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Kyoto University
CAP automorphic representations of low rank groups *

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Abstract

In this talk, I report my recent joint work with K. Konno on non-tempered automorphic representations on low rank groups [KK]. We obtain a fairly complete classification of such automorphic representations for the quasisplit unitary groups in four variables.

1 CAP forms

The term CAP in the title is a short hand for the phrase “Cuspidal but Associated to Parabolic subgroups”. This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. More precisely, let $G$ be a connected reductive group defined over a number field $F$, and $G^*$ be its quasisplit inner form. We write $\mathbb{A} = \mathbb{A}_F$ for the adele ring of $F$. An irreducible cuspidal representation $\pi = \bigotimes_v \pi_v$ is a CAP form if there exists a residual discrete automorphic representation $\pi^* = \bigotimes_v \pi_v^*$ such that, at all but finite number of $v$, $\pi_v$ and $\pi_v^*$ share the same absolute values of Hecke eigenvalues.

It is a consequence of the result of Jacquet-Shalika [JS81a], [JS81b] and Moeglin-Waldspurger [MW89] that there are no CAP forms on the general linear groups. On the other hand, for a central division algebra $D$ of dimension $n^2$ over $F^*$, the trivial representation of $D^*(\mathbb{A})$ is clearly a CAP form which shares the same local component, at any place $v$ where $D$ is unramified, with the residual representation $1_{GL(n,\mathbb{A})}$. On the other hand, a quasisplit unitary group $U_{E/F}(3)$ of 3-variables already have non-trivial CAP forms, which can be obtained as $\theta$-lifts of some automorphic characters of $U_{E/F}(1)$ [GR90], [GR91]. But the first and the most well-known example of CAP forms are the analogues of the $\theta_{10}$ representation by Howe-Piatetski-Shapiro [Sou88] and the Saito-Kurokawa representations of $Sp_4$ [PS83]. Also Gan-Gurevich-Jiang obtained very interesting example

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of CAP forms on the split group of type $G_2$ [GGJ02] (see also the article by Gan in this volume).

In any case, the local components of CAP forms at almost all places are non-trivial Langlands quotients by definition, and hence non-tempered in an apparent way. To put such forms into the framework of Langlands' conjecture, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is through the so-called $A$-parameters, homomorphisms $\psi$ from the direct product of the hypothetical Langlands group $\mathcal{L}_F$ of $F$ with $SL(2, \mathbb{C})$ to the $L$-group $^L G$ of $G$ [Bor79]:

$$\psi : \mathcal{L}_F \times SL(2, \mathbb{C}) \longrightarrow ^L G,$$

considered modulo $\hat{G}$-conjugation. We write $\Psi(G)$ for the set of $\hat{G}$-conjugacy classes of $A$-parameters for $G$. By restriction, we obtain the local component

$$\psi_v : \mathcal{L}_{F_v} \times SL(2, \mathbb{C}) \rightarrow ^L G_v$$

of $\psi$ at each place $v$. Here the local Langlands group $\mathcal{L}_{F_v}$ is defined in [Kot84, §12], and $^L G_v$ is the $L$-group of the scalar extension $G_v = G \otimes_F F_v$. The local conjecture, among other things, associates to each $\psi_v$ a finite set $\Pi_{\psi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an $A$-packet. At all but finite number of $v$, $\Pi_{\psi_v}(G_v)$ is expected to contain a unique unramified element $\pi_v^1$. Using such elements, we can form the global $A$-packet associated to $\psi$

$$\Pi_{\psi}(G) := \left\{ \bigotimes_v \pi_v | \begin{array}{l} (i) \quad \pi_v \in \Pi_{\psi_v}(G_v), \forall v; \vphantom{^L G} \\ (ii) \quad \pi_v = \pi_v^1, \forall v \end{array} \right\}.$$

Arthur's conjecture predicts the multiplicity of each element in $\Pi_{\psi}(G)$ in the discrete spectrum of the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\mathcal{A}_G \backslash G(\mathbb{A}))$. Here $\mathcal{A}_G$ is the maximal $\mathbb{R}$-vector subgroup in the center of the infinite component $G(\mathbb{A}_{\infty})$ of $G(\mathbb{A})$.

We say an $A$-parameter $\psi$ is of CAP type if

(i) $\psi$ is elliptic. This is the condition for $\Pi_{\psi}(G)$ to contain an element which occurs in the discrete spectrum.

