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Regular non-additive measure and Choquet integral

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1 Introduction

The Choquet integral with respect to a non-additive measure proposed by Murofushi and Sugeno [6] is a basic tool for multicriteria decision making, image processing and recognition [4, 5]. Most of these applications are restricted on a finite set, and we need the theory which can also treat an infinite set.

Generally, considering an infinite set, if nothing is assumed, it is too general and is sometimes inconvenient. Then we assume the universal set X to be a locally compact Hausdorff space, whose example is the set R of the real number.

Narukawa et al. [9, 10, 11] propose the notion of a regular non-additive measure, that is a extension of classical regular measure, and show the usefulness in the point of representation of some functional.

In this paper, new results about the outer regular non-additive measure and the regular non-additive measure are introduced.

Basic properties of the non-additive measure and the Choquet integral are shown in

In section 3, we define an outer regular non-additive measure and show its properties. We have one of the monotone convergence theorem.

In section 4, we define a regular non-additive measure, and show its properties. In this section, we show the assumption of the result in [9] can be reduced. We also show that the Choquet integral of any measurable function can be approximated by the Choquet integral of continuous function with compact support if the non-additive measure is regular. This is the main theorem in this paper.

2 Preliminaries

In this section, we define a non-additive measure and the Choquet integral, and show their basic properties.

Throughout this paper, we assume that X is a locally compact Hausdorff space, $\mathcal{B}$ is the class of Borel sets, $\mathcal{C}$ is the class of compact sets, and $\mathcal{O}$ is the class of open sets. .

Definition 2.1. [13] A *non-additive measure* μ is an extended real valued set function, $\mu : \mathcal{B} \longrightarrow \overline{R^+}$ with the following properties; (1) $\mu(\emptyset) = 0$, (2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in \mathcal{B}$, where $\overline{R^+} = [0, \infty]$ is the set of extended nonnegative real numbers.

When $\mu(X) < \infty$, we define *the conjugate* μ^c *of* μ by $\mu^c(A) = \mu(X) - \mu(A^C)$ for $A \in \mathcal{B}$.

The class of measurable functions is denoted by M and the class of non-negative measurable functions is denoted by M^+.

Definition 2.2. [1, 6] Let μ be a non-additive measure on $(X, \mathcal{B})$.

(1) *The Choquet integral* of $f \in M^+$ with respect to μ is defined by

$$(C)\int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{x|f(x) \geq r\})$.

(2) Suppose $\mu(X) < \infty$. The Choquet integral of $f \in M$ with respect to μ is defined by

$$(C)\int f d\mu = (C)\int f^+ d\mu - (C)\int f^- d\mu^c,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. When the right hand side is $\infty - \infty$, the Choquet integral is not defined.

$L_1^+(\mu)$ denotes the class of nonnegative Choquet integrable functions. That is,

$$L_1^+(\mu) := \{f | f \in M^+, (C)\int f d\mu < \infty\}.$$

Definition 2.3. [3] Let $f, g \in M$. We say that f and g are *comonotonic* if $f(x) < f(x') \Rightarrow g(x) \leq g(x')$ for $x, x' \in X$.

$f \sim g$ denotes that f and g are comonotonic.

The Choquet integral of $f \in M$ with respect to a non-additive measure have the next basic properties.

Theorem 2.4. [2, 7] *Let $f, g \in M$.*

(1) *If $f \leq g$, then*

$$(C)\int f\, d\mu \ \leq \ (C)\int g\, d\mu$$

(2) *If a is a nonnegative real number, then*

$$(C)\int af\, d\mu \ = \ a\,(C)\int f\, d\mu.$$

(3) *If $f \sim g$, then*

$$(C)\int (f+g)\, d\mu = (C)\int f\, d\mu + (C)\int g\, d\mu.$$

The class of continuous functions with compact support is denoted by K and the class of non-negative continuous functions with compact support is denoted by K^+.

Next, we define upper and lower semi-continuity of functions.

