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<td>Cho, Nak Eun; Owa, Shigeyoshi</td>
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Kyoto University
ON THE FEKETE-SZEGŐ PROBLEM FOR STRONGLY 
$\alpha$-LOGARITHMIC QUASICONVEX FUNCTIONS

NAK EUN CHO AND SHIGEYOSHI OWA

ABSTRACT. The purpose of the present paper is to introduce the classes $\mathcal{M}^\alpha(\beta)$ and 
$Q^\alpha(\beta)$, respectively, of normalized strongly $\alpha$-logarithmic convex and quasiconvex func-
tions of order $\beta$ in the open unit disk and to obtain sharp Fekete-Szegő inequalities for functions belonging to the classes be the class $\mathcal{M}^\alpha(\beta)$ and $Q^\alpha(\beta)$.

1. Introduction

Let $S$ denote the class of analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are univalent in the open unit disk $U = \{z : |z| < 1\}$. A classical theorem of Fekete and Szegő [8] states that for $f \in S$ given by (1.1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu & \text{if } \mu \leq 0, \\
1 + 2e^{-2\mu/(1-\mu)} & \text{if } 0 \leq \mu \leq 1, \\
4\mu - 3 & \text{if } \mu \geq 1.
\end{cases}$$

This inequality is sharp in the sense that for each $\mu$ there exists a function in $S$ such that equality holds. Recently, Pfluger [17,18] has considered the problem when $\mu$ is complex. In the case of $\mathcal{C}$, $S^*$ and $\mathcal{K}$, the subclasses of convex, starlike and close-to-convex functions, respectively, the above inequality can be improved [10,11]. Also, Darus and Thomas [5] studied the class $\mathcal{M}^\alpha$ of $\alpha$-logarithmic convex functions and they also have solved the Fekete-Szegő problem for the class $\mathcal{M}^\alpha$. Furthermore, London [14] have extended the results of Abdel-Gawad and Thomas [1], Keogh and

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Merkes [10] and Koepf [11,12] to the class \( \mathcal{K}(\beta) \) of strongly close-to-convex functions of order \( \beta \). Now we introduce new classes which incorporate well-known classes of univalent functions.

**Definition 1.1.** A function \( f \in S \) given by (1.1) is said to be strongly logarithmic \( \alpha \)-convex of order \( \beta \) if

\[
\left| \arg \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))'}{f'(z)} \right)^\alpha \right\} \right| \leq \frac{\pi}{2} \beta \quad (\alpha \geq 0; 0 < \beta \leq 1; z \in \mathcal{U}).
\]

(1.2)

Denote by \( \mathcal{M}^\alpha(\beta) \) the class of strongly \( \alpha \)-logarithmic convex functions of order \( \beta \). The class \( \mathcal{M}^\alpha(\beta) \) was introduced by Chiang [4]. In particular, the classes \( \mathcal{M}^\alpha(1) = \mathcal{M}^\alpha \) and \( \mathcal{M}^0(\beta) \) have been extensively studied by Lewandowski, Miller and Zlotkiewicz [13] and Bramman and Kirwan [2] (also, see [7,20]), respectively.

**Definition 1.2.** A function \( f \in S \) given by (1.1) is said to be \( \alpha \)-logarithmic quasiconvex of order \( \beta \) if there exists a function \( g \in \mathcal{C} \) such that

\[
\left| \arg \left\{ \left( \frac{f'(z)}{g'(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))'}{g'(z)} \right)^\alpha \right\} \right| \leq \frac{\pi}{2} \beta \quad (\alpha, \beta \geq 0; z \in \mathcal{U}).
\]

(1.3)

We denote by \( Q^\alpha(\beta) \) the class of strongly \( \alpha \)-logarithmic quasiconvex functions of order \( \beta \). Clearly, \( Q^0(1) \) and \( Q^1(1) \) are the classes of close-to-convex functions and quasiconvex functions introduced by Kaplan [9] and Noor [15] (also, see [16]), respectively. Also we note that \( Q^0(\beta) = \mathcal{K}(\beta) \).

In the present paper, we derive sharp Fekete-Szegö inequalities for functions belonging to the classes \( \mathcal{M}^\alpha(\beta) \) and \( Q^\alpha(\beta) \), which imply the results obtained by Abdel-Gawad and Thomas [1], Darus and Thomas [5], Keogh and Merkes [10], Koepf [11,12], and London [14].