(ii) $\psi|_{SL(2, \mathbb{C})}$ is non-trivial.

According to the conjecture, the CAP automorphic representations of $G(\mathbb{A})$ is contained in some of the global $A$-packets associated to such $A$-parameters. In this talk, we shall classify the CAP forms by such parameters along the line of Arthur's conjecture, in the case of the quasisplit unitary group $U_{E/F}(4)$ of four variables. Although our description of such forms tells nothing about the character relations conjectured in [Art89], it is quite explicit and fairly complete. We hope to apply this to certain analysis of the cohomology of the Shimura variety attached to $GU_{E/F}(4)$.

2 Parameter consideration

Global case Take a quadratic extension $E/F$ of number fields and write $\sigma$ for the generator of the Galois group of this extension. Let $G = G_n := U_{E/F}(n)$ be the quasisplit
unitary groups in $n$ variables associated to $E/F$. Later we shall mainly be concerned with the case $n = 4$. The $L$-group $^L G$ is the semi-direct product of $\hat{G} = GL(n, \mathbb{C})$ by the absolute Weil group $W_F$ of $F$, where $W_F$ acts through $W_F/W_E \simeq \text{Gal}(E/F)$ by

$$\rho_G(\sigma)g = \text{Ad}(I_n)^tg^{-1}, \quad I_n := \begin{pmatrix} 1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ (-1)^{n-1} & \cdots & 1 \end{pmatrix}.$$ 

Thus an $A$-parameter $\psi$ for $G$ is determined by its restriction to $\mathcal{L}_E \times SL(2, \mathbb{C})$, which is just a completely reducible representation:

$$\psi|_{\mathcal{L}_E \times SL(2, \mathbb{C})} = \bigoplus_{i=1}^r \varphi_{\Pi_i} \otimes \rho_d.$$ 

Here $\Pi_i$ is an irreducible cuspidal representation of $GL(m_i, A_F)$ enjoying the following properties:

- $\sigma(\Pi_i) := \Pi_i \circ \sigma$ is isomorphic to the contragredient $\Pi'_i$.
- Its central character $\omega_{\Pi_i}$ restricted to $\mathbb{A}^\times$ equals $\omega_{E/F}^{n-d_i-m_i+1}$, where $\omega_{E/F}$ is the quadratic character associated to $E/F$ by the classfield theory.
- Some condition on the order of its twisted Asai $L$-functions at $s = 1$.

$\rho_d$ is the $d$-dimensional irreducible representation of $SL(2, \mathbb{C})$. We note that $\psi$ is elliptic if and only if its irreducible components $\varphi_{\Pi_i} \otimes \rho_d$, are distinct to each other. The $S$-group

$$S_\psi(G) := \pi_0(\text{Cent}(\psi, \hat{G})/Z(\hat{G}))$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r-1}$, where $\pi_0(\bullet)$ stands for the group of connected components. This plays a central role in the conjectural multiplicity formula.

**Local case** Similar description for the $A$-packets of the unitary group $G = G_n$ associated to a quadratic extension $E/F$ of local fields is also valid. For each $A$-parameter $\psi$, we have the associated non-tempered Langlands parameter

$$\phi_\psi : \mathcal{L}_F \ni w \mapsto \psi(w, \left(\begin{array}{cc} |w|_{F}^{1/2} & 0 \\ 0 & |w|_{F}^{-1/2} \end{array}\right)) \in {}^L G.$$ 

Here the "absolute value" $|w|_F$ on $\mathcal{L}_F$ is the composite $|w|_F : \mathcal{L}_F \rightarrow W^\text{rec}_F \overset{\sim}{\rightarrow} F^\times \rightarrow \mathbb{R}_+^\times$. (rec denotes the reciprocity map in the local classfield theory.) In Arthur’s conjecture, it was imposed that the $L$-packet $\Pi_{\phi_\psi}(G)$ associated to $\phi_\psi$ should be contained in $\Pi_{\psi}(G)$. We also have the $S$-group $S_\psi(G)$ as in the global case. We postulate the following:

**Assumption 2.1.** There exists a bijection $\Pi_{\phi_\psi}(G) \ni \pi \mapsto (\bar{s}, \pi|_\psi) \in \Pi(S_\psi(G))$. Here $\Pi(S_\psi(G))$ is the set of isomorphism classes of irreducible representations of $S_\psi(G)$. 

