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Three classes of nonextensive entropies characterized by Shannon additivity and pseudoadditivity

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Abstract

Nonextensive entropies are divided into three classes, each of which is characterized by Shannon additivity and pseudoadditivity. One of the three classes has properties of both additivities. The remaining classes have only one property of the two additivities, respectively. An example of nonextensive entropy is shown concretely for each class. In particular, one class is found to consist of only Tsallis entropy. More precisely, Tsallis entropy is proved to be uniquely determined by only these two additivities. The present classification using these two distinct additivities reveals unique characteristics of Tsallis entropy.

1 Introduction

Shannon additivity and pseudoadditivity are characteristic properties of Tsallis entropy [1]. Both of these additivities are nonextensive generalizations of the standard additivity in extensive systems. Although these properties appear to be similar in the sense that both additivities are given by the formulations of the entropy for the composite systems, A and B, they are actually different. When two systems, A and B, are mutually independent, in extensive systems, the Shannon additivity coincides with the pseudoadditivity, but in nonextensive systems, this coincidence generally does not hold.

In the present paper, these two additivities are applied to characterization of three classes of nonextensive entropies. One of the three classes is characterized by satisfying both of these additivities, and the remaining two classes are characterized by satisfying only one or the other of the two additivities. An example of nonextensive entropy is shown concretely for each class. In addition, Tsallis entropy is uniquely determined by only these two additivities. In the present paper, both the original Tsallis entropy and the normalized Tsallis entropy are discussed.

2 Shannon additivity and pseudoadditivity of the original Tsallis Entropy

The *original Tsallis entropy* S_q [2] is given by

$$S_q(p_1,\cdots,p_n) := \frac{1-\sum_{i=1}^{n} p_i^q}{q-1} \tag{2.1}$$

where $q \in R^+$. This nonextensive entropy is one-parameter generalization of Shannon entropy in the sense that

$$\lim_{q \to 1} S_q = S_1 := -\sum_{i=1}^{n} p_i \ln p_i. \tag{2.2}$$

The Shannon additivity [3] and the pseudoadditivity [1] of the original Tsallis entropy are given by followings:

(1) Shannon additivity:

$$\forall i = 1, \cdots, n, \forall j = 1, \cdots, m_i : p_{ij} \geq 0, \quad p_i = \sum_{j=1}^{m_i} p_{ij}, \quad \sum_{i=1}^{n} p_i = 1, \tag{2.3}$$

$$S_q\left(p_{11}, \cdots, p_{nm_n}\right) = S_q\left(p_1, \cdots, p_n\right) + \sum_{i=1}^{n} p_i^q S_q\left(\frac{p_{i1}}{p_i}, \cdots, \frac{p_{im_i}}{p_i}\right). \tag{2.4}$$

(2) pseudoadditivity:

$$S_q(A, B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B) \tag{2.5}$$

where A and B are mutually independent finite event systems:

$$A = \begin{pmatrix} A_1 & \cdots & A_n \\ p_1^A & \cdots & p_n^A \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & \cdots & B_m \\ p_1^B & \cdots & p_m^B \end{pmatrix}. \tag{2.6}$$

These two additivities, Eqs.(2.4) and (2.5), can be proven in a straightforward manner using the definition given in Eq.(2.1). If the condition of independence given by Eq.(2.6):

$$\forall i = 1, \cdots, n, \ \forall j = 1, \cdots, m \ : \ p_{ij} = p_i \cdot p_j, \tag{2.7}$$

is applied to the Shannon additivity given by Eq.(2.4), then the equation is simplified to

$$S_q(p_{11}, \cdots, p_{nm}) = S_q(p_1, \cdots, p_n) + \left(\sum_{i=1}^{n} p_i^q\right) S_q(p_1, \cdots, p_m). \tag{2.8}$$

