PROPERTIES OF CERTAIN INTEGRAL OPERATOR

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Abstract

Let \( A(p) \) denote the class of functions \( f(z) \) which are analytic and \( p \)-valent in the unit disk \( U \). A new subclass \( \Omega(\alpha, \beta; \gamma) \) of \( A(p) \) consisting of analytic and \( p \)-valent functions \( f(z) \) associated with the certain integral operator \( Q_\beta^\alpha \) which is the generalization of the integral operator studied by I.B.Jung, Y.C.Kim and H.M.Srivastava (J. Math. Anal. Appl. 248(2000), 475 - 481) is introduced. Some interesting properties of the operator \( Q_\beta^\alpha \) for functions \( f(z) \) belonging to \( A(p) \) are investigated.

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1. Introduction.

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\})$$

which are analytic and $p$-valent in the unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $S_p^*(\gamma)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the condition

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > p\gamma$$

for $0 \leq \gamma < 1$ and $z \in U$. A function in $S_p^*(\gamma)$ is called $p$-valent starlike of order $\gamma$ in $U$.

Let $f(z)$ and $g(z)$ be analytic in $U$. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in $U$ such that $|w(z)| < 1 (z \in U)$ and $g(z) = f(w(z))$. For this relation the symbol $g(z) \prec f(z)$ is used. In case $f(z)$ is univalent in $U$ we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

Recently, Jung, Kim and Srivastava [3] introduced the following integral operator:

$$Q_\beta^\alpha f(z) = \left( \frac{\alpha + \beta}{\beta} \right) \frac{\alpha}{z^\beta} \int_0^z (1 - \frac{t}{z})^{\alpha-1} t^{\beta-1} f(t) dt$$

$$(\alpha > 0, \beta > -1; f \in A(1))$$

They also showed that

$$Q_\beta^\alpha f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)}{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)} a_{p+n} z^{p+n}$$

It follows from (1.3) that one can define the operator $Q_\beta^\alpha$ for $\alpha \geq 0$ and $\beta > -1$. Some interesting subclasses of analytic function, associated with the operator $Q_\beta^\alpha$, have been considered recently by Jung et al.[3], Aouf et al.[1], Li[5], Liu[6] and others.

Motivated by Jung, Kim and Srivastava's work [3], we now consider a linear operator $Q_\beta^\alpha : A(p) \rightarrow A(p)$ as following:

$$Q_\beta^\alpha f(z) = \left( \frac{p + \alpha + \beta - 1}{p + \beta - 1} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{p-1} t^{\beta-1} f(t) dt$$

$$(\alpha \geq 0, \beta > -1; f \in A(p))$$

We note that

$$Q_\beta^\alpha f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)}{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)} a_{p+n} z^{p+n}$$
\[
\alpha \geq 0, \beta > -1; f \in A(p).
\]

(1.4)

It is easily verified from the definition (1.4) that

\[
z(Q_\beta^\alpha f(z))' = (\alpha + \beta + p - 1)Q_\beta^{\alpha-1}f(z) - (\alpha + \beta - 1)Q_\beta^\alpha f(z).
\]

(1.5)

When \( p = 1 \), the identity (1.5) is given in [3]. One can easily see that the operator \( Q_\beta^\alpha \) has an inverse operator \( Q_{\beta+\alpha}^{-\alpha} \) and \( Q_\beta^0 \) is an unit operator.

A function \( f(z) \in A(p) \) is said to be in the class \( \Omega(\alpha, \beta; \gamma) \) if it satisfies the condition

\[
\frac{z(Q_\beta^\alpha f(z))}{Q_\beta^\alpha f(z)} + \frac{pz^p}{1-z^p} \prec \frac{p+p(1-2\gamma)z}{1-z}
\]

for all \( z \in U \) and \( 0 \leq \gamma < 1 \).

In this paper, we shall show the extreme points of the closed convex hull of the class \( \Omega(\alpha, \beta; \gamma) \). It is then used to determine the coefficient bounds.

In the sequel, we denote the closed convex hull of a class \( H \) by \( coH \). Also, let \( E(coH) \) denote the set of all extreme points of \( H \).

2. Main Results.

In order to derive our main results, we shall need the following lemmas.

