RESTRICTION OF HERMITIAN MAASS LIFTS AND
THE GROSS-PRASAD CONJECTURE
(JOINT WITH T. IKEDA)

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This note is a report on a joint work with Tamotsu Ikeda [12].
After the discovery of the integral representation of triple product $L$-functions by Garrett [5], Harris and Kudla [10] determined the transcendental parts of the central critical values of triple product $L$-functions. The transcendental parts behaves differently according to whether the weights are "balanced" or not. In the "balanced" case, the critical values of triple product $L$-functions have also been studied by Garrett [5], Orloff [18], Satoh [20], Garrett and Harris [6], Gross and Kudla [7], Böcherer and Schulze-Pillot [4], and so on. By contrast, in the "imbalanced" case, there are no results on the critical values of triple product $L$-functions except [10] to our knowledge. We express certain period integrals of Maass lifts which appear in the Gross-Prasad conjecture [8], [9], as the algebraic parts of the central critical values in the "imbalanced" case.

1. THE GROSS-PRASAD CONJECTURE

In [8], [9], Gross and Prasad suggested that the central values of certain $L$-functions control a global obstruction of blanching rules for automorphic representations of special orthogonal groups. Let $V$ be a non-degenerate quadratic space of dimension $n$ over a number field $k$ and $H = \text{SO}(V)$ the special orthogonal group of $V$. Take a non-degenerate quadratic subspace $V'$ of $V$ of dimension $n-1$ and regard $H' = \text{SO}(V')$ as a subgroup of $H$. Let $\tau \simeq \bigotimes_v \tau_v$ (resp. $\tau' \simeq \bigotimes_v \tau'_v$) be an irreducible cuspidal automorphic representation of $H(A_k)$ (resp. $H'(A_k)$).

**Conjecture 1.1** (Gross-Prasad). Assume that $\tau$ and $\tau'$ are both tempered. Then the period integral

$$\langle G|_{H'}, F \rangle = \int_{H'(k) \backslash H'(A_k)} G(h)\overline{F(h)} \, dh$$

does not vanish for some $G \in \tau$ and some $F \in \tau'$ if and only if

(i) $\text{Hom}_{H'(k)}(\tau_v, \tau'_v) \neq 0$ for all places $v$ of $k$.

(ii) $L(1/2, \tau \times \tau') \neq 0$.  

Remark that a meromorphic continuation of the $L$-function $L(s, \tau \times \tau')$ has not been established in general, however, it could be described in terms of $L$-functions of general linear groups by the functoriality. We also note that the conjecture is supported by the results of Waldspurger [22] for $n = 3$, Harris and Kudla [10], [11] for $n = 4$, Böcherer, Furusawa, and Schulze-Pillot [3] for $n = 5$.

Gross and Prasad restricted their conjecture to the tempered cases. According to the Arthur conjecture [2], non-tempered cuspidal automorphic representations exist, and if $\tau$ or $\tau'$ is non-tempered, then the $L$-function $L(s, \tau \times \tau')$ could have a pole at $s = 1/2$. Hence a modification to the condition (ii) would be inevitable if one consider the Gross-Prasad conjecture in general (see [3] for $n = 5$). Our result provides an example for $n = 6$ when $\tau, \tau'$ are both non-tempered. Remark that the triple product $L$-function considered in this note is only of degree 8 and is a part of the $L$-function $L(s, \tau \times \tau')$ of degree 24.

2. SAITO-KUROKAWA LIFTS

First, we review the notion of Saito-Kurokawa lifts [16], [17], [1], [23]. Let $k$ be a positive even integer. Let

$$F(Z) = \sum_{B > 0} A(B)e^{2\pi \sqrt{-1} \text{tr}(BZ)} \in S_k(\text{Sp}_2(\mathbb{Z})), \quad Z \in \mathfrak{h}_2$$

be a Siegel modular form of degree 2. Here $\mathfrak{h}_2$ is the Siegel upper half plane given by

$$\mathfrak{h}_2 = \{Z = {}^tZ \in M_2(\mathbb{C}) \mid \text{Im}(Z) > 0\}.$$

We say that $F$ satisfies the Maass relation if there exists a function $\beta_F^* : \mathbb{N} \to \mathbb{C}$ such that

$$A \left( \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \right) = \sum_{d|(n,r,m)} d^{k-1} \beta_F^* \left( \frac{4nm - r^2}{d^2} \right).$$

We denote by $S_k^\text{Maass}(\text{Sp}_2(\mathbb{Z}))$ the space of Siegel cusp forms which satisfy the Maass relation.