Now for $n=4$, the possibilities of $\{(d_i, m_i)\}$ for elliptic $A$-parameters with non-trivial $SL(2, \mathbb{C})$-component are given as follows.

(1) Stable cases. $\{(4, 1)\}, \{(2, 2)\}$.

(2) Endoscopic cases.
   (a) $\{(3, 1), (1, 1)\}$;
   (b) $\{(2, 1), (1, 2)\}$;
   (c) $\{(2, 1), (2, 1)\}$;
   (d) $\{(2, 1), (1, 1), (1, 1)\}$.

In the cases (1), (2.a), it follows from Assumption 2.1 that $\Pi_{\phi}(G) = \Pi_{\psi}(G)$, and we know from [KOn98] that all the contribution of the corresponding global $A$-packets belong to the residual spectrum. On the other hand, $\Pi_{\phi}(G) \backslash \Pi_{\psi}(G)$ is expected to be non-empty in the rest cases. We shall use the local $\theta$-correspondence to construct the missing members.

3 Local $\theta$-correspondence

Local Howe duality First let us recall the local $\theta$-correspondence. We consider an $m$-dimensional (non-degenerate) hermitian space $(V, (, ))$ and $n$-dimensional skew-hermitian space $(W, \langle, \rangle)$ over $E$. We write $G(V)$ and $G(W)$ for the unitary groups of $V$ and $W$, respectively. If we define the symplectic space $(\mathcal{W}, \langle\langle, \rangle\rangle)$ by

$$\mathcal{W} := V \otimes_{E} W, \quad \langle\langle v \otimes w, v' \otimes w'\rangle\rangle := \frac{1}{2}i \mathrm{R}_{E/F}[\langle v, v'\rangle \sigma(\langle w, w'\rangle)],$$

Then $(G(V), G(W))$ form a so-called dual reductive pair in the symplectic group $Sp(\mathcal{W})$ of this symplectic space:

$$\iota_{V,W} : G(V) \times G(W) \ni (g, g') \mapsto g \otimes g' \in Sp(\mathcal{W}).$$

Fixing a non-trivial character $\psi_F$ of $F$, we have the metaplectic group of $\mathcal{W}$ which is a central extension

$$1 \rightarrow \mathbb{C}^1 \rightarrow Mp_{\psi_F}(\mathcal{W}) \rightarrow Sp(\mathcal{W}) \rightarrow 1.$$

This admits a unique Weil representation $\omega_{\psi_F}$ on which $\mathbb{C}^1$ acts by the multiplication [RR93]. For each pair $\xi = (\xi, \xi')$ of characters of $E^\times$ satisfying $\xi|_{F^\times} = \omega_{E/F}^{m}, \xi'|_{F^\times} = \omega_{E/F}^{m}$, we have the corresponding lifting $\tilde{\iota}_{V,W,\xi} : G(V) \times G(W) \rightarrow Mp_{\psi_F}(\mathcal{W})$ of $\iota_{V,W}$:

$$G(V) \times G(W) \xrightarrow{\tilde{\iota}_{V,W,\xi}} Mp_{\psi_F}(\mathcal{W})$$

The composite $\omega_{V,W,\xi} := \omega_{\psi} \circ \tilde{\iota}_{V,W,\xi}$ is the Weil representation of the dual reductive pair $(G(V), G(W))$ associated to $\xi$. It is the product of the Weil representations $\omega_{W,\xi}$ of $G(V)$ and $\omega_{V,\xi}$ of $G(W)$.
We write $\mathcal{R}(G(V), \omega_{W, \xi})$ for the set of isomorphism classes of irreducible admissible representations of $G(V)$ which appear as quotients of $\omega_{W, \xi}$. For $\pi_V \in \mathcal{R}(G(V), \omega_{W, \xi})$, the maximal $\pi_V$-isotypic quotient of $\omega_{V, W, \xi}$ is of the form $\pi_V \otimes \Theta_{\xi}(\pi_V, W)$ for some smooth representation $\Theta_{\xi}(\pi_V, W)$ of $G(W)$. Similarly we have $\mathcal{R}(G(W), \omega_{V, \xi})$ and $\Theta_{\xi}(\pi_W, V)$ for each $\pi_W \in \mathcal{R}(G(W), \omega_{V, \xi})$. The local Howe duality conjecture, which was proved by R. Howe himself if $F$ is archimedean [How89] and by Waldspurger if $F$ is a non-archimedean local field of odd residual characteristic [Wal90], asserts the following:

(i) $\Theta_{\xi}(\pi_V, W)$ (resp. $\Theta_{\xi}(\pi_W, V)$) is an admissible representation of finite length of $G(W)$ (resp. $G(V)$), so that it admits an irreducible quotient.