Definition 2.5. *We say that the function $f : X \longrightarrow R$ is upper semi-continuous if $\{x|f \geq a\}$ is closed for all $a \in R$, and the function $f : X \longrightarrow R$ is lower semi-continuous if $\{x|f > a\}$ is open for all $a \in R$.*

The class of non-negative upper semi-continuous functions with compact support is denoted by $USCC^+$ and the class of non-negative lower semi-continuous functions is denoted by LSC^+. We define some property for continuity of non-additive measures.

Definition 2.6. Let μ be a non-additive measure on the measurable space $(X, \mathcal{B})$.

μ is said to be o-continuous from below if

$$O_n \uparrow O \implies \mu(O_n) \uparrow \mu(O)$$

where $n = 1, 2, 3, \ldots$ and both O_n and O are open sets. μ is said to be c-continuous from above if

$$C_n \downarrow C \implies \mu(C_n) \downarrow \mu(C)$$

where $n = 1, 2, 3, \ldots$ and both C_n and C are compact sets.

3 Outer regular non-additive measures

First, we define the outer regular non-additive measures, and show their properties.

Definition 3.1. Let μ be a non-additive measure on measurable space $(X, \mathcal{B})$. μ is said to be *outer regular* if

$$\mu(B) = \inf\{\mu(O)|O \in \mathcal{O}, O \supset B\}$$

for all $B \in \mathcal{B}$.

The next proposition is shown in [9].

Proposition 3.2. *Let μ be an outer regular non-additive measure. μ is c-continuous from above.*

Let $f_n \in USCC^+$ for $n = 1, 2, 3, \cdots$ and $f_n \downarrow f$. Since

$$\bigcap_{n=1}^{\infty} \{x | f_n(x) \geq a\} = \{x | f(x) \geq a\},$$

we have the next theorem from Proposition 3.2.

Theorem 3.3. *Let μ be an outer regular non-additive measure. Suppose that $f_n \in USCC^+$ for $n = 1, 2, 3, \cdots$ and $f_n \downarrow f$. Then we have*

$$\lim_{n \to \infty} (C) \int f_n d\mu = (C) \int f d\mu$$

Let $C \in \mathcal{C}$. It follows from Definition 3.1 that

$$\mu(C) = \inf\{\mu(C) | C \subset O, O \in \mathcal{O}\}.$$

Suppose that $C \subset O$. Since X is locally compact Hausdorff space, there exists an open set U such that its closure $cl(U)$ is compact, satisfying

$$C \subset U \subset cl(U) \subset O.$$

Applying Urysohn's lemma, there exists $f \in K^+$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin cl(U). \end{cases}$$

Therefore we have the next theorem.

Theorem 3.4. *Let μ be an outer regular non-additive measure and C be a compact set. Then we have*

$$\mu(C) = \inf\{(C) \int f d\mu | 1_C \leq f, f \in K^+\}.$$

4 Regular non-additive measures

We define the regular non-additive measure by adding a condition to the outer regular non-additive measure.

Definition 4.1. *Let μ be an outer regular non-additive measure. μ is said to be regular, if for all $O \in \mathcal{O}$*

$$\mu(O) = \sup\{\mu(C) | C \in \mathcal{C}, C \subset O\}.$$

The next proposition is obvious from the definition.

Proposition 4.2. *Let μ be a regular non-additive measure. μ is o-continuous from below.*

The next monotone convergence theorem follows immediately from Proposition 4.2.

Theorem 4.3. *Let μ be a regular non-additive measure. Suppose that $f_n \in LSC^+$ for $n = 1, 2, 3, \cdots$ and $f_n \uparrow f$. Then we have*

$$\lim_{n\to\infty} (C) \int f_n d\mu = (C) \int f d\mu$$

Applying Theorem 3.4 and Theorem 4.3, we have the next theorem.

Theorem 4.4. [9] *Let μ_1 and μ_2 be regular non-additive measures. If*

$$(C) \int f \, d\mu_1 \;=\; (C) \int f d\mu_2$$

for all $f \in K^+$, then $\mu_1(A) \;=\; \mu_2(A)$ for all $A \in \mathcal{B}$.

This theorem means that any two regular non-additive measures which assign the same Choquet integral to each $f \in K^+$ are necessary identical.

In [9], we proved this theorem under the assumption of X to be separable. Using Theorem 3.4, we can prove this theorem without this assumption.