Kohnen [13] introduced the plus subspace $S_{k-1/2}^+(\Gamma_0(4))$ given by

$$S_{k-1/2}^+(\Gamma_0(4)) = \{h(\tau) = \sum_{N > 0} c(N)q^N \in S_{k-1/2}(\Gamma_0(4)) \mid c(N) = 0 \text{ if } -N \not\equiv 0, 1 \mod 4\}.$$
For $F \in S_k^\text{Maass}(\text{Sp}_2(\mathbb{Z}))$, put

$$\Omega_{SK}^k(F)(\tau) = \sum_{N \geq 0} \beta_F^*(N) q^N.$$

Then $\Omega_{SK}^k(F) \in S_{k-1/2}^+(\Gamma_0(4))$, and the linear map

$$\Omega_{SK} : S_k^\text{Maass}(\text{Sp}_2(\mathbb{Z})) \rightarrow S_{k-1/2}^+(\Gamma_0(4))$$

is an isomorphism.

### 3. Hermitian Maass Lifts

Next, we recall an analogue of Saito-Kurokawa lifts for hermitian modular forms by Kojima [14], Sugano [21], and Krieg [15]. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D < 0$, $\mathcal{O}$ the ring of integers of $K$, $w_K$ the number of roots of unity contained in $K$, and $\chi$ be the primitive Dirichlet character corresponding to $K/\mathbb{Q}$.

Write

$$\chi = \prod_{q \in \mathcal{Q}_D} \chi_q,$$

where $\mathcal{Q}_D$ is the set of all primes dividing $D$ and $\chi_q$ is a primitive Dirichlet character mod $q^{ord_q D}$ for each $q \in \mathcal{Q}_D$.

Let $k$ be a positive integer such that $w_K | k$. Let

$$G(Z) = \sum_{H \in \Lambda_2(\mathcal{O})^+} A(H) e^{2\pi \sqrt{-1} \text{tr}(HZ)} \in S_k(U(2, 2)), \quad Z \in \mathcal{H}_2$$

be a hermitian modular form of degree 2. Here $\mathcal{H}_2$ is the hermitian upper half plane given by

$$\mathcal{H}_2 = \left\{ Z \in M_2(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - {}^t \overline{Z}) > 0 \right\},$$

and

$$\Lambda_2(\mathcal{O})^+ = \left\{ H = {}^t \overline{H} \in \frac{1}{\sqrt{-D}} M_2(\mathcal{O}) \mid \text{diag}(H) \in \mathbb{Z}^2, \ H > 0 \right\}.$$

We say that $G$ satisfies the Maass relation if there exists a function $\alpha_G^* : \mathbb{N} \rightarrow \mathbb{C}$ such that

$$A(H) = \sum_{d | \varepsilon(H)} d^{k-1} \alpha_G^* \left( \frac{D \det(H)}{d^2} \right),$$

where

$$\varepsilon(H) = \max\{n \in \mathbb{N} \mid n^{-1} H \in \Lambda_2(\mathcal{O})^+\}.$$
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We denote by $S^\text{Maass}_k(U(2,2))$ the space of hermitian cusp forms which satisfy the Maass relation.

Krieg [15] introduced the space $S^*_k(\Gamma_0(D), \chi)$ which is an analogue of the Kohnen plus subspace and is given by

$$S^*_k(\Gamma_0(D), \chi) = \{g^*(\tau) = \sum_{N>0} a_{g^*}(N) q^N \in S_{k-1}(\Gamma_0(D), \chi) \mid a_{g^*}(N) = 0 \text{ if } a_D(N) = 0\},$$

where

$$a_D(N) = \prod_{q \in Q_D} (1 + \chi_q(-N)).$$

Let

$$g(\tau) = \sum_{N>0} a_g(N) q^N \in S_{k-1}(\Gamma_0(D), \chi)$$

be a primitive form. For each $Q \subset Q_D$, set

$$\chi_Q = \prod_{q \in Q} \chi_q, \quad \chi'_Q = \prod_{q \in Q_D-Q} \chi_q.$$

Then there exists a primitive form

$$g_Q(\tau) = \sum_{N \geq 0} a_{g_Q}(N) q^N \in S_{k-1}(\Gamma_0(D), \chi)$$

such that

$$a_{g_Q}(p) = \begin{cases} \chi_Q(p) a_g(p) & \text{if } p \notin Q, \\ \chi'_Q(p) a_g(p) & \text{if } p \in Q, \end{cases}$$

for each prime $p$. Put

$$g^* = \sum_{Q \subset Q_D} \chi_Q(-1) g_Q.$$

Then $g^* \in S^*_k(\Gamma_0(D), \chi)$. When $g$ runs over primitive forms in $S_{k-1}(\Gamma_0(D), \chi)$, the forms $g^*$ span $S^*_k(\Gamma_0(D), \chi)$.

