Title: PROPERTIES OF CERTAIN INTEGRAL OPERATOR

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PROPERTIES OF CERTAIN INTEGRAL OPERATOR

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Abstract

Let $A(p)$ denote the class of functions $f(z)$ which are analytic and $p$-valent in the unit disk $U$. A new subclass $\Omega(\alpha, \beta; \gamma)$ of $A(p)$ consisting of analytic and $p$-valent functions $f(z)$ associated with the certain integral operator $Q_\beta^\alpha$ which is the generalization of the integral operator studied by I.B.Jung, Y.C.Kim and H.M.Srivastava (J. Math. Anal. Appl. 248(2000), 475 - 481) is introduced. Some interesting properties of the operator $Q_\beta^\alpha$ for functions $f(z)$ belonging to $A(p)$ are investigated.

Key Words and phrases: Integral operator, extreme point, multivalent.

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1. Introduction.

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad (p \in N = \{1, 2, 3, \cdots \})$$ (1.1)

which are analytic and $p$-valent in the unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. Let $S_p^*(\gamma)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the condition

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > p\gamma$$

for $0 \leq \gamma < 1$ and $z \in U$. A function in $S_p^*(\gamma)$ is called $p$-valent starlike of order $\gamma$ in $U$.

Let $f(z)$ and $g(z)$ be analytic in $U$. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in $U$ such that $|w(z)| < 1 (z \in U)$ and $g(z) = f(w(z))$. For this relation the symbol $g(z) \prec f(z)$ is used. In case $f(z)$ is univalent in $U$ we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

Recently, Jung, Kim and Srivastava [3] introduced the following integral operator:

$$Q_{\alpha}^{\beta}f(z) = \left( \begin{array}{c} \alpha + \beta \\ \beta \end{array} \right) \frac{\alpha}{z^\beta} \int_{0}^{z} (1 - \frac{t}{z})^{\alpha-1} t^{\beta-1} f(t) dt$$

(\alpha > 0, \beta > -1; f \in A(1)). \quad (1.2)

They also showed that

$$Q_{\alpha}^{\beta}f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta + n)\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + \alpha + n)\Gamma(\beta + 1)} a_n z^n.$$

It follows from (1.3) that one can define the operator $Q_{\alpha}^{\beta}$ for $\alpha \geq 0$ and $\beta > -1$. Some interesting subclasses of analytic function, associated with the operator $Q_{\alpha}^{\beta}$, have been considered recently by Jung et al.[3], Aouf et al.[1], Li[5], Liu[6] and others.

Motivated by Jung, Kim and Srivastava's work [3], we now consider a linear operator $Q_{\alpha}^{\beta} : A(p) \rightarrow A(p)$ as following:

$$Q_{\alpha}^{\beta}f(z) = \left( \begin{array}{c} p + \alpha + \beta - 1 \\ p + \beta - 1 \end{array} \right) \frac{\alpha}{z^{p+\beta-1}} \int_{0}^{z} \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt$$

(\alpha \geq 0, \beta > -1; f \in A(p)). \quad (1.3)

We note that

$$Q_{\alpha}^{\beta}f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p + n + \beta)\Gamma(p + \alpha + \beta)}{\Gamma(p + n + \alpha + \beta)\Gamma(p + \beta)} a_{p+n}z^{p+n}$$
It is easily verified from the definition (1.4) that

$$z(Q_{\beta}^{\alpha}f(z))' = (\alpha + \beta + p - 1)Q_{\beta}^{\alpha-1}f(z) - (\alpha + \beta - 1)Q_{\beta}^{\alpha}f(z).$$

(1.5)

When $p = 1$, the identity (1.5) is given in [3]. One can easily see that the operator $Q_{\beta}^{\alpha}$ has an inverse operator $Q_{\beta+\alpha}^{-\alpha}$ and $Q_{\beta}^{0}$ is an unit operator.

A function $f(z) \in A(p)$ is said to be in the class $\Omega(\alpha, \beta; \gamma)$ if it satisfies the condition

$$\frac{z(Q_{\beta}^{\alpha}f(z))}{Q_{\beta}^{\alpha}f(z)} + \frac{pz^{p}}{1-z^{p}} \prec \frac{p+p(1-2\gamma)z}{1-z},$$

(1.6)

for all $z \in U$ and $0 \leq \gamma < 1$.

