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Kyoto University
PROPERTIES OF CERTAIN INTEGRAL OPERATOR

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Abstract

Let $A(p)$ denote the class of functions $f(z)$ which are analytic and $p$-valent in the unit disk $U$. A new subclass $\Omega(\alpha, \beta; \gamma)$ of $A(p)$ consisting of analytic and $p$-valent functions $f(z)$ associated with the certain integral operator $Q_\beta^\alpha$ which is the generalization of the integral operator studied by I.B.Jung, Y.C.Kim and H.M.Srivastava (J. Math. Anal. Appl. 248(2000), 475-481) is introduced. Some interesting properties of the operator $Q_\beta^\alpha$ for functions $f(z)$ belonging to $A(p)$ are investigated.

Key Words and phrases: Integral operator, extreme point, multivalent.

2000 Mathematics Subject Classification: Primary 30C45.
1. Introduction.

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^\infty a_{p+n}z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots \})$$

which are analytic and $p$-valent in the unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $S_p^*(\gamma)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the condition

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > p\gamma$$

for $0 \leq \gamma < 1$ and $z \in U$. A function in $S_p^*(\gamma)$ is called $p$-valent starlike of order $\gamma$ in $U$.

Recently, Jung, Kim and Srivastava [3] introduced the following integral operator:

$$Q_{\beta}^\alpha f(z) = \left( \begin{array}{c} \alpha + \beta \\ \beta \end{array} \right) \frac{\alpha}{z^\beta} \int_0^z (1 - \frac{t}{z})^{\alpha-1} t^{\beta-1} f(t) dt$$

$$= \sum_{n=2}^\infty \frac{\Gamma(\beta + n)\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + \alpha + n)\Gamma(\beta + 1)} a_n z^n.$$ (1.2)

They also showed that

$$Q_{\beta}^\alpha f(z) = z^p + \sum_{n=1}^\infty \frac{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)}{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)} a_{p+n}z^{p+n}.$$ (1.3)

It follows from (1.3) that one can define the operator $Q_{\beta}^\alpha$ for $\alpha \geq 0$ and $\beta > -1$. Some interesting subclasses of analytic function, associated with the operator $Q_{\beta}^\alpha$, have been considered recently by Jung et al.[3], Aouf et al.[1], Li[5], Liu[6] and others.

Motivated by Jung, Kim and Srivastava’s work [3], we now consider a linear operator $Q_{\beta}^\alpha : A(p) \to A(p)$ as following:

$$Q_{\beta}^\alpha f(z) = \left( \begin{array}{c} p + \alpha + \beta - 1 \\ p + \beta - 1 \end{array} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt$$

$$= \sum_{n=1}^\infty \frac{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)}{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)} a_{p+n}z^{p+n}.$$ (1.3)

We note that
(\alpha \geq 0, \beta > -1; f \in A(p)). \tag{1.4}

It is easily verified from the definition (1.4) that

\[ z(Q_\beta^\alpha f(z))' = (\alpha + \beta + p - 1)Q_\beta^{\alpha-1}f(z) - (\alpha + \beta - 1)Q_\beta^\alpha f(z). \tag{1.5} \]

When \( p = 1 \), the identity (1.5) is given in [3]. One can easily see that the operator \( Q_\beta^\alpha \) has an inverse operator \( Q_{\beta+\alpha}^{-\alpha} \) and \( Q_\beta^0 \) is an unit operator.

A function \( f(z) \in A(p) \) is said to be in the class \( \Omega(\alpha, \beta; \gamma) \) if it satisfies the condition

\[ \frac{z(Q_\beta^\alpha f(z))}{Q_\beta^\alpha f(z)} + \frac{pz^p}{1-z^p} < \frac{p+p(1-2\gamma)z}{1-z} \tag{1.6} \]

for all \( z \in U \) and \( 0 \leq \gamma < 1 \).

In this paper, we shall show the extreme points of the closed convex hull of the class \( \Omega(\alpha, \beta; \gamma) \). It is then used to determine the coefficient bounds.

In the sequel, we denote the closed convex hull of a class \( H \) by \( coH \). Also, let \( E(coH) \) denote the set of all extreme points of \( H \).

2. Main Results.

In order to derive our main results, we shall need the following lemmas.

