j}(z) = \sum_{l=1}^{\infty} c_{j}(l)e^{2\pi ilz}
\]

be holomorphic primitive cusp forms of weight 1 for \( \Gamma_{0}(N_{j}) \) with odd characters \( \chi_{j} \) where \( N_{j} \in \mathbb{Z}_{>0} \) and \( j = 1, \ldots, n \). Suppose \( n \geq 3 \). Then by Kurokawa [3, Theorem 5], the Dirichlet series

\[
\sum_{l=1}^{\infty} c_{1}(l) \cdots c_{n}(l) l^{-s}
\]

has meromorphic continuation in the region \( \text{Re}(s) > 0 \) but has the line \( \text{Re}(s) = 0 \) as a natural boundary. (Cf. also [4, Theorem 8].) Thus it is a nontrivial problem to find a series associated with more than two elliptic modular forms which has analytic continuation to the whole \( s \)-plane.

(ii) In case \( n = 1 \) we have

\[
D(s; f_{1}; g_{1}) = 2\xi(2s)(4\pi)^{1-k_{1}-s}\Gamma(s + k_{1} - 1)D(s + k_{1} - 1, f_{1}, g_{1})
\]

for \( \text{Re}(s) > (k_{1} + 1)/2 \), where

\[
D(s, f_{1}, g_{1}) := \sum_{m=1}^{\infty} a_{1}(m)\overline{b_{1}(m)m^{-s}}.
\]

Thus in this case Theorem 1 states nothing but the well-known properties of the Rankin series \( D(s, f_{1}, g_{1}) \).
(iii) In case $n = 2$ we have

$$D(s; f_1, f_2; g_1, g_2) = 2^{6-2|k|} \pi^{-2-|k|} (2\pi)^{-2\epsilon} \frac{\Gamma(s)\Gamma(s+|k|-2)\Gamma(s+k_1-1)\Gamma(s+k_2-1)}{\Gamma(2s+|k|-2)} \sum_{m_1, m_2 \in \mathbb{Z}_{>0}} a_1(m_1)a_2(m_2)\overline{b_1(m_1)b_2(m_2)}m_1^{1-k_1-\epsilon}m_2^{1-k_2}$$

for $\Re(s) > \max(k_1, k_2) + 1$, where $F = \text{}_2F_1$ is the hypergeometric function.

(iv) The function $Q_k^{(n)}(m, s)$ has another representation:

$$Q_k^{(n)}(m, s) = 2^{3n-|k|+1} \pi^{\frac{n-|k|}{2}} \sum_{\lambda_1, \ldots, \lambda_n \in \mathbb{Z}_{>0}} \left( \prod_{j=1}^{n} \lambda_j^{k_j-1} \right) \int_{0}^{\infty} t^{2s-1+|k|-n} \prod_{j=1}^{n} K_{k_j-1}(4\sqrt{\pi m_j \lambda_j} t) dt$$

for $\Re(s) > n/2$, where $K_\nu$ is the modified Bessel function of order $\nu$.

References


