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Kyoto University
On Siegel modular forms of degree 2 with square-free level

RALF SCHMIDT

Introduction

For representations of GL(2) over a $p$-adic field $F$ there is a well-known theory of local newforms due to CASSELMAN, see [Cas]. This local theory together with the global strong multiplicity one theorem for cuspidal automorphic representations of GL(2) is reflected in the classical Atkin–Lehner theory for elliptic modular forms.

In contrast to this situation, there is currently no satisfactory theory of local newforms for the group GSp(2, $F$). As a consequence, there is no analogue of Atkin–Lehner theory for Siegel modular forms of degree 2. In this paper we shall present such a theory for the “square-free” case. In the local context this means that the representations in question are assumed to have non-trivial Iwahori–invariant vectors. In the global context it means that we are considering congruence subgroups of square-free level.

We shall begin by reviewing some well known facts from the classical theory of elliptic modular forms. Then we shall give a definition of a space $S_k(\Gamma_0(N)^{(2)})^{\text{new}}$ of newforms in degree 2, where $N$ is a square-free positive integer. Table 1 on page 8 lies at the heart of our theory. It contains the dimensions of the spaces of fixed vectors under each parahoric subgroup in every irreducible Iwahori–spherical representation of GSp(2) over a $p$-adic field $F$.

Section 4 deals with a global tool, namely a suitable $L$–function theory for certain cuspidal automorphic representations of PGSp(2). Since none of the existing results on the spin $L$–function seems to fully serve our needs, we have to make certain assumptions at this point. Having done so, we shall present our main result in the final section 5. It essentially says that given a cusp form $f \in S_k(\Gamma_0(N))^{\text{new}}$, assumed to be an eigenform for almost all unramified Hecke algebras and also for certain Hecke operators at places $p | N$, we can attach a global $L$–packet $\pi_f$ of automorphic representations of PGSp(2, $\mathbb{A}_Q$) to $f$. This allows us to associate with $f$ a global (spin) $L$–function with a nice functional equation. We shall describe the local factors at the bad places explicitly in terms of certain Hecke eigenvalues.
1 Review of classical theory

We recall some well-known facts for classical holomorphic modular forms. Let \( f \in S_k(\Gamma_0(N)) \) be an elliptic cuspform, and let \( G = \text{GL}(2) \), considered as an algebraic \( \mathbb{Q} \)-group. It follows from strong approximation for \( \text{SL}(2) \) that there is a unique associated adelic function \( \Phi_f : G(\mathbb{A}) \to \mathbb{C} \) with the following properties:

1. \( \Phi_f(\rho gz) = \Phi_f(g) \) for all \( g \in G(\mathbb{A}) \), \( \rho \in G(\mathbb{Q}) \) and \( z \in Z(\mathbb{A}) \). Here \( Z \) is the center of \( \text{GL}(2) \).

2. \( \Phi_f(gh) - \Phi_f(g) \) for all \( g \in G(\mathbb{A}) \) and \( h \in \prod_{p<\infty} K_p(N) \). Here \( K_p(N) = \{ (a b \mid c d) \in \text{GL}(2, \mathbb{Z}_p) : c \in N\mathbb{Z}_p \} \) is the local analogue of \( \Gamma_0(N) \).

3. \( \Phi_f(g) = (f|_{k} g)(i) := \det(g)^{k/2}j(g,i)^{-k}f(g(i)) \) for all \( g \in \text{GL}(2, \mathbb{R})^+ \) (the identity component of \( \text{GL}(2, \mathbb{R}) \)).

Since \( f \) is a cusp form, \( \Phi_f \) is an element of \( L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{A})) \). Let \( \pi_f \) be the unitary \( \text{PGL}(2, \mathbb{A}) \)-subrepresentation of this \( L^2 \)-space generated by \( \Phi_f \).

1.1 Theorem. With the above notations, the representation \( \pi_f \) is irreducible if and only if \( f \) is an eigenform for the Hecke operators \( T(p) \) for almost all primes \( p \). If this is the case, then \( f \) is automatically an eigenform for \( T(p) \) for all \( p \mid N \).