**Lemma 1** ([4]). \( E(coS_p^*(\alpha)) \) consists of the functions given by

\[ \frac{z^p}{(1-xz)^{2p(1-\gamma)}} = z^p + \sum_{n=1}^{\infty} \frac{(2p-2p\gamma)_n}{n!} x^n z^{p+n} \quad (z \in U), \tag{2.1} \]

where \((a)_n = a(a+1) \cdots (a+n-1), x \in C \) and \(|x| = 1\).

**Lemma 2** ([9]). The function \((1-z)^\rho = e^{\rho \log(1-z)}, \rho \neq 0\), is univalent in \( U \) if and only if \( \rho \) is either in the closed disk \(|\rho - 1| \leq 1\) or in the closed disk \(|\rho + 1| \leq 1\).

**Lemma 3** ([7]). Let \( q(z) \) be univalent in \( U \) and let \( \theta(w) \) and \( \phi(w) \) be analytic in a domain \( D \) containing \( q(U) \) with \( \phi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = zq'(z)\phi(q(z)), h(z) = \theta(q(z)) + Q(z) \) and suppose that

(1) \( Q(z) \) is starlike (univalent) in \( U \);

(2) \( \text{Re} \left\{ \frac{zq'(z)}{Q(z)} \right\} = \text{Re} \left\{ \frac{\theta'(q(z)) + zQ'(z)}{Q(z)} \right\} \geq 0 \quad (z \in U). \)

If \( p(z) \) is analytic in \( U \), with \( p(0) = q(0), p(U) \subset D \) and

\[ \theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{2.2} \]

then \( p(z) < q(z) \) and \( q(z) \) is the best dominant.
**Theorem 1.** A function $f(z) \in A(p)$ is in $\Omega(\alpha, \beta; \gamma)$ if and only if $f(z)$ can be expressed as

$$f(z) = Q_{\beta + \alpha}^{-\alpha} \left\{ z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \right\},$$  \hfill (2.3)

where $\mu$ is a probability measure defined on the unit circle $X = \{x : |x| = 1\}$.

**Proof.** Let $f(z) \in \Omega(\alpha, \beta; \gamma)$. Then by Herglotz formula [2], we have

$$\frac{z(Q_{\beta}^{\alpha} f(z))'}{Q_{\beta}^{\alpha} f(z)} + \frac{pz^p}{1 - z^p} = p(1 - \gamma) \int_X \frac{1 + xz}{1 - xz} d\mu(x) + p\gamma,$$  \hfill (2.4)

where $\mu$ is a probability measure defined on the unit circle $X = \{x : |x| = 1\}$. By means of the identity

$$\frac{d}{dz} \log \frac{Q_{\beta}^{\alpha} f(z)}{z^p (1 - z^p)} = \frac{1}{z} \left[ \frac{z(Q_{\beta}^{\alpha} f(z))'}{Q_{\beta}^{\alpha} f(z)} + \frac{pz^p}{1 - z^p} - p \right],$$  \hfill (2.5)

(2.4) yields

$$Q_{\beta}^{\alpha} f(z) = z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)].$$  \hfill (2.6)

Thus

$$f(z) = Q_{\beta + \alpha}^{-\alpha} \left\{ z^p (1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \right\}.$$  \hfill (2.7)

Now the proof is complete.

**Theorem 2.** Let $0 \leq \gamma_1 < \gamma_2 < 1$, then $\Omega(\alpha, \beta; \gamma_2) \subset \Omega(\alpha, \beta; \gamma_1)$.

**Proof.** We define a linear operator on $\Omega(\alpha, \beta; \gamma)$ as following:

$$T_{\gamma}(f) = \frac{Q_{\beta}^{\alpha} f(z)}{1 - z^p} \quad (z \in U).$$  \hfill (2.7)

Then $T_{\gamma}$ is a linear homeomorphism from $\Omega(\alpha, \beta; \gamma)$ to $S_p^*(\gamma)$. It is well-known that $S_p^*(\gamma_2) \subset S_p^*(\gamma_1)$ for $0 \leq \gamma_1 < \gamma_2 < 1$. The result follows immediately.

**Theorem 3.** (i) The extreme points of $co\Omega(\alpha, \beta; \gamma)$ are given by the functions

$$f_x(z) = Q_{\beta + \alpha}^{-\alpha} \left\{ \frac{z^p(1 - z^p)}{(1 - xz)^{2p(1 - \gamma)}} \right\}$$  \hfill (2.8)

$$\quad (x \in C, |x| = 1; z \in U).$$
\[(i) \quad \Omega(\alpha, \beta; \gamma) = \{ f : f(z) = \int_X f_x(z) d\mu(x) \}, \tag{2.9} \]

where \( \mu \) varies over the probability measures defined on the unit circle \( X \).