2. Results

To prove our main results, we need the following

**Lemma 2.1.** Let \( p \) be analytic in \( \mathcal{U} \) and satisfy \( \text{Re} \{p(z)\} > 0 \) for \( z \in \mathcal{U} \), with \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \). Then

\[
|p_n| \leq 2 \quad (n \geq 1)
\]

(2.1)

and

\[
\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.
\]

(2.2)
The inequality (2.1) was first proved by Carathéodory [3](also, see Duren [6, p. 41]) and the inequality (2.2) can be found in [19, p.166].

With the help of Lemma 2.1, we now derive

**Theorem 2.1.** Let \( f \in \mathcal{M}^a(\beta) \) and be given by (1.1). Then for complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{\beta}{1 + 2\alpha} \max \left\{ 1, \frac{|3(1 + 3\alpha) - 4\mu(1 + 2\alpha)|\beta}{(1 + \alpha)^2} \right\}.
\]

For each \( \mu \), there is a function in \( \mathcal{M}^a(\beta) \) such that equality holds.

**Proof.** From (1.2), we can write

\[
\left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))'}{f(z)} \right)^{\alpha} = p^\beta(z),
\]

where \( p \) is given by Lemma 2.1. Equating coefficients, we obtain

\[
a_2 = \frac{\beta}{1 + \alpha} p_1
\]

and

\[
a_3 = \frac{1}{4(1 + 2\alpha)} \left( \beta(\beta - 1)p_1^2 + 2\beta p_2 - (\alpha^2 - 7\alpha - 2) \left( \frac{\beta p_1}{1 + \alpha} \right)^2 \right).
\]

Then we have

\[
a_3 - \mu a_2^2 = \frac{\beta}{2(1 + 2\alpha)} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{3 + 9\alpha - 4\mu(1 + 2\alpha)\beta^2 p_1^2}{4(1 + 2\alpha)(1 + \alpha)^2}.
\]

Hence (2.4) and Lemma 2.1 give

\[
|a_3 - \mu a_2^2| \leq \frac{\beta}{2(1 + 2\alpha)} \left( 2 - \frac{|p_1|^2}{2} \right) + \frac{|3 + 9\alpha - 4\mu(1 + 2\alpha)|\beta^2 |p_1|^2}{4(1 + 2\alpha)(1 + \alpha)^2}
\]

\[
\leq \frac{\beta}{1 + 2\alpha} + \left\{ \frac{|3 + 9\alpha - 4\mu(1 + 2\alpha)|\beta^2 - (1 + \alpha)^2 \beta}{4(1 + 2\alpha)(1 + \alpha)^2} \right\} |p_1|^2.
\]

Therefore, by using \( |p_1| \leq 2 \), we have

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{\beta}{1 + 2\alpha}, & \text{if } k(\alpha) \leq \frac{(1 + \alpha)^2}{\beta}, \\
\frac{|3 + 9\alpha - 4\mu(1 + 2\alpha)|\beta^2}{(1 + 2\alpha)(1 + \alpha)^2}, & \text{if } k(\alpha) \geq \frac{(1 + \alpha)^2}{\beta},
\end{array} \right.
\]
where
\[ k(\alpha) = |3(1 + 3\alpha) - 4\mu(1 + 2\alpha)|. \]

Equality is attained for functions in \( \mathcal{M}^\alpha(\beta) \), respectively, given by
\[ (zf'(z))^{1-\alpha} \left( \frac{(zf'(z))'}{f(z)} \right)^\alpha = \left( \frac{1+z^2}{1-z^2} \right)^\beta \]  
(2.5)
and
\[ (zf'(z))^{1-\alpha} \left( \frac{(zf'(z))'}{f(z)} \right)^\alpha = \left( \frac{1+z}{1-z} \right)^\beta \]  
(2.6)

**Remark 2.1.** It follows at once from (2.3) that \( |a_2| \leq 2\beta/(1+\alpha) \) and Theorem 2.1 gives
\[ |a_3| \leq \begin{cases} \frac{\beta}{1+2\alpha}, & \text{if } (1+\alpha)^2 \geq 3(1+3\alpha)\beta, \\ \frac{3(1+3\alpha)^2\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } (1+\alpha)^2 \leq 3(1+3\alpha)\beta, \end{cases} \]
The inequality for \( |a_2| \) is sharp when \( f \) is defined by (2.6) and the inequalities for \( |a_3| \) are sharp when \( f \) is defined by (2.5) and (2.6), respectively.