In this paper, we shall show the extreme points of the closed convex hull of the class $\Omega(\alpha, \beta; \gamma)$. It is then used to determine the coefficient bounds.

In the sequel, we denote the closed convex hull of a class $H$ by $coH$. Also, let $E(coH)$ denote the set of all extreme points of $H$.

2. Main Results.

In order to derive our main results, we shall need the following lemmas.

**Lemma 1** ([4]). $E(coS_{p}^{\alpha}(\alpha))$ consists of the functions given by

$$\frac{z^{p}}{(1-xz)^{2p(1-\gamma)}} = z^{p} + \sum_{n=1}^{\infty} \frac{(2p-2p\gamma)_{n}}{n!} x^{n} z^{p+n} (z \in U),$$

(2.1)

where $(a)_{n} = a(a+1) \cdots (a+n-1), x \in C$ and $|x| = 1$.

**Lemma 2** ([9]). The function $(1-z)^{\rho} \equiv e^{\rho \log(1-z)}$, $\rho \neq 0$, is univalent in $U$ if and only if $\rho$ is either in the closed disk $|\rho-1| \leq 1$ or in the closed disk $|\rho+1| \leq 1$.

**Lemma 3** ([7]). Let $q(z)$ be univalent in $U$ and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z)), h(z) = \theta(q(z)) + Q(z)$ and suppose that

(1) $Q(z)$ is starlike (univalent) in $U$;

(2) $Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} = Re \left\{ \frac{\theta'(q(z)) + zQ'(z)}{\phi(q(z)) + zQ(z)} \right\} > 0 \quad (z \in U)$.

If $p(z)$ is analytic in $U$, with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

(2.2)

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.
Theorem 1. A function \( f(z) \in A(p) \) is in \( \Omega(\alpha, \beta; \gamma) \) if and only if \( f(z) \) can be expressed as

\[
f(z) = Q_{\beta + \alpha}^{-\alpha} \left\{ z^p(1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \right\},
\]

(2.3)

where \( \mu \) is a probability measure defined on the unit circle \( X = \{ x : |x| = 1 \} \).

Proof. Let \( f(z) \in \Omega(\alpha, \beta; \gamma) \). Then by Herglotz formula [2], we have

\[
\frac{z(Q_{\beta}^\alpha f(z))'}{Q_{\beta}^\alpha f(z)} + \frac{pz^p}{1 - z^p} = p(1 - \gamma) \int_X \frac{1 + xz}{1 - xz} d\mu(x) + p\gamma,
\]

(2.4)

where \( \mu \) is a probability measure defined on the unit circle \( X = \{ x : |x| = 1 \} \). By means of the identity

\[
\frac{d}{dz} \log \frac{Q_{\beta}^\alpha f(z)}{z^p(1 - z^p)} = \frac{1}{z} \left[ \frac{z(Q_{\beta}^\alpha f(z))'}{Q_{\beta}^\alpha f(z)} + \frac{pz^p}{1 - z^p} - p \right],
\]

(2.5)

(2.4) yields

\[
Q_{\beta}^\alpha f(z) = z^p(1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)].
\]

(2.6)

Thus

\[
f(z) = Q_{\beta + \alpha}^{-\alpha} \left\{ z^p(1 - z^p) \exp[-2p(1 - \gamma) \int_X \log(1 - xz) d\mu(x)] \right\}.
\]

Now the proof is complete.

Theorem 2. Let \( 0 \leq \gamma_1 < \gamma_2 < 1 \), then \( \Omega(\alpha, \beta; \gamma_2) \subset \Omega(\alpha, \beta; \gamma_1) \).

Proof. We define a linear operator on \( \Omega(\alpha, \beta; \gamma) \) as following:

\[
T_\gamma(f) = \frac{Q_{\beta}^\alpha f(z)}{1 - z^p} \quad (z \in U).
\]

(2.7)

Then \( T_\gamma \) is a linear homeomorphism from \( \Omega(\alpha, \beta; \gamma) \) to \( S_{p}^*(\gamma) \). It is well-known that \( S_{p}^*(\gamma_2) \subset S_{p}^*(\gamma_1) \) for \( 0 \leq \gamma_1 < \gamma_2 < 1 \). The result follows immediately.