**Lemma 1** ([4]). \( E(coS_\beta^*(\alpha)) \) consists of the functions given by

\[
z^p \frac{(1-zx)^{2p(1-\gamma)}}{(1-xz)^{2p(1-\gamma)}} = z^p + \sum_{n=1}^{\infty} \frac{(2p-2p\gamma)_n}{n!} x^n z^{p+n} \quad (z \in U),
\]

(2.1)

where \( (a)_n = a(a+1) \cdots (a+n-1), x \in C \) and \( |x| = 1 \).

**Lemma 2** ([9]). The function \( (1-z)^\rho \equiv e^{\rho \log(1-z)}, \rho \neq 0 \), is univalent in \( U \) if and only if \( \rho \) is either in the closed disk \( |\rho - 1| \leq 1 \) or in the closed disk \( |\rho + 1| \leq 1 \).

**Lemma 3** ([7]). Let \( q(z) \) be univalent in \( U \) and let \( \theta(w) \) and \( \phi(w) \) be analytic in a domain \( D \) containing \( q(U) \) with \( \phi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = zq'(z)\phi(q(z)) \), \( h(z) = \theta(q(z)) + Q(z) \) and suppose that

1. \( Q(z) \) is starlike (univalent) in \( U \);
2. \( Re \left\{ \frac{zh(q(z))}{Q(z)} \right\} = Re \left\{ \frac{\theta'(q(z)) + zQ'(z)}{\phi(q(z)) + zQ(z)} \right\} > 0 \quad (z \in U) \).

If \( p(z) \) is analytic in \( U \), with \( p(0) = q(0), p(U) \subset D \) and

\[
\theta(p(z)) +zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),
\]

(2.2)

then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.
Theorem 1. A function \( f(z) \in A(p) \) is in \( \Omega(\alpha,\beta;\gamma) \) if and only if \( f(z) \) can be expressed as

\[
f(z) = Q_{\beta+\alpha}^{-\alpha} \left\{ z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \right\},
\]  

(2.3)

where \( \mu \) is a probability measure defined on the unit circle \( X = \{ x : |x| = 1 \} \).

Proof. Let \( f(z) \in \Omega(\alpha,\beta;\gamma) \). Then by Herglotz formula [2], we have

\[
\frac{z(Q_{\beta}^\alpha f(z))'}{Q_{\beta}^\alpha f(z)} + \frac{pz^p}{1-z^p} = p(1-\gamma) \int_X \frac{1+xz}{1-xz} d\mu(x) + p\gamma,
\]  

(2.4)

where \( \mu \) is a probability measure defined on the unit circle \( X = \{ x : |x| = 1 \} \). By means of the identity

\[
\frac{d}{dz} \log \frac{Q_{\beta}^\alpha f(z)}{z^p (1-z^p)} = \frac{1}{z} \left[ \frac{z(Q_{\beta}^\alpha f(z))'}{Q_{\beta}^\alpha f(z)} + \frac{pz^p}{1-z^p} - p \right],
\]  

(2.5)

(2.4) yields

\[
Q_{\beta}^\alpha f(z) = z^p (1-z^p) \exp[-2p(1-\gamma) \int_X \log(1-xz) d\mu(x)].
\]  

(2.6)

Thus

\[
f(z) = Q_{\beta+\alpha}^{-\alpha} \left\{ z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \right\}.
\]

Now the proof is complete.

Theorem 2. Let \( 0 \leq \gamma_1 < \gamma_2 < 1 \), then \( \Omega(\alpha,\beta;\gamma_2) \subset \Omega(\alpha,\beta;\gamma_1) \).

Proof. We define a linear operator on \( \Omega(\alpha,\beta;\gamma) \) as following:

\[
T_\gamma(f) = \frac{Q_{\beta}^\alpha f(z)}{1-z^p} \quad (z \in U).
\]  

(2.7)

Then \( T_\gamma \) is a linear homeomorphism from \( \Omega(\alpha,\beta;\gamma) \) to \( S_p^*(\gamma) \). It is well-known that \( S_p^*(\gamma_2) \subset S_p^*(\gamma_1) \) for \( 0 \leq \gamma_1 < \gamma_2 < 1 \). The result follows immediately.