Next, we consider the real number \( \mu \) as follows.

**Theorem 2.2.** Let \( f \in \mathcal{M}^\alpha(\beta) \) and be given by (1.1). Then for real number \( \mu \),
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{(3(1+3\alpha)-4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \leq \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta} \leq \mu \leq \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{(4(1+2\alpha)\mu-3(1+3\alpha))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \geq \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}. \end{cases} \]

For each \( \mu \), there is a function in \( C^\alpha(\beta) \) such that equality holds in all cases.

**Proof.** We consider two cases. At first, we suppose that \( \mu \leq 3(1+3\alpha)/(4(1+2\alpha)) \).

Then (2.3) and Lemma 2.1 give
\[ |a_3 - \mu a_2^2| \leq \frac{\beta}{2(1+2\alpha)} \left( 2 - \frac{|p_1|^2}{2} \right) + \frac{(3+9\alpha-4\mu(1+2\alpha))\beta^2|p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \]
\[ \leq \frac{\beta}{1+2\alpha} + \frac{((3+9\alpha-4\mu(1+2\alpha))\beta^2-(1+\alpha)^2\beta)|p_1|^2}{4(1+2\alpha)(1+\alpha)^2}. \]
So, by using the fact that \( |p_1| \leq 2 \), we obtain...
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\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{(3(1+3\alpha)-4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \leq \frac{3(1+3\alpha)(1+\alpha)}{4(1+2\alpha)}, \\
\frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)(1+\alpha)}{4(1+2\alpha)} \leq \mu \leq \frac{3(1+3\alpha)}{4(1+2\alpha)}. 
\end{cases} \]

Equality is attained by choosing \( p_1 = p_2 = 2 \) and \( p_1 = 0, p_2 = 2 \), respectively, in \( (2.3) \).

Next, we suppose that \( \mu \geq \frac{3(1+3\alpha)}{4(1+2\alpha)} \). In this case, it follows again from \( (2.3) \) and Lemma 2.1 that

\[ |a_3 - \mu a_2^2| \leq \frac{\beta}{2(1+2\alpha)} \left( 2 - \frac{|p_1|^2}{2} \right) + \frac{(4\mu(1+2\alpha)-(3+9\alpha))\beta^2|p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \]

\[ \leq \frac{\beta}{1+2\alpha} + \frac{(4\mu(1+2\alpha)-(3+9\alpha))\beta^2 - \beta(1+\alpha)^2|p_1|^2}{4(1+2\alpha)(1+\alpha)^2}, \]

and so, as in the first case, we have

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)(1+\alpha)}{4(1+2\alpha)} \leq \mu \leq \frac{3(1+3\alpha)(1+\alpha)}{4(1+2\alpha)}, \\
\frac{(4\mu(1+2\alpha)-3(1+3\alpha))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \geq \frac{3(1+3\alpha)(1+\alpha)}{4(1+2\alpha)}. 
\end{cases} \]

The results are sharp by choosing \( p_1 = 0, p_2 = 2 \) and \( p_1 = 2i, p_2 = -2 \), respectively, in \( (2.3) \).

\textbf{Remark 2.2.} If we take \( \beta = 1 \) in Theorem 2.1 and Theorem 2.2, then we obtain the results by Darus and Thomas [5].

Finally, we prove

\textbf{Theorem 2.3.} Let \( f \in Q^\alpha(\beta) \) and be given by \((1.1)\). Then for \( \alpha \geq 0 \) and \( \beta \geq 0 \), we have

\[ 3(2\alpha+1)|a_3 - \mu a_2^2| \leq \begin{cases} 
1 + \frac{(1+\beta)^2(2(3\alpha+1)-3(2\alpha+1)\mu)}{(\alpha+1)^2}, & \text{if } \mu \leq \frac{2\alpha(1-\alpha)+2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)}, \\
1 + 2\beta + \frac{2(3\alpha+1)-3(2\alpha+1)\mu}{2(\alpha+1)^2-\beta(2(3\alpha+1)-3(2\alpha+1)\mu)}, & \text{if } \frac{2\alpha(1-\alpha)+2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)} \leq \mu \leq \frac{2(3\alpha+1)}{3(2\alpha+1)}, \\
1 + 2\beta, & \text{if } \frac{2(3\alpha+1)}{3(2\alpha+1)} \leq \mu \leq \frac{2(\alpha+1)^2+2(3\alpha+1)(1+\beta)}{3(3\alpha+1)(1+\beta)}, \\
-1 + \frac{(1+\beta)^2(2(3\alpha+1)-3(2\alpha+1)\mu)}{(\alpha+1)^2}, & \text{if } \mu \geq \frac{2(\alpha+1)^2+2(3\alpha+1)(1+\beta)}{3(3\alpha+1)(1+\beta)}. 
\end{cases} \]