Idea of Proof: We decompose the representation \( \pi_f \) into irreducibles, \( \pi_f = \bigoplus_i \pi_i \). Each \( \pi_i \) can be written as a restricted tensor product of local representations,

\[
\pi_i \simeq \bigotimes_{p \leq \infty} \pi_{i,p}, \quad \pi_{i,p} \text{ a representation of } \text{PGL}(2, \mathbb{Q}_p).
\]

Assuming that \( f \) is an eigenform, one can show easily that for almost all \( p \) we have \( \pi_{i,p} \simeq \pi_{j,p} \). But Strong Multiplicity One for \( \text{GL}(2) \) says that two cuspidal automorphic representations coincide (as spaces of automorphic forms) if their local components are isomorphic at almost every place. It follows that \( \pi_f \) must be irreducible.

Thus to each eigenform \( f \) we can attach an automorphic representation \( \pi_f = \otimes \pi_p \). A natural problem is to identify the local representations \( \pi_p \) given only the classical function \( f \). This is easy at the archimedean place: \( \pi_\infty \) is the discrete series representation of \( \text{PGL}(2, \mathbb{R}) \) with a lowest weight.
vector of weight $k$. It is also easy for finite primes $p$ not dividing $N$. At such places $\pi_p$ is an unramified principal series representation, i.e., $\pi_p$ is an infinite-dimensional representation containing a non-zero $GL(2, \mathbb{Z}_p)$-fixed vector. These representations are characterized by their Satake parameter $\alpha \in \mathbb{C}^*$, and the relationship between $\alpha$ and the Hecke-eigenvalue $\lambda_p$ is $\lambda_p = p^{(k-1)/2}(\alpha + \alpha^{-1})$.

In general it is not easy to identify the local components $\pi_p$ at places $p | N$. But if $N$ is square-free, we have the following result.

1.2 Theorem. Assume that $N$ is a square-free positive integer, and let $f \in S_k(\Gamma_0(N))$ be an eigenform. Further assume that $f$ is a newform. Then the local component $\pi_p$ of the associated automorphic representation $\pi_f$ at a place $p | N$ is given as follows:

\[ \pi_p = \begin{cases} 
\text{St}_{GL(2)} & \text{if } a_1f = -f, \\
\xi \text{St}_{GL(2)} & \text{if } a_1f = f.
\end{cases} \]

Here $\text{St}_{GL(2)}$ is the Steinberg representation of $GL(2, \mathbb{Q}_p)$, and $\xi$ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^*$. The operator $a_1$ is the Atkin-Lehner involution at $p$.

Idea of Proof: It follows from the fact that $f$ is a modular form for $\Gamma_0(N)$ that $\pi_p$ contains non-trivial vectors invariant under the Iwahori subgroup

\[ I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}_p) : c \in p\mathbb{Z}_p \right\}. \]

The following is a complete list of all such Iwahori-spherical representations together with the dimensions of their spaces of fixed vectors under $I$ and under $K = GL(2, \mathbb{Z}_p)$.

<table>
<thead>
<tr>
<th>representation</th>
<th>$K$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi(\chi, \chi^{-1})$, $\chi$ unramified, $\chi^2 \neq \pm 1$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$\text{St}<em>{GL(2)}$ or $\xi \text{St}</em>{GL(2)}$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

We recall the definition of newforms, for notational simplicity assuming that $N = p$. We have two operators

\[ T_0, T_1 : S_k(SL(2, \mathbb{Z})) \rightarrow S_k(\Gamma_0(p)), \]
where $T_0$ is simply the inclusion and $T_1$ is given by $(T_1 f)(\tau) = f(p \tau)$. Then the space of oldforms is defined as

$$S_k(\Gamma_0(p))^{\text{old}} = \text{im}(T_0) \cap \text{im}(T_1),$$

and the space of newforms $S_k(\Gamma_0(p))^{\text{new}}$ is by definition the orthogonal complement of $S_k(\Gamma_0(p))^{\text{old}}$ with respect to the Petersson inner product. Now it is easily checked that locally, in an unramified principal series representation $\pi(\chi, \chi^{-1})$ realized on a space $V$, we have

$$V^{I} = T_0 V^{K} + T_1 V^{K}.$$  

Hence the fact that $f$ is a newform means precisely that $\pi_p$ cannot be an unramified principal series representation $\pi(\chi, \chi^{-1})$. Therefore $\pi_p = \text{St}_{\text{GL}(2)}$ or $\pi_p = \xi \text{St}_{\text{GL}(2)}$, and easy computations show the connection with the Atkin–Lehner eigenvalue (cf. [Sch], section 3).