In the case of regular non-additive measure, the Choquet integral of any measurable function can be approximated by the Choquet integral of continuous function with compact support. In the following, we state this fact.

The next lemma follows from the definition of the regular non-additive measure.

Lemma 4.5. *Let μ be a regular non-additive measure on $(X, \mathcal{B})$. For every $M \in \mathcal{B}$ such that $\mu(M) < \infty$ and for every $\epsilon > 0$, there exist $f \in K^+$ such that*

$$|\mu(M) - (C)\int f d\mu| < \epsilon.$$

Applying Lemma 4.5, Urysohn's lemma and comonotonic additivity of Choquet integral, we have the next lemma.

Lemma 4.6. *Let μ be a regular non-additive measure on $(X, \mathcal{B})$, $M_1, M_2 \in \mathcal{B}$ such that $M_1 \subset M_2$ and $\mu(M_2) < \infty$ and $f := a_1 1_{M_1} + a_2 1_{M_2}, a_1 > 0, a_2 > 0$. For every $\epsilon > 0$, there exist $g \in K^+$ such that*

$$|(C)\int f d\mu - (C)\int g d\mu| < \epsilon.$$

Applying Lemma 4.6, we have the next approximation theorem. This is the main theorem in this paper.

Theorem 4.7. *Let μ be a regular non-additive measure on $(X, \mathcal{B})$. For every $\epsilon > 0$ and $f \in L_1^+(\mu)$, there exists $g \in K^+$ such that*

$$|(C)\int f d\mu - (C)\int g d\mu| < \epsilon.$$

References

[1] G.Choquet . Theory of capacities. *Ann. Inst. Fourier, Grenoble.* 5 (1955) 131-295.

[2] D. Denneberg, *Non-additive measure and Integral,* Kluwer Academic Publishers, 1994.

[3] C. Dellacherie, Quelques commentaires sur les prolongements de capacités, *Séminaire de Probabilités 1969/1970, Strasbourg, Lecture Notes in Mathematics,* 191, Springer, 1971, 77- 81.

[4] M. Grabisch, H.T. Nguyen and E. A. Walker, *Fundamentals of uncertainty calculi with applications to fuzzy inference,* Kluwer Academic Publishers, 1995.

[5] M. Grabisch, T. Murofushi, M. Sugeno, eds. *Fuzzy Measures and Integrals: Theory and Applications* Phisica Verlag, 2000.

[6] T. Murofushi and M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, *Fuzzy Sets and Systems,* 29 (1989) 201–227.

[7] T. Murofushi and M. Sugeno, A Theory of Fuzzy Measures: Representations, the Choquet integral and null sets, *J. Math. Anal. Appl.* 159 (1991) 532–549.

[8] Y. Narukawa, T. Murofushi, and M. Sugeno, The comonotonically additive functional on the class of continuous functions with compact support, *Proc. FUZZ-IEEE'97* (1997) 845-852.

[9] Y. Narukawa, T. Murofushi, and M. Sugeno, Regular fuzzy measure and representation of comonotonically additive functionals, *Fuzzy Sets and Systems,* 112 (2) , (2000), 177-186.

[10] Y. Narukawa, T. Murofushi, and M. Sugeno, Boundedness and Symmetry of Comonotonically Additive Functionals, *Fuzzy Sets and Systems* 118,(3), (2001) 539-545.

[11] Y. Narukawa, T. Murofushi, and M. Sugeno, Representation of Comonotonically Additive Functional by Choquet Integral, B. Bouchon-Meunier, R.R.Yager and L. A. Zadeh (eds), *Information, Uncertainty, Fusion,* Kluwer Academic Publisher, 2000, 93-104.

[12] J. Šipoš, Non linear integral, *Math. Slovaca,* 29 (3) (1979) 257-270.

[13] M. Sugeno, *Theory of fuzzy integrals and its applications,* Doctoral Thesis, Tokyo Institute of Technology, (1974).

[14] M. Sugeno, Y. Narukawa and T. Murofushi, Choquet integral and fuzzy measures on locally compact space, *Fuzzy sets and Systems,* 99 (2),(1998) 205–211.