(ii) Moreover its irreducible quotient $\theta_{\xi}(\pi_V, W)$ (resp. $\theta_{\xi}(\pi_W, V)$) is unique.

(iii) $\pi_V \mapsto \theta_{\xi}(\pi_V, W)$, $\pi_W \mapsto \theta_{\xi}(\pi_W, V)$ are bijections between $\mathcal{R}(G(V), \omega_{W, \xi})$ and $\mathcal{R}(G(W), \omega_{V, \xi})$ converse to each other.

Adams' conjecture A link between the local $\theta$-correspondence and $A$-packets is given by the following conjecture of J. Adams [Ada89]. Suppose $n \geq m$. Then we have an $L$-embedding $i_{V, W, \xi} : \mathcal{L}G(V) \to \mathcal{L}G(W)$ given by

$$i_{V, W, \xi}(g \times w) := \begin{cases} 
\xi' \xi^{-1}(w) \begin{pmatrix} g & 1_{n-m} \\
-1 & g \\
1_{n-m} & 1_{n-m} 
\end{pmatrix} \times w & \text{if } w \in W_F, \\
J_n \times w_\sigma & \text{if } w = w_\sigma,
\end{cases}$$

where $w_\sigma$ is a fixed element in $W_F \setminus W_E$ and

$$J_n := \begin{pmatrix} 1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & (-1)^{n-1}
\end{pmatrix}$$

Let $T : SL(2, \mathbb{C}) \to \text{Cent}(i_{V, W, \xi})$, $\tilde{G}(W)$ be the homomorphism which corresponds to a regular unipotent element in $\text{Cent}(i_{V, W, \xi}) \simeq GL(n - m, \mathbb{C})$ (the tail representation of $SL(2, \mathbb{C})$). Using this, we define the $\theta$-lifting of $A$-parameters by

$$\theta_{V, W, \xi} : \Psi(G(V)) \ni \psi \mapsto (i_{V, W, \xi} \circ \psi^\vee) \cdot T \in \Psi(G(W)).$$

Conjecture 3.1 ([Ada89] Conj.A). The local $\theta$-correspondence should be subordinated to the map of $A$-packets: $\Pi_{\psi}(G(V)) \mapsto \Pi_{\theta_{V, W, \xi}(\psi)}(G(W))$.

Here we have said subordinated because $\mathcal{R}(G(V), \omega_{W, \xi})$ is not compatible with $A$-packets, that is, $\Pi_{\psi}(G(V)) \cap \mathcal{R}(G(V), \omega_{W, \xi})$ is often strictly smaller than $\Pi_{\psi}(G(V))$. But when these two are assured to coincide, we can expect more:
Conjecture 3.2 ([Ada89] Conj.B). For $V$, $W$ in the stable range, that is, the Witt index of $W$ is larger than $m$, we have

$$\Pi_{\theta_{V,W}(\psi)}(G(W)) = \bigcup_{V: \dim_{E}V = m} \theta_{\xi}(\Pi_{\psi}(G(V)), W).$$

Now we note that our situation is precisely that of Conj. 3.2 with $m = 2$ and $W = V \oplus -V$. Moreover, we find that the $A$-parameters in the cases (2.b), (2.c), (2.d) in § 2 are exactly those of the form

$$\theta_{V,W}(\psi), \quad \psi \in \Psi(G(V)).$$

$\varepsilon$-dichotomy We explain the construction of the $A$-packets when $F$ is non-archimedean. We need one more ingredient.

Proposition 3.3 (\varepsilon-dichotomy). Suppose $\dim_{E}V = 2$ and write $W_{1}$ for the hyperbolic skew-hermitian space $(E^{2}, (0, 1))$. Take an $L$-packet $\Pi$ of $G_{2}(F) = G(W)$ and $\tau \in \Pi$ ([Rog90, Ch.11]).