Theorem 3. (i) The extreme points of \( \text{co}\Omega(\alpha,\beta;\gamma) \) are given by the functions

\[
f_x(z) = Q_{\beta+\alpha}^{-\alpha} \left\{ \frac{z^p (1 - z^p)}{(1-xz)^{2p(1-\gamma)}} \right\}
\]  

\[
(x \in C, |x| = 1; z \in U).
\]  

(2.8)
(ii) \( \Omega(\alpha, \beta; \gamma) = \{ f : f(z) = \int_X f_x(z) d\mu(x) \} \), \hspace{1cm} (2.9)

where \( \mu \) varies over the probability measures defined on the unit circle \( X \).

Proof. Since \( T_\gamma \) defined by (2.7) is a linear homeomorphism from \( \Omega(\alpha, \beta; \gamma) \) to \( S^*_p(\gamma) \), it preserves extreme points. By making use of Lemma 1, the results follow at once.

According to Theorem 3 and Lemma 1, we have the following corollaries.

**Corollary 1.** Let \( f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \Omega(\alpha, \beta; \gamma) \). Then

\[
|a_{p+n}| \leq \left\{ \begin{array}{ll}
\frac{(2p-2p\gamma)_{n-p}}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & 1 \leq n < p, \\
\frac{(2p-2p\gamma)_{n-p} \prod_{k=1}^{p} (2p-2p\gamma+n-k) - \prod_{k=1}^{p} (n-p+k) \Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{n! \Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & n \geq p.
\end{array} \right.
\]

The result is sharp.

**Corollary 2.** Let \( f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \Omega(\alpha, \beta; \gamma) \). Then for \( |z| = r < 1 \).

\[
|f(z)| \leq r^p + \sum_{n=1}^{p} \frac{(2p-2p\gamma)_{n}}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)} r^{p+n} + \sum_{n=p}^{\infty} \frac{(2p-2p\gamma)_{n-p} \prod_{k=1}^{p} (2p-2p\gamma+n-k) - \prod_{k=1}^{p} (n-p+k)}{n! \Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)} r^{p+n}.
\]

The result is sharp.

**Theorem 4.** Let \( f(z) \in \Omega(\alpha, \beta; \gamma) \). Let \( \rho \) be a complex number with \( \rho \neq 0 \) and satisfy either \( |2p\rho(1-\gamma) + 1| \leq 1 \) or \( |2p\rho(1-\gamma) - 1| \leq 1 \). Then

\[
\left( \frac{Q_\beta^\alpha f(z)}{z^p (1-z^p)} \right)^\rho < \frac{1}{(1-z)^{2p\rho(1-\gamma)}} = q(z) \quad (z \in U),
\]

where \( q(z) \) is the best dominant.

Proof. Let

\[
p(z) = \left( \frac{Q_\beta^\alpha f(z)}{z^p (1-z^p)} \right)^\rho,
\]

then \( p(z) \) in analytic is \( U \) with \( p(0) = 1 \). Differentiating (2.11) logarithmically we have

\[
\frac{zp'(z)}{p(z)} = \rho \left( \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} + \frac{pz^p}{1-z^p - p} \right).
\]
Since $f(z) \in \Omega(\alpha, \beta; \gamma)$, (2.12) is equivalent to

$$p + \frac{zp'(z)}{\rho p(z)} < \frac{p + p(1 - 2\gamma)z}{1 - z} = h(z).$$

(2.13)

If we take

$$q(z) = \frac{1}{(1 - z)^{2\rho(1-\gamma)}}, \theta(w) = p \quad \text{and} \quad \phi(w) = \frac{1}{\rho w},$$

(2.14)

then $q(z)$ is univalent by the condition of the theorem and Lemma 2. It is easy to show that $q(z), \theta(w)$ and $\phi(w)$ satisfy the conditions of Lemma 3. Since

$$Q(z) = zq'(z) \phi(q(z)) = \frac{2p(1 - \gamma)z}{1 - z}$$

(2.15)

is univalent starlike in $U$ and

$$h(z) = \theta(q(z)) + Q(z) = \frac{p + p(1 - 2\gamma)z}{1 - z},$$

(2.16)

it may be readily checked that the conditions (1) and (2) of Lemma 3 are satisfied. Thus the result follows from (2.13) immediately.

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**References**