Here, we take $m := m_i$ for all $i = 1, \cdots, n$. In terms of the following notations:

$$\forall i = 1, \cdots, n : p_i := p_i^A, \ \forall j = 1, \cdots, m : p_j := p_j^B, \tag{2.9}$$

$$S_q(A) = S_q(p_1, \cdots, p_n), \ S_q(B) = S_q(p_1, \cdots, p_m) \tag{2.10}$$

the obtained Shannon additivity, Eq.(2.8), is written as

$$S_q(A, B) = S_q(A) + \left(\sum_{i=1}^{n} p_i^q\right) S_q(B). \tag{2.11}$$

Note that in the case of $q = 1$ (extensive systems), the Shannon additivity finally reaches

$$S_1(A, B) = S_1(A) + S_1(B), \tag{2.12}$$

which coincides with the pseudoadditivity of Eq.(2.5). In this way, when two systems, A and B, are mutually independent in the extensive systems, the Shannon additivity coincides with the pseudoadditivity.

However, under the above condition of independence, Eq.(2.6), in nonextensive systems, these two additivities differ from each other in the sense that one additivity cannot be derived from the other. Comparing Eqs.(2.5) and (2.11), these additivities do not coincide with each other.

Therefore, the nonextensive entropies are divided into the following three classes:

Class 1: Nonextensive entropy $S_q^{(1)}$ satisfying both additivities

Class 2: Nonextensive entropy $S_q^{(2)}$ satisfying only the Shannon additivity

Class 3: Nonextensive entropy $S_q^{(3)}$ satisfying only the pseudoadditivity

Here, we need a condition that all nonextensive entropies should satisfy Eq.(2.2).

Examples of each class of nonextensive entropy will be shown concretely in order to reveal the difference between the two additivities.

First, we uniquely derive the nonextensive entropy $S_q^{(1)}$ of Class 1 from only these two additivities, and the obtained entropy is found to coincide with the original Tsallis entropy.

If two systems, A and B, are mutually independent in the nonextensive systems, then the two additivities are given by Eqs.(2.5) and (2.11), respectively. Eliminating $S_q(A,B)$ from both Eqs.(2.5) and (2.11) yields

$$S_q^{(1)}(A) + S_q^{(1)}(B) + (1-q)\, S_q^{(1)}(A)\, S_q(B) = S_q^{(1)}(A) + \left(\sum_{i=1}^{n} p_i^q\right) S_q^{(1)}(B). \tag{2.13}$$

Therefore, the original Tsallis entropy, Eq.(2.1), is directly derived from Eq.(2.13), as follows:

$$S_q^{(1)}(A) = \frac{1 - \sum_{i=1}^{n} p_i^q}{q-1} \tag{2.14}$$

Note that in this derivation only the Shannon additivity and pseudoadditivity are used. In other words, these two additivities uniquely determine the original Tsallis entropy.

We must note here that our derivation is not consistent with the result reported in [4]. As reported in [4], the four conditions: (i) continuity, (ii) increasing monotonicity, (iii) pseudoadditivity, Eq.(2.5), and (iv) Shannon additivity, Eq.(2.4), are given as axioms of the original Tsallis entropy and the uniqueness theorem thereof is proven. Note that the Shannon additivity in [4] is a special case of Eq.(2.4). The generalization of the Shannon additivity in [4] to Eq.(2.4) is straightforward [1, 5]. In the present derivation, we uniquely determine Tsallis entropy using only two axioms, (iii) and (iv) described above. Therefore, the above four conditions in [4] are ***redundant*** axioms of Tsallis entropy. Thus, these two additivities become self-consistent axioms of the original Tsallis entropy, but complete parallelism does not exist between the Shannon-Khinchin axioms and the two additivities [6].