(i) $\tau \in \mathcal{R}(G(W), \omega_{V,\xi})$ if and only if

$$\varepsilon(1/2, \Pi \times \xi^{-1}, \psi_{F})\omega_{1}(-1)\lambda(E/F, \psi_{F})^{-2} = \omega_{E/F}(-\det V).$$

Here the $\varepsilon$-factor on the right hand side is the standard $\varepsilon$-factor for $G_{2}$ twisted by $\xi^{-1}$ defined by the Langlands-Shahidi theory [Sha90]. $\omega_{1}$ is the central character of the elements of $\Pi$ and $\lambda(E/F, \psi_{F})$ is Langlands’ $\lambda$-factor [Lan70].

(ii) If this is the case, we have $\theta_{\xi}(\tau, V) = (\xi^{-1}\xi')_{G(V)}\tau_{V}$. Here $(\xi^{-1}\xi')_{G(V)}$ denotes the character of $G(V)$ given by the composite

$$G(V) \ni \tau \to U_{E/F}(1, F) \ni z/\sigma(z) \mapsto \xi^{-1}\xi'(z) \in C^{\times}.$$

$\tau_{V}$ stands for the Jacquet-Langlands correspondent of $\tau$.

This is a special case of the $\varepsilon$-dichotomy of the local $\theta$-correspondence for unitary groups over $p$-adic fields, which was proved for general unitary groups (at least for supercuspidal representations) in [HKS96]. But since we need to combine this with our description of the residual spectrum [Kou98], we have to use the Langlands-Shahidi $\varepsilon$-factors instead of Piatetski-Shapiro-Rallis’s doubling $\varepsilon$-factors adopted by them. By this reason, we deduced this proposition from the analogous result for the unitary similitude groups [Har93] combined with the following description of the base change for $G_{2}$.

Lemma 3.4. Let $\overline{\pi} = \omega \otimes \pi'$ be an irreducible admissible representation of the unitary similitude group $GU_{E/F}(2) \simeq (E^{\times} \times GL(2, F))/\Delta F^{\times}$, and write $\Pi(\overline{\pi})$ for the associated $L$-packet of $G_{2}(F)$ consisting of the irreducible components of $\overline{\pi}|_{G_{2}(F)}$. Then the standard base change of $\Pi(\overline{\pi})$ to $GL(2, E)$ ([Rog90, 11.4]) is given by $\omega(\det)\pi_{E}^{\times}$, where $\pi_{E}$ is the base change lift of $\pi'$ to $GL(2, E)$ [Lan80].

\footnote{In fact, the Jacquet-Langlands correspondence for unitary groups in two variables is defined only for $L$-packets and not for each member of the packets [LL79]. We know that $\tau \mapsto \tau_{V}$ certainly defines a bijection between $\Pi$ and its Jacquet-Langlands correspondent. But we do not specify the bijection explicitly here. See Rem. 3.8 also.}
Now we construct the $A$-packets. Our construction is summarized in the following picture.

Each $A$-parameter of our concern is of the form

$$\psi|_{\mathfrak{L} \times SL(2, \mathbb{C})} = \psi_1|_{\mathfrak{L} \times SL(2, \mathbb{C})} \oplus (\xi^\prime \xi^{-1} \otimes \rho_2),$$

where $\psi_1$ is some $A$-parameter for $G_2$. Take $\tau \in \Pi_{\psi_1}(G_2)$ and let $(V, (, ))$ be the 2-dimensional hermitian space such that the condition of Prop. 3.3 (i) holds. If we write $\pi_V := \theta_\tau(\tau, V) \simeq (\xi^\prime \xi^{-1})_{G(V)} \tau_V^\vee$, then the result of [Kud86] tells us $\pi_+ := \theta_\tau(\tau, W_2)$, $(\tau \in \Pi_{\psi_1}(G_2))$ form the local residual $L$-packet $\Pi_{\psi_1}(G_4)$. We now suppose that there exists a Jacquet-Langlands correspondent $\pi_{V'} \simeq (\xi^\prime \xi^{-1})_{G(V')} \tau_{V'}^\vee$, of $\pi_V$ on the unitary group $G(V')$ of the other (isometry class of) 2-dimensional hermitian space. Then Prop. 3.3 (i) tells us that $\pi_{V'} \notin \mathscr{R}(G(V'), \omega_{W_1, \xi})$. Yet its local $\vartheta$-lifting $\pi_{-} := \theta_\tau(\tau, W_2)$ to the larger group $G_4(F)$ still exists. This is the so-called early lift or the first occurrence. Following Conj. 3.2, we define

$$\Pi_{\psi}(G_4) := \{\pi_{\pm} | \tau \in \Pi_{\psi}(G_2)\}.$$

This gives sufficiently many members of the packet as predicted by Assumption 2.1.