Knowing the local components $\pi_p$ allows to correctly attach local factors to the modular form $f$. For example, if $f$ is a newform as in Theorem 1.2, one would define for $p|N$

$$L_p(s, f) = L_p(s, \pi_p) = \begin{cases} (1 - p^{-1/2-s})^{-1} & \text{if } a_1 f = -f, \\ (1 + p^{-1/2-s})^{-1} & \text{if } a_1 f = f. \end{cases}$$

$$\varepsilon_p(s, f) = \varepsilon_p(s, \pi_p) = \begin{cases} -p^{1/2-s} & \text{if } a_1 f = -f, \\ p^{1/2-s} & \text{if } a_1 f = f. \end{cases}$$

With these definitions, and unramified and archimedean factors as usual, the functional equation $L(s, f) = \varepsilon(s, f)L(1 - s, f)$ holds for $L(s, f) = \Pi_p L_p(s, f)$ and $\varepsilon(s, f) = \Pi_p \varepsilon_p(s, f)$.

2 Newforms in degree 2

It is our goal to develop a similar theory as outlined in the previous section for the space of Siegel cusp forms $S_k(\Gamma_0(N)^{(2)})$ of degree 2 and square-free level $N$. Here we are facing several difficulties.

- Strong multiplicity one fails for the underlying group $\text{GSp}(2)$, and even weak multiplicity one is presently not known. Thus it is not clear how to attach an automorphic representation of $\text{GSp}(2, \mathbb{A})$ to a classical cusp form $f$. 
The local representation theory of $\text{GSp}(2, \mathbb{Q}_p)$ is much more complicated than that of $\text{GL}(2, \mathbb{Q}_p)$. In particular, there are 13 different types of infinite-dimensional representations containing non-trivial vectors fixed under the local Siegel congruence subgroup, while in the $\text{GL}(2)$ case we had only 2 (see Table (1)).

There is currently no generally accepted notion of newforms for Siegel modular forms of degree 2.

The last two problems are of course related. Let $P_1$ be the Siegel congruence subgroup of level $p$, i.e.,

$$P_1 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}(2, \mathbb{Z}_p) : C \equiv 0 \mod p \right\}.$$

Every classical definition of newforms with respect to $P_1$ must in particular be designed to exclude $K$-spherical representations, where $K = \text{GSp}(2, \mathbb{Z}_p)$. Since an unramified principal series representation of $\text{GSp}(2, \mathbb{Q}_p)$ contains a four-dimensional space of $P_1$-invariant vectors (see Table 1 below), we expect four operators

$$T_0, T_1, T_2, T_3 : S_k(\text{Sp}(2, \mathbb{Z})) \to S_k(\Gamma_0(p))$$

whose images would span the space of oldforms. (From now on, when we write $\Gamma_0(N)$, we mean groups of $4 \times 4$-matrices.) For this purpose we are now going to introduce four endomorphisms $T_0(p), \ldots, T_3(p)$ of the space $S_k(\Gamma_0(N))$, where $N$ is square-free and $p \mid N$.

- $T_0(p)$ is simply the identity map.

- $T_1(p)$ is the Atkin–Lehner involution at $p$, defined as follows. Choose integers $\alpha, \beta$ such that $p\alpha - \frac{N}{p}\beta = 1$. Then the matrix

$$\eta_p = \begin{pmatrix} p\alpha & 1 \\ p\alpha & 1 \\ N\beta & p \\ N\beta & p \end{pmatrix}$$

is in $\text{GSp}(2, \mathbb{R})^+$ with multiplier $p$. It normalizes $\Gamma_0(N)$, hence the map $f \mapsto f|_k\eta_p$ defines an endomorphism of $S_k(\Gamma_0(N))$. Since $\eta_p^2 \in p\Gamma_0(N)$, this endomorphism is an involution (we always normalize the slash operator as

$$(f|_k g)(Z) = \mu(g)^k j(g, Z)^{-k} f(g(Z)) \quad (\mu \text{ is the multiplier}),$$

which makes the center of $\text{GSp}(2, \mathbb{R})^+$ act trivially). This is the Atkin–Lehner involution at $p$. It is independent of the choice of $\alpha$ and $\beta$. 