Next, we consider the following example of the nonextensive entropy $S_q^{(2)}$ of Class 2,

$$S_q^{(2)}(p_1, \cdots, p_n) := \frac{1 - \sum_{i=1}^{n} p_i^q}{\varphi(q)} \tag{2.15}$$

where $\varphi(q)$ is a differentiable function with respect to any $q \in R^+$, satisfying

$$\lim_{q\to1} \frac{d\varphi(q)}{dq} = 1, \quad \lim_{q\to1} \varphi(q) = \varphi(1) = 0, \quad \varphi(q) \neq 0 \ (q \neq 1), \quad \text{and} \quad \varphi(q) \neq q-1. \tag{2.16}$$

The difference in $S_q^{(2)}$ from that of the original Tsallis entropy, Eq.(2.1), is only the denominator $\varphi(q)$. Hence, the last condition of Eq.(2.16) is required in order to belong to Class 2. The following equation for $\varphi(q)$ is an example satisfying all conditions of Eq.(2.16),

$$\varphi(q) = \frac{(q-1)\left(q^2+1\right)}{2}. \tag{2.17}$$

We must confirm that the nonextensive entropy $S_q^{(2)}$ belongs to Class 2. Using l'Hospital's rule, we have

$$\lim_{q\to 1} S_q^{(2)} = \lim_{q\to 1} \frac{1-\sum_{i=1}^{n} p_i^q}{\varphi(q)} = \lim_{q\to 1} \frac{-\sum_{i=1}^{n} p_i^q \ln p_i}{\frac{d\varphi(q)}{dq}} = -\sum_{i=1}^{n} p_i \ln p_i = S_1. \tag{2.18}$$

Thus, $S_q^{(2)}$ satisfies condition of Eq.(2.2). $S_q^{(2)}$ is found to have the property of the Shannon additivity, Eq.(2.4), as follows:

$$\begin{aligned} S_q^{(2)}(p_1,\cdots,p_n) + \sum_{i=1}^{n} p_i^q S_q^{(2)}\left(\frac{p_{i1}}{p_i},\cdots,\frac{p_{im_i}}{p_i}\right) &= \frac{1-\sum_{i=1}^{n} p_i^q}{\varphi(q)} + \sum_{i=1}^{n}\left(p_i^q \cdot \frac{1-\sum_{j=1}^{m_i}\left(\frac{p_{ij}}{p_i}\right)^q}{\varphi(q)}\right) \\ &= S_q^{(2)}(p_{11},\cdots,p_{nm_n}). \end{aligned}$$

However, $S_q^{(2)}$ does *not* satisfy the pseudoadditivity of Eq.(2.5).

$$S_q^{(2)}(A,B) \neq S_q^{(2)}(A) + S_q^{(2)}(B) + (1-q)\, S_q^{(2)}(A)\, S_q^{(2)}(B) \tag{2.19}$$

Thus, $S_q^{(2)}$ belongs to Class 2.

Finally, we consider the nonextensive entropies of Class 3 in the following example:

$$S_q^{(3)}(p_1,\cdots,p_n) := \frac{\sum_{i=1}^{n} p_i^{q^{-1}}\left(p_i^{q-1}-1\right)}{(1-q)\sum_{i=1}^{n} p_i^{q^{-1}}} = \frac{\sum_{i=1}^{n} p_i^{q+q^{-1}-1} - \sum_{i=1}^{n} p_i^{q^{-1}}}{(1-q)\sum_{i=1}^{n} p_i^{q^{-1}}} \tag{2.20}$$

Again, using l'Hospital's rule, we have

$$\begin{aligned} \lim_{q\to 1} S_q^{(3)} &= \lim_{q\to 1} \frac{\sum_{i=1}^{n} p_i^{q+q^{-1}-1} - \sum_{i=1}^{n} p_i^{q^{-1}}}{(1-q)\sum_{i=1}^{n} p_i^{q^{-1}}} = \lim_{q\to 1} \frac{\left(1-\frac{1}{q^2}\right)\sum_{i=1}^{n} p_i^q \ln p_i + \frac{1}{q^2}\sum_{i=1}^{n} p_i^{q^{-1}} \ln p_i}{-\sum_{i=1}^{n} p_i^{q^{-1}} - (1-q)\frac{1}{q^2}\sum_{i=1}^{n} p_i^{q^{-1}} \ln p_i} \\ &= \frac{\sum_{i=1}^{n} p_i \ln p_i}{-1} = -\sum_{i=1}^{n} p_i \ln p_i = S_1. \end{aligned} \tag{2.21}$$

Thus, $S_q^{(3)}$ satisfies condition of Eq.(2.2). $S_q^{(3)}$ can be easily found to satisfy the pseudoadditivity of Eq.(2.5).

$$S_q^{(3)}(A,B) = S_q^{(3)}(A) + S_q^{(3)}(B) + (1-q)\, S_q^{(3)}(A)\, S_q^{(3)}(B) \tag{2.22}$$

However, the Shannon additivity of Eq.(2.4) does not hold for $S_q^{(3)}$. Accordingly, $S_q^{(3)}$ belongs to Class 3.