Proof. Since \( T_\gamma \) defined by (2.7) is a linear homeomorphism from \( \Omega(\alpha, \beta; \gamma) \) to \( S_p^*(\gamma) \), it preserves extreme points. By making use of Lemma 1, the results follow at once. According to Theorem 3 and Lemma 1, we have the following corollaries.

**Corollary 1.** Let \( f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \Omega(\alpha, \beta; \gamma) \). Then

\[
|a_{p+n}| \leq \left\{ \begin{array}{ll}
(2p - 2p\gamma) \frac{\prod_{k=1}^{n} (2p - 2p\gamma + n - k) - \prod_{k=1}^{n} (n - p + k)}{n!} \frac{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)}{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)}, & 1 \leq n < p, \\
(2p - 2p\gamma) \frac{\prod_{k=1}^{n} (2p - 2p\gamma + n - k)}{n!} \frac{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)}{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)}, & n \geq p.
\end{array} \right.
\]

The result is sharp.

**Corollary 2.** Let \( f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \Omega(\alpha, \beta; \gamma) \). Then for \( |z| = r < 1 \).

\[
|f(z)| \leq r^p + \sum_{n=1}^{p-1} \frac{(2p - 2p\gamma) \prod_{k=1}^{p} (2p - 2p\gamma + n - k) - \prod_{k=1}^{p} (n - p + k)}{n!} \frac{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)}{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)} r^{p+n} + \sum_{n=p}^{\infty} \frac{(2p - 2p\gamma) \prod_{k=1}^{n} (2p - 2p\gamma + n - k - p) - \prod_{k=1}^{n} (n - p + k)}{n!} \frac{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)}{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)} r^{p+n}.
\]

The result is sharp.

**Theorem 4.** Let \( f(z) \in \Omega(\alpha, \beta; \gamma) \). Let \( \rho \) be a complex number with \( \rho \neq 0 \) and satisfy either \( |2p\rho(1 - \gamma) + 1| \leq 1 \) or \( |2p\rho(1 - \gamma) - 1| \leq 1 \). Then

\[
\left( \frac{Q^\rho f(z)}{z^p(1 - z^p)} \right)^\rho < \frac{1}{(1 - z)^{2p\rho(1 - \gamma)}} = q(z) \quad (z \in U), \tag{2.10}
\]

where \( q(z) \) is the best dominant.

Proof. Let

\[
p(z) = \left( \frac{Q^\rho f(z)}{z^p(1 - z^p)} \right)^\rho, \tag{2.11}
\]

then \( p(z) \) in analytic is \( U \) with \( p(0) = 1 \). Differentiating (2.11) logarithmically we have

\[
\frac{zp'(z)}{p(z)} = \rho \left( \frac{z(Q^\rho f(z))'}{Q^\rho f(z)} + \frac{pz^p}{1 - z^p} - p \right). \tag{2.12}
\]
Since $f(z) \in \Omega(\alpha, \beta; \gamma)$, (2.12) is equivalent to
\[
p + \frac{zp'(z)}{\rho p(z)} < \frac{p + p(1 - 2\gamma)z}{1 - z} = h(z).
\] (2.13)

If we take
\[
q(z) = \frac{1}{(1 - z)^{2p\rho(1-\gamma)}}, \theta(w) = p \quad \text{and} \quad \phi(w) = \frac{1}{\rho w},
\] (2.14)

then $q(z)$ is univalent by the condition of the theorem and Lemma 2. It is easy to show that $q(z), \theta(w)$ and $\phi(w)$ satisfy the conditions of Lemma 3. Since
\[
Q(z) = zq'(z)\phi(q(z)) = \frac{2p(1 - \gamma)z}{1 - z}
\] (2.15)
is univalent starlike in $U$ and
\[
h(z) = \theta(q(z)) + Q(z) = \frac{p + p(1 - 2\gamma)z}{1 - z},
\] (2.16)

it may be readily checked that the conditions (1) and (2) of Lemma 3 are satisfied. Thus the result follows from (2.13) immediately.

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**References**