We define $T_2(p)$ by

$$(T_2(p)f)(Z) = \sum_{g \in \Gamma_0(N) \backslash \Gamma_0(N)} (f|_k g)(Z)$$

$$= \sum_{x, \mu, \kappa \in \mathbb{Z}/p\mathbb{Z}} \left( f|_k \begin{pmatrix} 1 & 1 & \mu & x \\ \mu & \kappa & 1 & 1 \\ p & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right)(Z). \quad (6)$$

This is a well-known operator in the classical theory. In terms of Fourier expansions, if $f(Z) = \sum_{n, r, m} c(n, r, m) e^{2\pi i (n\tau + rz + mz')}$ with $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$, then

$$(T_2(p)f)(Z) = \sum_{n, r, m} c(np, rp, mp) e^{2\pi i (n\tau + rz + mz')} \quad (7)$$

Finally, we define $T_3(p) := T_1(p) \circ T_2(p)$.

Now we are ready to define newforms in degree 2.

2.1 Definition. Let $N$ be a square-free positive integer. In $S_k(\Gamma_0(N))$ we define the subspace of oldforms $S_k(\Gamma_0(N))^{\text{old}}$ to be the sum of the spaces

$$T_i(p)S_k(\Gamma_0(Np^{-1})), \quad i = 0, 1, 2, 3, \quad p|N.$$ 

The subspace of newforms $S_k(\Gamma_0(N))^{\text{new}}$ is defined as the orthogonal complement of $S_k(\Gamma_0(N))^{\text{old}}$ inside $S_k(\Gamma_0(N))$ with respect to the Petersson scalar product.

Note that this definition is analogous to the definition of oldforms in the degree 1 case. The operator $T_1$ given in (2) has the same effect as the Atkin-Lehner involution on modular forms for $\text{SL}(2, \mathbb{Z})$.

See [Ib] for more comments on the topic of old and new Siegel modular forms.

3 Local newforms

Let us realize $G = \mathrm{GSp}(2)$ using the symplectic form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In this section we shall consider $G$ as an algebraic group over a $p$-adic field $F$. Let $\mathfrak{o}$ be the ring of integers of $F$ and $p$ its maximal ideal. Let $K = G(\mathfrak{o})$ be
the standard special maximal compact subgroup of $G(F)$. As an Iwahori subgroup we choose

$$I = \left\{ g \in K : g \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \mod p \right\}$$

The parahoric subgroups of $G(F)$ correspond to subsets of the simple Weyl group elements in the Dynkin diagram of the affine Weyl group $C_2$:

$$s_0 \quad s_1 \quad s_2$$

The Iwahori subgroup corresponds to the empty subset of $\{s_0, s_1, s_2\}$. The numbering is such that $s_1$ and $s_2$ generate the usual 8-element Weyl group of $\text{GSp}(2)$. The corresponding parahoric subgroup is $P_{12} = K$. The Atkin–Lehner element

$$\eta = \begin{pmatrix} 1 \\ \varpi \\ \varpi \end{pmatrix} \in \text{GSp}(2, F) \quad (\varpi \text{ a uniformizer}) \quad (8)$$

induces an automorphism of the Dynkin diagram. The parahoric subgroup $P_{01}$ corresponding to $\{s_0, s_1\}$ is therefore conjugate to $K$ via $\eta$. We further have the Siegel congruence subgroup $P_1$ (see (5)), the Klingen congruence subgroup $P_2$, its conjugate $P_0 = \eta P_2 \eta^{-1}$, and the paramodular group

$$P_{02} = \left\{ g \in G(F) : g, g^{-1} \in \begin{pmatrix} o & p & o & o \\ o & o & o & p^{-1} \\ o & p & o & o \\ p & p & p & o \end{pmatrix} \right\}.$$