**Example 3.5.** (i) Suppose $\Pi_{\psi_1}(G_2)$ is an $L$-packet consisting of supercuspidal elements. For $\tau \in \Pi_{\psi_1}(G_2)$, $\pi_+$ is the Langlands quotient $J_{P_2}^{G_4}(\xi^\prime \xi^{-1} || \psi_1^2 \otimes \tau)$, where $P_2$ is a parabolic subgroup with the Levi factor $\mathbb{R}_E F G_m \times G_2$. On the other hand the early lift $\pi_-$ of the supercuspidal $\tau$ is again supercuspidal. Thus $\Pi_{\psi}(G_4)$ consists of non-tempered members and supercuspidal elements.

(ii) On the contrary, we take $\xi = \xi^\prime$ and consider $\Pi_{\psi_1}(G_2)$ consists of either the Steinberg representation $\delta_{G_2}$ or the trivial representation $1_{G_2}$.

- $\delta_{G_2}$ lifts to $\pi_V = 1_{G(V)}$, where $V$ is anisotropic. $\pi_{V'} = \delta_{G_2}$. $\pi_+ = J_{P_2}^{G_4}(|| \psi_1^2 \otimes \delta_{G_2})$ and $\pi_-$ is an irreducible tempered but not square integrable representation.

- $1_{G_2}$ lifts to $\pi_V = 1_{G(V)}$ but $V$ is hyperbolic this time. $\pi_{V'}$ is again $1_{G(V')}$ but this should be viewed as the Jacquet-Langlands correspondent of the $A$-packet $\{1_{G(V)}\}$. We have $\pi_+ = J_{P_2}^{G_4}(I_{B}^{GL(2)_{E}}(1 \otimes 1) | \det |^{1/2})$, where $P_2$ is the so-called Siegel parabolic subgroup with the Levi factor $GL(2, E)$. Obviously $\pi_- = J_{P_2}^{G_4}(|| \psi_1^2 \otimes \delta_{G_2})$. This last lift is shared by the two packets considered here.
Real case  We end this section by some comments on the case $E/F = \mathbb{C}/\mathbb{R}$. Similar results are obtained by applying the argument of Adams-Barbasch [AB95]. In fact, the local $\theta$-correspondence between unitary groups of the same size is described quite explicitly and in full generality in [Pau98]. Their argument also works in the present case. Let me explain some example.

We write $G_{p,q} = U(p, q)$. For a regular integral infinitesimal character $\lambda = (\lambda_1, \lambda_2)$ for $G_{1,1}$, consider the extended $L$-packet:

$$\Pi_{\lambda} = \{\delta_{1,1}^{+}, \delta_{1,1}^{-}, \delta_{2,0}, \delta_{0,2}\}$$

consisting of the discrete series representation of various $G_{p,q}$ with the infinitesimal character $\lambda$. The subscript $p, q$ indicates that $\delta_{p,q}^\ast$ lives on $G_{p,q}$. We can write $\xi'\xi^{-1}(z) = (z/\bar{z})^n$, $\forall z \in \mathbb{C}$ for some $n \in \mathbb{Z}$. An analogue of Prop. 3.3 in the real case asserts that the local $\theta$-correspondence under the Weil representation $\omega_{\psi}$ gives a bijection

$$\theta_{\xi} : \Pi_{\lambda} \longrightarrow \Pi_{n-\lambda},$$

where $n - \lambda = (n - \lambda_2, n - \lambda_1)$.

If $\lambda$ is sufficiently regular, by which we mean $|\lambda_i - n| > 1$, then it is proved by J.-S. Li [Li90] that $\theta_{\xi}(\xi(\delta_{1,1}^{\pm}), W_2)$ is a non-tempered cohomological representation $A_q(\lambda')$, where the Levi factor of the $\theta$-stable parabolic subalgebra $q$ is $u(1,1) \oplus u(1)^2$. As for the other elements $\delta_{p,q} \in \Pi_{n-\lambda}$, $\theta_{\xi}(\delta_{p,q}, W_2)$ is a discrete series representation $A_q(\lambda')$. This time $q$ has the Levi factor $u(2) \oplus u(1)^2$. The resulting $A$-packet $\theta_{\xi}(\Pi_{n-\lambda})$ is exactly the cohomological $A$-packet defined by Adams-Johnson [AJ87].