3 Shannon additivity and pseudoadditivity of the normalized Tsallis Entropy

The *normalized Tsallis entropy* $\hat{S}_q$ was first introduced in [8] as one candidate of the generalized nonextensive entropies. The requirement for the normalized Tsallis entropy is presented based on the principle of the form invariance of Kullback-Leibler entropy [9]. or on that of the pseudoadditivity [?], respectively. The *normalized Tsallis entropy* $\hat{S}_q$ [8, 9, 10] is given by

$$\hat{S}_q\left(p_1,\cdots,p_n\right) := \frac{1-\sum\limits_{i=1}^{n} p_i^q}{(q-1)\sum\limits_{j=1}^{n} p_j^q} \tag{3.23}$$

where $q \in R^+$. This nonextensive entropy is also a one-parameter generalization of Shannon entropy in the sense that

$$\lim_{q\to 1} \hat{S}_q = S_1 = -\sum_{i=1}^{n} p_i \ln p_i. \tag{3.24}$$

The Shannon additivity and the pseudoadditivity [1] of the normalized Tsallis entropy are given by the followings:

(1) Shannon additivity:

$$\forall i=1,\cdots,n, \forall j=1,\cdots,m_i : p_{ij} \geq 0, \quad p_i = \sum_{j=1}^{m_i} p_{ij}, \quad \sum_{i=1}^{n} p_i = 1, \tag{3.25}$$

$$\begin{aligned} &\left(\sum_{i=1}^{n}\sum_{j=1}^{m_i} p_{ij}^q\right) \hat{S}_q(p_{11},\,\cdots,\,p_{1m_1},\,\cdots,\,p_{n1},\,\cdots,\,p_{nm_n}) \\ &\quad = \left(\sum_{i=1}^{n} p_i^q\right) \hat{S}_q(p_1,\,\cdots,\,p_n) + \sum_{i=1}^{n}\sum_{j=1}^{m_i} p_{ij}^q \hat{S}_q\left(\frac{p_{i1}}{p_i},\,\cdots,\,\frac{p_{im_i}}{p_i}\right) \end{aligned} \tag{3.26}$$

(2) pseudoadditivity:

$$\hat{S}_q\left(A,B\right) = \hat{S}_q\left(A\right) + \hat{S}_q\left(B\right) + (q-1)\,\hat{S}_q\left(A\right)\hat{S}_q\left(B\right) \tag{3.27}$$

where A and B are mutually independent finite event systems given by Eq.(2.6).

The two additivities given by Eqs.(3.26) and (3.27) are proven using the definition given in Eq.(3.23). Note that the difference between the two Shannon additivities given by Eqs.(2.4) and (3.26) is due to the normalization of Tsallis entropy. The two Tsallis entropies S_q and $\hat{S}_q$ defined by Eqs.(2.1) and (3.23) yield the following relation:

$$S_q(p_1,\,\cdots,\,p_n) = \left(\sum_{j=1}^{n} p_j^q\right) \hat{S}_q(p_1,\,\cdots,\,p_n)\,. \tag{3.28}$$

Thus, substituting Eq.(3.28) into Eq.(2.4) yields the Shannon additivity of Eq.(3.26) for the normalized Tsallis entropy $\hat{S}_q$. Another difference in pseudoadditivity is that the coefficient "$1-q$" in front of the cross term in the right-hand side of the pseudoadditivity of Eq.(2.5) is the inverse of "$q-1$" in Eq.(3.27). The reason for this difference is the normalization of the original Tsallis entropy, which is discussed in detail in our paper [10].