Theorem 3. (i) The extreme points of \( \text{co} \Omega(\alpha, \beta; \gamma) \) are given by the functions

\[
f_z(z) = Q_{\beta + \alpha}^{-\alpha} \left\{ \frac{z^p(1 - z^p)}{(1 - xz)^{2p(1 - \gamma)}} \right\}
\]

(2.8)
(ii) \( Co \Omega(\alpha, \beta; \gamma) = \{ f : f(z) = \int_X f_x(z) d\mu(x) \} \), \hspace{1cm} (2.9)

where \( \mu \) varies over the probability measures defined on the unit circle \( X \).

Proof. Since \( T_\gamma \) defined by (2.7) is a linear homeomorphism from \( \Omega(\alpha, \beta; \gamma) \) to \( S^*_p(\gamma) \), it preserves extreme points. By making use of Lemma 1, the results follow at once.

According to Theorem 3 and Lemma 1, we have the following corollaries.

**Corollary 1.** Let \( f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \in \Omega(\alpha, \beta; \gamma) \). Then

\[
|a_{p+n}| \leq \left\{ \begin{array}{ll}
\frac{(2p-2p\gamma)_n}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & 1 \leq n < p, \\
\frac{(2p-2p\gamma)_{n-p}}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & n \geq p.
\end{array} \right.
\]

The result is sharp.

**Corollary 2.** Let \( f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \in \Omega(\alpha, \beta; \gamma) \). Then for \( |z| = r < 1 \).

\[
|f(z)| \leq r^p + \sum_{n=1}^{p-1} \frac{(2p-2p\gamma)_n}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)} r^{p+n}
+ \sum_{n=p}^{\infty} \frac{(2p-2p\gamma)_{n-p}}{n!} \cdot \frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)} r^{p+n}.
\]

The result is sharp.

**Theorem 4.** Let \( f(z) \in \Omega(\alpha, \beta; \gamma) \). Let \( \rho \) be a complex number with \( \rho \neq 0 \) and satisfy either \( |2p\rho(1-\gamma) + 1| \leq 1 \) or \( |2p\rho(1-\gamma) - 1| \leq 1 \). Then

\[
\left( \frac{Q_\beta^\alpha f(z)}{z^p(1-z^p)} \right)^\rho < \frac{1}{(1-z)^{2p\rho(1-\gamma)}} = q(z) \hspace{0.5cm} (z \in U),
\]

where \( q(z) \) is the best dominant.

Proof. Let

\[
p(z) = \left( \frac{Q_\beta^\alpha f(z)}{z^p(1-z^p)} \right)^\rho,
\]

then \( p(z) \) in analytic is \( U \) with \( p(0) = 1 \). Differentiating (2.11) logarithmically we have

\[
\frac{zp'(z)}{p(z)} = \rho \left( \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} + \frac{pz^p}{1-z^p} - p \right).
\]

(2.12)
Since $f(z) \in \Omega(\alpha, \beta; \gamma)$, (2.12) is equivalent to
\[ p + \frac{zp'(z)}{pp(z)} < \frac{p + p(1 - 2\gamma)z}{1 - z} = h(z). \] (2.13)

If we take
\[ q(z) = \frac{1}{(1 - z)^{2p\rho(1-\gamma)}}, \quad \theta(w) = p \quad \text{and} \quad \phi(w) = \frac{1}{\rho w}, \] (2.14)
then $q(z)$ is univalent by the condition of the theorem and Lemma 2. It is easy to show that $q(z), \theta(w)$ and $\phi(w)$ satisfy the conditions of Lemma 3. Since
\[ Q(z) = zq'(z)\phi(q(z)) = \frac{2p(1 - \gamma)z}{1 - z} \] (2.15)
is univalent starlike in $U$ and
\[ h(z) = \theta(q(z)) + Q(z) = \frac{p + p(1 - 2\gamma)z}{1 - z}, \] (2.16)
it may be readily checked that the conditions (1) and (2) of Lemma 3 are satisfied. Thus the result follows from (2.13) immediately.

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**References**