For $G \in S^\text{Maass}_k(U(2,2))$, put

$$\Omega(G)(\tau) = \sum_{N>0} a_D(N) a^*_G(N) q^N.$$

Then $\Omega(G) \in S^*_k(\Gamma_0(D), \chi)$, and the linear map

$$\Omega : S^\text{Maass}_k(U(2,2)) \to S^*_k(\Gamma_0(D), \chi)$$

is an isomorphism.
Let $k$ be a positive integer such that $w_K | k$. Let $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ be a primitive form and $h(\tau) = \sum_{N>0} c(N)q^N \in S_{k-1/2}^+(\Gamma_0(4))$ a Hecke eigenform which corresponds to $f$ by the Shimura correspondence. Note that $h$ is unique up to scalars. Let $F = \left(\Omega^{8k}\right)^{-1}(h) \in S_k^\text{Maass}(\text{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of $f$. Define the Petersson norms of $f$ and $F$ by

$$\langle f, f \rangle = \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}_1} |f(\tau)|^2 y^{2k-4} d\tau,$$

$$\langle F, F \rangle = \int_{\text{Sp}_2(\mathbb{Z}) \backslash \mathfrak{h}_2} |F(Z)|^2 \det \text{Im}(Z)^{k-3} dZ,$$

respectively.

Let $g(\tau) = \sum_{N>0} a_g(N)q^N \in S_{k-1}(\Gamma_0(\mathbb{D}), \chi)$ be a primitive form and $G = \Omega^{-1}(g^*) \in S_k^\text{Maass}(\text{U}(2, 2))$ the hermitian Maass lift of $g$, where $g^* \in S_k^*(\Gamma(\mathbb{D}), \chi)$ is given by (3.1). Observe that $\mathfrak{h}_2 \subset \mathfrak{h}_2$, and by [15], the restriction $G|_{\mathfrak{h}_2}$ belongs to $S_k^\text{Maass}(\text{Sp}_2(\mathbb{Z}))$.

The completed triple product $L$-function $\Lambda(s, g \times g \times f)$ is given by

$$\Lambda(s, g \times g \times f) = (2\pi)^{-4s+4k-8}\Gamma(s)\Gamma(s-2k+4)\Gamma(s-k+2)^2 L(s, g \times g \times f),$$

and satisfies a functional equation which replaces $s$ with $4k - 6 - s$.

Our main result is as follows.

**Theorem 4.1.**

$$\frac{\Lambda(2k-3, g \times g \times f)}{\langle f, f \rangle^2} = -2^{4k-6}D^{-2k+3}c(D)^2 \frac{\langle G|_{\mathfrak{h}_2}, F \rangle^2}{\langle F, F \rangle^2}$$

**5. Proof**

Theorem 4.1 follows from the following seesaws.

(5.1) \[ \begin{array}{ccc}
O(4, 2) & \times & \widetilde{\text{SL}}_2 \times \widetilde{\text{SL}}_2 \\
\times & & \times \\
O(3, 2) \times O(1) & \times & O(2, 1) \times O(1)
\end{array} \]

(5.2) \[ \begin{array}{ccc}
\text{Sp}_6 & \times & O(2, 2)^3 \\
\times & & \times \\
\text{SL}_2^3 & \times & O(2, 2)
\end{array} \]
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To explain these seesaws more precisely, we introduce some notation. In [13], Kohnen defined a linear map

$$S_{-D}^{+} : S_{k-1/2}^{+}(\Gamma_0(4)) \to S_{2k-2}(\text{SL}_2(\mathbb{Z})), $$

$$\sum_{N>0} c(N)q^{N} \mapsto \sum_{N>0} \sum_{d|N} \chi(d)d^{k-2}c\left(\frac{N^2}{d^2}D\right)q^{N}. $$

If $h(\tau) = \sum_{N>0} c(N)q^{N} \in S_{k-1/2}^{+}(\Gamma_0(4))$ is a Hecke eigenform and corresponds to $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ by the Shimura correspondence, then

$$S_{-D}^{+}(h) = c(D)f. $$

Let $\text{Tr}^{P}_1$ denote the trace operator given by

$$\text{Tr}^{P}_1 : S_{2k-2}(\Gamma_0(D)) \to S_{2k-2}(\text{SL}_2(\mathbb{Z})), $$

$$f \mapsto \sum_{\gamma \in \Gamma_0(D) \backslash \text{SL}_2(\mathbb{Z})} f|_{\gamma}. $$

The seesaw (5.1) accounts for the following identity.

**Proposition 5.1.**

$$S_{-D}^{+}(\Omega^{SK}(G|_{b_2})) = a_g(D)^2 \text{Tr}^{P}_1(g^2). $$

This identity is proved by computing the Fourier coefficients of the both sides explicitly.

The seesaw (5.2) accounts for the following refinement of the main identity by Harris and Kudla [10].

**Proposition 5.2.**

$$\Lambda(2k-3, g \times g \times f) = -2^{4k-6}D^{-2k+3}a_g(D)^4\langle \text{Tr}^{P}_1(g^2), f \rangle^2 $$

This identity is proved by computing the local zeta integrals which arise in the integral representation of triple product $L$-functions by Garrett [5], Piatetski-Shapiro and Rallis [19] at bad primes.

Now Theorem 4.1 follows from Propositions 5.1 and 5.2.

**REFERENCES**


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