Similar to the case of the original Tsallis entropy, the condition of independence given by Eq.(2.6) is applied to the Shannon additivity of Eq.(3.26), which yields

$$\left(\sum_{j=1}^{m} p_j^q\right) \hat{S}_q(A,B) = \hat{S}_q(A) + \left(\sum_{j=1}^{m} p_j^q\right) \hat{S}_q(B)\,. \tag{3.29}$$

Here, we take $m := m_i$ for all $i = 1,\cdots,n$. Note that in the case of $q = 1$ (extensive systems) the Shannon additivity becomes the standard additivity of Eq.(2.12), which coincides with the pseudoadditivity of

Eq.(3.27). In this way, when two systems, A and B, are mutually independent in the extensive systems, the Shannon additivity coincides with the pseudoadditivity.

However, under the above condition of independence in the nonextensive systems, these two additivities differ from each other in the sense that one additivity cannot be derived from the other. Comparing Eqs.(3.27) and (3.29), these additivities do not coincide with each other.

Therefore, the normalized nonextensive entropies are divided into the following three classes:

Class 1: Nonextensive entropies $\hat{S}_q^{(1)}$ satisfying both additivities

Class 2: Nonextensive entropies $\hat{S}_q^{(2)}$ satisfying only the Shannon additivity

Class 3: Nonextensive entropies $\hat{S}_q^{(3)}$ satisfying only the pseudoadditivity

Here, we need a condition that all nonextensive entropies should satisfy Eq.(3.24).

Examples of each class of nonextensive entropy in each class will be shown concretely in order to reveal the difference between the two additivities.

First, we uniquely derive the nonextensive entropy $\hat{S}_q^{(1)}$ of Class 1 as in the previous section. If two systems, A and B, are mutually independent in the extensive systems, then the two additivities are given by Eqs.(3.27) and (3.29), respectively. Eliminating $S_q(A,B)$ from both Eqs.(3.27) and (3.29) yields

$$\left(\sum_{j=1}^{m} p_j^q\right)\left\{\hat{S}_q(A)+\hat{S}_q(B)+(q-1)\,\hat{S}_q(A)\,\hat{S}_q(B)\right\}=\hat{S}_q(A)+\left(\sum_{j=1}^{m} p_j^q\right)\hat{S}_q(B)\,. \qquad (3.30)$$

Therefore, the normalized Tsallis entropy of Eq.(3.23) is directly derived from Eq.(3.30) as follows:

$$\hat{S}_q^{(1)}(B)=\frac{1-\sum_{i=1}^{m} p_i^q}{(q-1)\sum_{j=1}^{m} p_j^q} \qquad (3.31)$$

Note that in this derivation only the Shannon additivity and pseudoadditivity are used. In other words, these two additivities uniquely determine the normalized Tsallis entropy, similar to the case of the original Tsallis entropy.

Next, we consider the following example of the nonextensive entropy $\hat{S}_q^{(2)}$ of Class 2,

$$\hat{S}_q^{(2)}(p_1,\cdots,p_n):=\frac{1-\sum_{i=1}^{n} p_i^q}{\varphi(q)\sum_{j=1}^{n} p_j^q} \qquad (3.32)$$

where $\varphi(q)$ is a differentiable function with respect to any $q\in R^+$, satisfying the conditions of Eq.(2.16). Here, we must verify that the nonextensive entropy $\hat{S}_q^{(2)}$ belongs to Class 2. Using l'Hospital's rule, we have

$$\lim_{q\to1}\hat{S}_q^{(2)}=\lim_{q\to1}\frac{1-\sum_{i=1}^{n} p_i^q}{\varphi(q)\sum_{j=1}^{n} p_j^q}=\lim_{q\to1}\frac{-\sum_{i=1}^{n} p_i^q\ln p_i}{\frac{d\varphi(q)}{dq}\sum_{j=1}^{n} p_j^q+\varphi(q)\sum_{j=1}^{n} p_j^q\ln p_j}=-\sum_{i=1}^{n} p_i\ln p_i=S_1\,. \qquad (3.33)$$

Thus, $\hat{S}_q^{(2)}$ satisfies the condition of Eq.(3.24). $\hat{S}_q^{(2)}$ is found to have the property of the Shannon additivity Eq.(3.26).