$K$ and $P_{02}$ represent the two conjugacy classes of maximal compact subgroups of $\text{GSp}(2, F)$. By a well-known result of BOREL (see [Bo]) the Iwahori–spherical irreducible representations are precisely the constituents of representations induced from an unramified character of the Borel subgroup. For $\text{GSp}(2)$, such representations were first classified by RODIER, see [Rod], but in the following we shall use the notation of SALLY–TADIC [ST]. The following Table 1 gives a complete list of all the irreducible representations of $\text{GSp}(2, F)$ with non-trivial $I$–invariant vectors. Behind each representation we have listed the dimension of the spaces of vectors fixed under each parahoric subgroup (modulo conjugacy). The last column gives the exponent of the conductor of the local parameter of each representation.
<table>
<thead>
<tr>
<th>I</th>
<th>representation</th>
<th>K</th>
<th>P_{02}</th>
<th>P_{2}</th>
<th>P_{1}</th>
<th>I</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\chi_1 \times \chi_2 \times \sigma$ (irreducible)</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>$\chi \text{St}_{GL(2)} \times \sigma$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\chi \text{1}_{GL(2)} \times \sigma$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>III</td>
<td>$\chi \times \sigma \text{St}_{GSp(1)}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$\chi \times \sigma \text{1}_{GSp(1)}$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>IV</td>
<td>$\sigma \text{St}_{GSp(2)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$L((\nu^2, \nu^{-1}\sigma \text{St}_{GSp(1)}))$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$L((\nu^{3/2}\text{St}_{GL(2)}, \nu^{-3/2}\sigma))$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\sigma \text{1}_{GSp(2)}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>V</td>
<td>$\delta([\xi_0, \nu \xi_0], \nu^{-1/2}\sigma)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$L((\nu^{1/2}\xi_0 \text{St}_{GL(2)}, \nu^{-1/2}\sigma))$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$L((\xi_0 \text{St}_{GL(2)}, \xi_0 \nu^{-1/2}\sigma))$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$L((\nu \xi_0, \xi_0 \times \nu^{-1/2}\sigma))$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>VI</td>
<td>$\tau(S, \nu^{-1/2}\sigma)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$\tau(T, \nu^{-1/2}\sigma)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$L((\nu^{1/2}\text{St}_{GL(2)}, \nu^{-1/2}\sigma))$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$L((\nu, \text{1}_{F^*} \times \nu^{-1/2}\sigma))$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Dimensions of spaces of invariant vectors in Iwahori–spherical representations of GSp(2, F).
The signs under the entries for the "symmetric" subgroups $P_{02}$, $P_1$ and $I$ indicate how these spaces of fixed vectors split into Atkin–Lehner eigenspaces, provided the central character of the representation is trivial. The signs listed in Table 1 are correct if one assumes that

- in Group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.

- in Groups IV, V and VI, where the central character is $\sigma^2$, the character $\sigma$ itself is trivial.

If these assumptions are not met, then one has to interchange the plus and minus signs in Table 3 to get the correct dimensions.

The information in Table 1 is essentially obtained by computations in the standard models of these induced representations. Details will appear elsewhere.

Imitating the classical theory, one can define oldforms by introducing natural operators from fixed vectors for bigger to fixed vectors for smaller parahoric subgroups. Here "bigger" not always means inclusion, since we also consider $K$ "bigger" than $P_{02}$. More precisely, we consider $R'$ bigger than $R$, and shall write $R' \succ R$, if there is an arrow from $R'$ to $R$ in the following diagram.

Whenever $R' \succ R$, one can define natural operators from $V^{R'}$ to $V^R$, where $V$ is any representation space. For example, our previously defined global operators $T_0(p)$ and $T_2(p)$ correspond to two natural maps $V^K \to V^{P_1}$. Our $T_1(p)$ and $T_3(p)$ correspond to two natural maps $V^{P_{01}} \to V^{P_1}$, composed with the Atkin–Lehner element $V^K \to V^{P_{01}}$.

This can be done for any parahoric subgroup, and it is natural to call any fixed vector that can be obtained from any bigger parahoric subgroup an oldform. Everything else would naturally be called a newform, but the meaning of
"everything else" has to be made precise. Let it suffice to say that if the
representation is unitary one can work with orthogonal complements as in
the classical theory.

Once these notions of oldforms and newforms are defined, one can verify the
decisive fact that each space of fixed vectors listed in Table 1 consists either
completely of oldforms or completely of newforms. If this were not true, our
notions of oldforms and newforms would make little sense. In Table 1 we
have indicated the spaces of newforms by writing their dimensions in bold
face. We see that they are not always one-dimensional.