For each \( \mu \), there is a function in \( Q^\alpha(\beta) \) such that equality holds in all cases.
Proof. Let \( f \in Q^\alpha(\beta) \). Then it follows from (1.2) that we may write
\[
\left( \frac{f'(z)}{g'(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))'}{g'(z)} \right)^{\alpha} = p^\beta(z),
\]
where \( g \) is convex and \( p \) has positive real part. Let \( g(z) = z + b_2z^2 + b_3z^3 + \cdots \) and let \( p \) be given in the Lemma above. Then by comparing the coefficients of both sides of (2.7), we obtain
\[
2(\alpha + 1)a_2 = \beta p_1 + 2b_2
\]
and
\[
3(2\alpha + 1)a_3 = 3b_3 + \frac{2\alpha(1-\alpha)}{(\alpha + 1)^2} b_2^2 + \beta \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{\beta^2(3\alpha + 1)}{2(\alpha + 1)^2} p_1^2 + \frac{2\beta(3\alpha + 1)}{(\alpha + 1)^2} p_1 b_2.
\]
So, with
\[
x = \frac{2(3\alpha + 1) - 3(2\alpha + 1)\mu}{(\alpha + 1)^2},
\]
we have
\[
3(2\alpha + 1)(a_3 - \mu a_2^2) = 3 \left( b_3 + \frac{1}{3}(x - 2)b_2^2 \right)
+ \beta \left( p_2 + \frac{1}{4}(\beta x - 2)p_1^2 \right) + \beta xp_1 b_2.
\]
Since rotations of \( f \) also belong to \( Q^\alpha(\beta) \), without loss of generality, we may assume that \( a_3 - \mu a_2^2 \) is positive. Thus we now estimate \( \text{Re}(a_3 - \mu a_2^2) \).

Since \( g \in C \), there exists \( h(z) = 1 + k_1 z + k_2 z^2 + \cdots \ (|z| < 1) \) with positive real part, such that \( g'(z) + zg''(z) = g'(z)h(z) \). Hence, by equating coefficients, we get that \( b_2 = k_1/2 \) and \( b_3 = (k_2 + k_1^2)/6 \). So, by Lemma 2.1,
\[
3\text{Re} \left( b_3 + \frac{1}{3}(x - 2)b_2^2 \right) = \frac{1}{2} \text{Re} \left( k_2 - \frac{1}{2} k_1^2 \right) + \frac{1}{4} (x + 1) \text{Re} k_1^2
\leq 1 - \rho^2 + (x + 1)\rho^2 \cos 2\phi,
\]
where \( b_2 = k_1/2 = \rho e^{i\phi} \) for some \( \rho \ (0 \leq \rho \leq 1) \). We also have
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\[
\beta \text{Re} \left( p_2 + \frac{1}{4}(\beta x - 2)p_1^2 \right) = \beta \text{Re} \left( p_2 - \frac{1}{2}p_1^2 \right) + \frac{1}{4}\beta^2 x \text{Re} p_1^2
\]
\[
\leq 2\beta(1 - r^2) + \beta^2 x r^2 \cos 2\theta, \tag{2.10}
\]

where \( p_1 = 2re^{i\theta} \) for some \( r \) (0 \leq r \leq 1). From (2.8-10), we obtain

\[
3\text{Re}(2\alpha + 1)(a_3 - \mu a_2^2) \leq 1 - \rho^2 + (x + 1)\rho^2 \cos 2\phi
\]
\[
+ 2\beta(1 - r^2) + \beta^2 x r^2 \cos 2\theta + 2\beta x r \cos(\theta + \phi), \tag{2.11}
\]

and we now proceed to maximize the right-hand of (2.11). This function will be denoted by \( \psi(x) \) whenever all the parameters except \( x \) are held constant.