However, $\hat{S}_q^{(2)}$ does *not* satisfy the pseudoadditivity of Eq.(3.27)

$$\hat{S}_q^{(2)}(A,B)\neq\hat{S}_q^{(2)}(A)+\hat{S}_q^{(2)}(B)+(q-1)\,\hat{S}_q^{(2)}(A)\,\hat{S}_q^{(2)}(B)\,. \qquad (3.34)$$

Thus, $\hat{S}_q^{(2)}$ belongs to Class 2.

Finally, we consider the nonextensive entropies of Class 3 in the following example:

$$\hat{S}_q^{(3)} := \frac{\sum_{i=1}^{n} p_i^{\frac{q^2+1}{2}} \left(p_i^{1-q} - 1\right)}{(q-1)\sum_{i=1}^{n} p_i^{\frac{q^2+1}{2}}} = \frac{\sum_{i=1}^{n} p_i^{\frac{q^2-2q+3}{2}} - \sum_{i=1}^{n} p_i^{\frac{q^2+1}{2}}}{(q-1)\sum_{i=1}^{n} p_i^{\frac{q^2+1}{2}}} \tag{3.35}$$

Again, using l'Hospital rule, we have

$$\begin{aligned} \lim_{q\to 1} \hat{S}_q^{(3)} &= \lim_{q\to 1} \frac{\sum_{i=1}^{n} p_i^{\frac{q^2-2q+3}{2}} - \sum_{i=1}^{n} p_i^{\frac{q^2+1}{2}}}{(q-1)\sum_{j=1}^{n} p_j^{\frac{q^2+1}{2}}} = \lim_{q\to 1} \frac{(q-1)\sum_{i=1}^{n} p_i^{\frac{q^2-2q+3}{2}} \ln p_i - q\sum_{i=1}^{n} p_i^{\frac{q^2+1}{2}} \ln p_i}{\sum_{j=1}^{n} p_j^{\frac{q^2+1}{2}} + q(q-1)\sum_{j=1}^{n} p_j^{\frac{q^2+1}{2}} \ln p_j} \\ &= \frac{-\sum_{i=1}^{n} p_i \ln p_i}{1} = -\sum_{i=1}^{n} p_i \ln p_i = S_1 \end{aligned} \tag{3.36}$$

Thus, $\hat{S}_q^{(3)}$ satisfies the condition of Eq.(3.24). In addition, $\hat{S}_q^{(3)}$ is verified to have the property of pseudoadditivity of Eq.(3.27).

$$\hat{S}_q^{(3)}(A,B) = \hat{S}_q^{(3)}(A) + \hat{S}_q^{(3)}(B) + (q-1)\,\hat{S}_q^{(3)}(A)\,\hat{S}_q^{(3)}(B) \tag{3.37}$$

However, the Shannon additivity of Eq.(3.26) does not hold for $\hat{S}_q^{(3)}$. Accordingly, $\hat{S}_q^{(3)}$ belongs to Class 3.

4 Conclusion

We have shown that the nonextensive entropies can be divided into three classes which are characterized by the Shannon additivity and the pseudoadditivity. An example of each class is shown concretely and these two distinct additivities reveal the following peculiarities of Tsallis entropy.

First, Tsallis entropy is uniquely determined by only these two additivities. Thus, these two additivities constitute the axioms of Tsallis entropy. However, these axioms are *not* a naturally generalization of the Shannon-Khinchin axioms in extensive systems due to the lack of complete parallelism between them.

Secondly, there exist nonextensive entropies satisfying only one of the two additivities. Accordingly, for the construction of the general axioms for Tsallis entropy, unsatisfactory axioms consisting of these two additivities are observed. In other words, axioms that include both additivities are redundant.

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