4 \textit{L–functions}

For the applications we have in mind we need the spin $L$–function of cuspidal
automorphic representations of $\text{GSp}(2, \mathbb{A})$ as a global tool. There are several
results on this $L$–function, see [No], [PS] or [An]. Unfortunately none of these
results fully serves our needs. What we need is the following.

4.1 \textit{L–Function Theory for GSp}(2).

i) To every cuspidal automorphic representation $\pi$ of $\text{PGSp}(2, \mathbb{A})$ is asso-
ciated a global $L$–function $L(s, \pi)$ and a global $\epsilon$–factor $\epsilon(s, \pi)$, both
defined as Euler products, such that $L(s, \pi)$ has meromorphic continu-
ation to all of $\mathbb{C}$ and such that a functional equation

$$L(s, \pi) = \epsilon(s, \pi)L(1 - s, \pi)$$

of the standard kind holds.

ii) For Iwahori–spherical representations, the local factors $L_v(s, \pi_v)$ and
$\epsilon_v(s, \pi_v, \psi_v)$ coincide with the spin local factors defined via the local
Langlands correspondence as in [KL].

Of course such an $L$–function theory is predicted by general conjectures over
any number field. For our classical applications we shall only need it over
$\mathbb{Q}$. Furthermore, we can restrict to the archimedean component being a
lowest weight representation with scalar minimal $K$–type (a discrete series
representation if the weight is $\geq 3$). All we need to know about $\epsilon$–factors is
in fact that they are of the form $cp^m$ with a constant $c \in \mathbb{C}^*$ and an integer
$m$.

The local Langlands correspondence is not yet a theorem for GSp(2) (but see
[Pr], [Rob]), but for Iwahori–spherical representations it is known by [KL].
In fact, the local parameters (four-dimensional representations of the Weil–Deligne group) of all the representations in Table 1 can easily be written down explicitly. Hence we know all their local factors. There is one case of $L$–indistinguishability in Table 1, namely, the representations VIa and VIb constitute an $L$–packet. The representation Va also lies in a two-element $L$–packet. Its partner is a $\theta_{10}$–type supercuspidal representation.

4.2 Theorem. We assume that an $L$–function theory as in 4.1 exists. Let $\pi_{1} = \otimes \pi_{1,p}$ and $\pi_{2} = \otimes \pi_{2,p}$ be two cuspidal automorphic representations of PGSp$(2, \mathbb{A})$. Let $S$ be a finite set of prime numbers such that the following holds:

i) $\pi_{1,p} \simeq \pi_{2,p}$ for each $p \notin S$.

ii) For each $p \in S$, both $\pi_{1,p}$ and $\pi_{2,p}$ possess non-trivial Iwahori–invariant vectors.

Then, for each $p \in S$, the representations $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation (from an unramified character of the Borel subgroup).

Idea of proof: We divide the two functional equations for $L(s, \pi_{1})$ and $L(s, \pi_{2})$ and obtain finite Euler products by hypothesis i). Since we are over $\mathbb{Q}$, and since the expressions $p^{-s}$ for different $p$ can be treated as independent variables, it follows that we get equalities

$$
\frac{L_{p}(s, \pi_{1,p})}{L_{p}(s, \pi_{2,p})} = cp^{ms} \frac{L_{p}(1 - s, \pi_{1,p})}{L_{p}(1 - s, \pi_{2,p})}, 
$$

$c \in \mathbb{C}^{*}$, $m \in \mathbb{Z}$,

for each $p \in S$. But we have the complete list of all possible local Euler factors. One can check that such a relation is only possible if $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation.

Remark: In Table 1, for two representations to be constituents of the same induced representation means that they are in the same group I–VI.

With some additional information on the representation this result sometimes allows to attach a unique equivalence class of automorphic representations to a classical cuspform $f$. For example, if $N$ is square-free and $f \in S_{k}(\Gamma_{0}(N))^{\text{new}}$ is an eigenform for almost all the unramified Hecke algebras and also an eigenvector for the Atkin–Lehner involutions for all $p|N$, then Theorem 4.2 together with the information in Table 1 show that the associated adelic function $\Phi_{f}$ generates a multiple of an automorphic representation $\pi_{f}$ of PGSp$(2, \mathbb{A})$. 