For the complete list of the packets both in the archimedean and non-archimedean case, see our paper [KK].

One can easily check that the $S$-groups in the cases (2.b), (2.c), (2.d) satisfy $S_{\psi}(G_4) \simeq S_{\psi_i}(G_2) \times \mathbb{Z}/2\mathbb{Z}$. Now we define the bijection in Assumption 2.1 by

- $\langle \bar{\delta}, \pi_{\pm}\rangle_{\psi_i} = \langle \bar{\delta}, \tau\rangle_{\psi_i}$ on $\bar{\delta} \in S_{\psi_i}(G_2)$;
- $\langle , \pi_{\pm}\rangle_{\psi}$ on $\mathbb{Z}/2\mathbb{Z}$ equals the sign character if $\pi_-$ and trivial character otherwise.

For the other cases, only the first one in this definition is enough to give a complete bijection. This finishes our local task.

**Remark 3.6.** In the above, we do not mention the definition of the pairing $\langle , \rangle_{\psi_i}$. There are several choices for this, and we can choose one by fixing a non-trivial character $\psi_F$ of $F$ [LL79]. Also we did not specify the correspondence $\pi_V \mapsto \pi_V^\ast$, which is again a subtle problem. In fact, we need to make a choice of (absolute) transfer factor as in [LL79] which again involves a choice of $\psi_F$ (appearing in $\lambda(E/F, \psi_F)$ in the transfer factor). Using this specific transfer, we label the members of endoscopic $L$-packets of anisotropic unitary group. The correspondence $\pi_V \mapsto \pi_V^\ast$ can be described in terms of these data, but we do not go into details here.
4 Multiplicity formula

We now go back to the global situation where $E/F$ is a quadratic extension of number fields. We note that there always exists a homomorphism $S_\psi(G_4) \ni \overline{s} \mapsto \overline{s}(v) \in S_\psi(G_{4,v})$. We can now state the main result of this talk. Although we treat only the number field case, we believe the result holds also over function fields of one variable over a finite field of odd characteristic.

**Theorem 4.1.** Let $\psi$ be an $A$-parameter of CAP type for $G_4 = U_{E/F}(4)$. As was explained in §1, we form the global $A$-packet $\Pi_\psi(G_4) := \bigotimes_v \Pi_{\psi_v}(G_{4,v})$. Then the multiplicity $m(\pi)$ of $\pi = \bigotimes_v \pi_v \in \Pi_\psi(G_4)$ in $L^2(G(F) \backslash G(A))$ is given by

$$m(\pi) = \frac{1}{|S_\psi(G_4)|} \sum_{\overline{s} \in S_\psi(G_4)} \epsilon_{\psi}(\overline{s}) \prod_v (\overline{s}(v), \pi_v)_{\psi_v},$$

where the sign character $\epsilon_{\psi}$ is defined by

$$\epsilon_{\psi} = \begin{cases} 
\text{sgn}_{S_\psi(G_4)} & \text{if } \psi_1 \text{ is a stable L-parameter} \\
\epsilon(1/2, \psi_1 \otimes \xi \xi^{-1}) = -1, & \text{and } \epsilon(1/2, \psi_1 \otimes \xi \xi^{-1}) = -1, \\
1 & \text{otherwise.}
\end{cases}$$

Here $\epsilon(s, \psi_1 \otimes \xi \xi^{-1})$ is the Artin root number attached to $\psi_1$, which equals the standard $\epsilon$-function for $\Pi_{\psi_1}(G_2) \times \xi \xi^{-1}$.

The proof divides into two parts. Our local construction together with the global $\theta$-correspondence shows that the multiplicity is no less than the right hand side. Note that we also relies on the multiplicity formula of Labesse-Langlands for unitary groups in two variables [LL79], [Rog90]. Then we prove a characterization of the image of such $\theta$-lifts by poles of certain $L$-functions, which gives the converse inequality. This also shows that all the CAP forms for $U_{E/F}(4)$ are obtained in the above as the contribution of the $A$-packets we constructed. In particular the $A$-packets contains the sufficiently many members at least for global purposes, so that our Assumption 2.1 is justified.

References