We consider first the case

\[
\frac{2\alpha(1 - \alpha) + 2\beta(3\alpha + 1)}{3(2\alpha + 1)(1 + \beta)} \leq \mu \leq \frac{2(3\alpha + 1)}{3(2\alpha + 1)},
\]

so that \( 0 \leq x \leq 2/(1 + \beta) \). Since the expression \(-2t^2 + t^2 \beta x \cos 2\theta + 2xt\) is the largest when \( t = x/(2 - \beta x \cos 2\theta) \), we have

\[
-2t^2 + t^2 \beta x \cos 2\theta + 2xt \leq \frac{x^2}{2 - \beta x \cos 2\theta} \leq \frac{x^2}{2 - \beta x}.
\]

Thus

\[
\psi(x) \leq x + 1 + \beta \left( 2 + \frac{x^2}{2 - \beta x} \right)
\]
\[
= 1 + 2\beta + \frac{2\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}{2(\alpha + 1)^2 - \beta\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}
\]

and with (2.11) this establishes the second inequality in the theorem. Equality occurs only if

\[
p_1 = \frac{2x}{2 - \beta x} = \frac{2\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}{2(\alpha + 1)^2 - \beta\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}, \quad p_2 = 2, \quad b_2 = b_3 = 1,
\]

and the corresponding function \( f \) is defined by

\[
(f'(z))^{1-\alpha}((zf'(z))')^\alpha = \frac{1}{(1-z)^2} \left( \lambda \frac{1+z}{1-z} + (1-\lambda) \frac{1-z}{1+z} \right)^\beta,
\]
\[
\lambda = \frac{2(\alpha + 1)^2 + (1 - \beta)(2(3\alpha + 1) - 3(2\alpha + 1)\mu)}{4(\alpha + 1)^2 - 2\beta(2(3\alpha + 1) - 3(2\alpha + 1)\mu)}.
\]

We now prove the first inequality. Let
\[
\mu \leq \frac{2\alpha(1 - \alpha) + 2\beta(3\alpha + 1)}{3(2\alpha + 1)(1 + \beta)},
\]
so that \(x \geq 2/(1 + \beta)\). With \(x_0 = 2/(1 + \beta)\), we have
\[
\psi(x) = \psi(x_0) + (x - x_0)(\rho^2 \cos 2\phi + \beta^2 \rho^2 \cos 2\theta + 2\beta \rho \cos(\theta + \phi))
\leq \psi(x_0) + (x - x_0)(1 + \beta)^2
\leq 1 + \frac{(1 + \beta)^2\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}{(\alpha + 1)^2}
\]
as required. Equality occurs only if \(p_1 = p_2 = 2\), \(b_2 = b_3 = 1\), and the corresponding function \(f\) is defined by
\[
(f'(z))^{1-\alpha}((zf'(z))')^\alpha = \frac{1}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^\beta.
\]

Let \(x_1 = -2/(1 + \beta)\). We shall find that \(\psi(x_1) = 1 + 2\beta\), and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain
\[
\psi(x) \leq \psi(x_1) + |x - x_1|(1 + \beta)^2
\leq -1 + \frac{(1 + \beta)^2\{3(2\alpha + 1)\mu - 2(3\alpha + 1)\}}{(\alpha + 1)^2}.
\]
if \(x \leq x_1\), that is,
\[
\mu \geq \frac{2(\alpha + 1)^2 + 2(3\alpha + 1)(1 + \beta)}{3(2\alpha + 1)(1 + \beta)}.
\]
Equality occurs only if \(p_1 = 2i\), \(b_2 = i\), \(p_2 = -2\), \(b_3 = -1\), and the corresponding function \(f\) is defined by
\[
(f'(z))^{1-\alpha}((zf'(z))')^\alpha = \frac{1}{(1-iz)^2} \left(\frac{1+iz}{1-iz}\right)^\beta.
\]

Also, for \(0 \leq \lambda \leq 1\), we note that
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\[ \psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda)\psi(0) \leq \lambda(1 + 2\beta) + (1 - \lambda)(1 + 2\beta) = 1 + 2\beta, \]

so \( \psi(x) \leq 1 + 2\beta \) for \( x_1 \leq x \leq 0 \), that is,

\[ \frac{2(3\alpha + 1)}{3(2\alpha + 1)} \leq \mu \leq \frac{2(\alpha + 1)^2 + 2(3\alpha + 1)(1 + \beta)}{3(2\alpha + 1)(1 + \beta)}. \]