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5 The main result

Let $N$ be a square-free positive integer. In the degree 1 case, given an eigenform $f \in S_k(\Gamma_0(N))^{\text{new}}$, knowing the Atkin–Lehner eigenvalues for $p|N$ was enough to identify the local representations and attach the correct local factors. In the degree 2 case, since there are more possibilities for the local representations, and since some of them have parameters, we need more information than just the Atkin–Lehner eigenvalues. For example, the representations IIa or IIIa, both of which have local newforms with respect to $P_1$, depend on characters $\chi$ and $\sigma$. Hence there are additional Satake parameters which enter into the $L$–factor. What we need are suitable Hecke operators on $S_k(\Gamma_0(N))^{\text{new}}$ to extract this information from the modular form $f$. It turns out that the previously defined operator $T_2(p)$ works well, but we need even more information. We are now going to define an additional endomorphism $T_4(p)$ of $S_k(\Gamma_0(N))^{\text{new}}$.

For notational simplicity assume $N = p$ is a prime and consider the following linear maps:

$$S_k(\Gamma_0(p))^{\text{new}} \overset{d_{02}}{\underset{d_{1}}{\longrightarrow}} S_k(\Gamma^{\text{para}}(p))^{\text{new}}$$  

Here $d_1$ and $d_{02}$ are trace operators which always exist between spaces of modular forms for commensurable groups. Explicitly,

$$d_{02} f = \frac{1}{(\Gamma^{\text{para}}(p) : \Gamma_0(p) \cap \Gamma^{\text{para}}(p))} \sum_{\gamma \in (\Gamma_0(p) \cap \Gamma^{\text{para}}(p)) \backslash \Gamma^{\text{para}}(p)} f|_k \gamma.$$  

It is obvious from Table 1 that these operators indeed map newforms to newforms. The additional endomorphism of $S_k(\Gamma_0(p))^{\text{new}}$ we require is

$$T_4(p) := (1 + p)^2 d_1 \circ d_{02}.$$  

Similarly we can define endomorphisms $T_4(p)$ of $S_k(\Gamma_0(N))^{\text{new}}$ for each $p|N$. Looking at local representations, the following is almost trivial.

5.1 Proposition. Let $N$ be square-free. The space $S_k(\Gamma_0(N))^{\text{new}}$ has a basis consisting of common eigenfunctions for the operators $T_2(p)$ and $T_4(p)$, all $p|N$, and for the unramified Hecke algebras at all good places $p \nmid N$.

We can now state our main result.
5.2 Theorem. We assume that an $L$–function theory as in 4.1 exists. Let $N$ be a square-free positive integer, and let $f \in S_k(\Gamma_0(N))^\text{new}$ be a newform in the sense of Definition 2.1. We assume that $f$ is an eigenform for the unramified local Hecke algebras $\mathcal{H}_p$ for almost all primes $p$. We further assume that $f$ is an eigenfunction for $T_2(p)$ and $T_4(p)$ for all $p | N$.

$$T_2(p)f = \lambda_p f, \quad T_4(p)f = \mu_p f$$
for $p | N$. (12)

Then:

i) $f$ is an eigenfunction for the local Hecke algebras $\mathcal{H}_p$ for all primes $p \nmid N$.

ii) Only the combinations of $\lambda_p$ and $\mu_p$ as given in the following table can occur. Here $\epsilon$ is $\pm 1$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>rep.</th>
<th>$L_p(s,f)^{-1}$</th>
<th>$\epsilon_p(s,f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\epsilon p \not\in {0,2p}$</td>
<td>0</td>
<td>IIa</td>
<td>$(1+\epsilon(p+1)(p-\mu)p^{-3/2-s}+p^{-2s})(1+\epsilon p^{-1/2-s})$</td>
<td>$\epsilon p^{1/2-s}$</td>
</tr>
<tr>
<td>$\not= \pm p$</td>
<td>0</td>
<td>IIIa</td>
<td>$(1-\lambda p^{-3/2-s})(1-\lambda^{-1} p^{1/2-s})$</td>
<td>$p^{1-2s}$</td>
</tr>
<tr>
<td>$-\epsilon p$</td>
<td>2p</td>
<td>Vb,c</td>
<td>$(1-\epsilon p^{1/2-s})(1-p^{-1/2-s})(1+p^{-1/2-s})$</td>
<td>$\epsilon p^{1/2-s}$</td>
</tr>
<tr>
<td>$-\epsilon p$</td>
<td>0</td>
<td>Vla,b</td>
<td>$(1+\epsilon p^{-1/2-s})^2$</td>
<td>$p^{1-2s}$</td>
</tr>
</tbody>
</table>