Equality occurs only if \( p_1 = b_2 = 0, \quad p_2 = 2, \quad b_3 = 1/3 \), and the corresponding function \( f \) is defined by

\[ (f'(z))^{1-\alpha}((zf'(z))')^{\alpha} = \frac{1}{1-z^2} \left( \frac{1+z^2}{1-z^2} \right)^{\beta} = \frac{(1+z^2)^{\beta}}{(1-z^2)^{1+\beta}}. \]

We now show that \( \psi(x_1) \leq 1 + 2\beta \). Since

\[ (-2 + \beta x_1 \cos 2\theta)t^2 + 2x_1 t \rho \cos(\theta + \phi) = (-2 + \beta x_1 \cos 2\theta) \left\{ t + \frac{x_1 \rho \cos(\theta + \phi)}{-2 + \sqrt{x_1 \cos 2\theta}} \right\}^2 + \frac{x_1^2 \rho^2 \cos^2(\theta + \phi)}{2 - \beta x_1 \cos 2\theta} \]

for all real \( t \) and

\[ 2 - \beta x_1 \cos 2\theta = 2 + \frac{2\beta}{1 + \beta} \cos 2\theta \geq 2 - \frac{2\beta}{1 + \beta} \geq 0, \]

we have

\[ \psi(x_1) - (1 + 2\beta) \leq \rho^2 \left( -1 + (x_1 + 1) \cos 2\phi + \frac{\beta x_1^2(1 + \cos 2(\theta + \phi))}{2(2 - \beta x_1 \cos 2\theta)} \right). \]

Thus we consider the inequality

\[ \beta x_1^2(1 + \cos 2(\theta + \phi)) + 2(2 - \beta x_1 \cos 2\theta)(-1 + (x_1 + 1) \cos 2\phi) \leq 0. \]

After some simplifications, this becomes

\[ 4(\beta^2(\cos 2\phi + 1)(\cos 2\phi - 1) - \beta(1 + \cos 2\theta + \sin 2\theta \sin 2\phi) - 1 - \cos 2\phi) \leq 0, \]

which is true if

\[ 2\beta^2 \cos^2 \theta \sin^2 \phi + 2\beta \cos \theta \sin \theta \cos \phi \sin \phi + \cos^2 \phi \geq 0. \]  (2.12)
Now, for all real $t$,
\[
2t^2 + 2t \sin \theta \cos \phi + \cos^2 \phi \geq 0,
\]
so, by taking $t = \beta \cos \theta \sin \phi$, we obtain (2.12). This completes the proof of the theorem.

Remark. Letting $\alpha = 0$ in Theorem 2.3, we have the corresponding result obtained by London [14], which extend the earlier results by several authors [1,5,10-12].

For $\alpha = 1$ in Theorem, we have the following

Corollary 2.1. Let $f \in Q^1(\beta)$ and be given by (1.1). Then for $\beta \geq 0$, we have
\[
9 |a_3 - \mu a_2^2| \leq \begin{cases} 1 + \frac{(1+\beta)^2(8-9\mu)}{4} & \text{if } \mu \leq \frac{8}{9(1+\beta)}, \\ 1 + 2\beta + \frac{2(8-9\mu)}{8-\beta(8-9\mu)} & \text{if } \frac{8}{9(1+\beta)} \leq \mu \leq \frac{8}{9}, \\ 1 + 2\beta & \text{if } \frac{8}{9} \leq \mu \leq \frac{8(2+\beta)}{9(1+\beta)}, \\ -1 + \frac{(1+\beta)^2(9\mu-8)}{4} & \text{if } \mu \geq \frac{8(2+\beta)}{9(1+\beta)}. \end{cases}
\]

For each $\mu$, there is a function in $Q^1(\beta)$ such that equality holds in all cases.

References

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NAK EUN CHO, DEPARTMENT OF APPLIED MATHEMATICS, PUKYONG NATIONAL UNIVERSITY, Pusan 608-737, KOREA (E-MAIL: necho@pknu.ac.kr)

SHIGEYOSHI OWA, DEPARTMENT OF MATHEMATICS, KINKI UNIVERSITY, HIGASHI-OsAKA, OSAKA 577-8502, JAPAN (E-MAIL: owa@math.kindai.ac.jp)