(We omit some indices $p$.)

iii) We define archimedean local factors according to our $L$–function theory and unramified spin Euler factors for $p | N$ as usual. For places $p | N$ we define $L$– and $\epsilon$–factors according to the table in ii). Then the resulting $L$–function has meromorphic continuation to the whole complex plane and satisfies the functional equation

$$L(s, f) = \epsilon(s, f) L(1 - s, f),$$

where $L(s, f) = \prod_{p \leq \infty} L_p(s, f)$ and $\epsilon(s, f) = \prod_{p | N \infty} \epsilon_p(s, f)$. (13)

Sketch of proof: Statement i) follows from Theorem 4.2. Statement ii) follows by explicitly computing the possible eigenvalues of $T_2(p)$ and $T_4(p)$ in local representations. In the present case we cannot conclude that in the global representation $\pi_f = \bigoplus \pi_i$ all the irreducible components $\pi_i$ must be isomorphic, because the eigenvalues in (12) cannot tell apart local representations Vla and Vlb. This is however the only ambiguity, so that we can at least associate a global $L$–packet with $f$. (As mentioned before, Vla
and VIb constitute a local $L$–packet.) The table in ii) indicates the possible representations depending on the Hecke eigenvalues.

The $L$–factors given in the table are those coming from the local Langlands correspondence. By hypothesis they coincide with the factors in our $L$–function theory. Hence the $L$–function in (13) coincides with the $L$–function of any one of the automorphic representations in our global $L$–packet. By our $L$–function theory we get the functional equation.

5.3 Corollary. If a cusp form $f \in S_k(\text{Sp}(2, \mathbb{Z}))$ is an eigenfunction for the unramified Hecke algebras $\mathcal{H}_p$ for almost all primes $p$, then it is an eigenfunction for those Hecke algebras for all $p$.

Remarks:

i) The corollary does not claim that $f$ generates an irreducible automorphic representation of $\text{PGSp}(2, \mathbb{A})$, but a multiple of such a representation. Without knowing multiplicity one for $\text{PGSp}(2)$ we cannot conclude that $f$ is determined by all its Hecke eigenvalues.

ii) The local factors given in Theorem 5.2 are the Langlands $L$– and $\varepsilon$–factors for the spin (degree 4) $L$–function. The following table lists the Langlands factors for the standard (degree 5) $L$–function.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>rep.</th>
<th>$L_p(s, f, st)^{-1}$</th>
<th>$\varepsilon_p(s, f, st)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-sp$</td>
<td>$\notin {0, 2p}$</td>
<td>IIa</td>
<td>$(1-(p+1)(p-\mu)p^{-2-s}+p^{-1-2s})(1-p^{-s})$</td>
<td>$p^{1-2s}$</td>
</tr>
<tr>
<td>$\neq p_0$</td>
<td>0</td>
<td>IIIa</td>
<td>$(1-\lambda^2 p^{-2s})(1-\lambda 2 p^2)(1-p^{-1-s})$</td>
<td>$p^{1-2s}$</td>
</tr>
<tr>
<td>$-sp$</td>
<td>$2p$</td>
<td>Vb,c</td>
<td>$(1+p^{-1-s})(1+p^{-s})(1-p^{-s})$</td>
<td>$p^{1-2s}$</td>
</tr>
<tr>
<td>$-sp$</td>
<td>0</td>
<td>VIa,b</td>
<td>$(1-p^{-s})^2(1-p^{-1-s})$</td>
<td>$p^{1-2s}$</td>
</tr>
</tbody>
</table>

iii) There is a statement analogous to Theorem 5.2 for modular forms with respect to the paramodular group $\Gamma_{\text{para}}(N)$. Instead of $T_q(p)$ as defined in (11) this result makes use of the “dual” endomorphism $T_p(p) := (1 + p)^2 d_{02} o d_1$ of $S_k(\Gamma_{\text{para}}(N))^{\text{new}}$.

References

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Ralf Schmidt  rschmidt@math.uni-sb.de
Universität des Saarlandes
Fachrichtung 6.1 Mathematik
Postfach 15 11 50
66041 Saarbrücken
Germany