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<td>Nunokawa, Mamoru; Owa, Shigeyoshi; Saitoh, Hitoshi; Pascu, Nicolae N.</td>
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Kyoto University
ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS

MAMORU NUNOKAWA, SHIGEYOSHI OWA, HITOSHI SAITO H
AND NICOLAE N. PASCU

ABSTRACT. Let $p(z)$ be analytic in the open unit disk $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. S.S. Miller and P.T. Mocanu (J. Math. Anal. Appl. 276(2002)) have shown some interesting subordination theorems for such functions $p(z)$. The object of the present paper is to discuss some sufficient conditions for arguments of $p(z)$ to be $|\arg p(z)| < \frac{\pi}{2}\rho$ for $z \in \mathbb{U}$.

1. INTRODUCTION

Let $p(z)$ be analytic in the open unit disk $\mathbb{U} = \{z\in \mathbb{C}: |z| < 1\}$ with $p(0) = 1$ and $p'(0) = 0$. For such functions $p(z)$, Miller and Mocanu [3] have shown some interesting subordination theorems.

Theorem A. ([3]) For $\frac{1}{2} < \rho \leq 1$ define the function $q(z)$ by

$$q(z) = q_{\rho}(z) = \left(\frac{1+z}{1-z}\right)^{\rho},$$

and let $t_0 \in (0,1)$ be the unique solution of

$$t^{\rho} \left\{ (1-\rho)t^2 \cos \left( \frac{\pi}{2}\rho \right) + t \sin \left( \frac{\pi}{2}\rho \right) - (1-\rho) \cos \left( \frac{\pi}{2}\rho \right) \right\} + t^2 - 1 = 0.$$

If $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1, p'(0) = 0$ and

$$|\arg(z p'(z) + p(z)^2 + p(z))| < \frac{\pi}{2}(\rho + 1) - \tan^{-1}\left( \frac{t_0}{1 + \rho - (1-\rho)t_0^2} \right),$$

then $p(z) \prec q_{\rho}(z)$, where the symbol $\prec$ means the subordinations.

To discuss our problems for functions $p(z)$, we need the following lemma due to Hal- lenbeck and Ruscheweyh [2] which is the same as one by Fukui and Sakaguchi [1].

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Lemma 1.1. Let $p(z)$ be analytic in $|z| < R$ and $p^{(k)}(0) = 0 (0 \leqq k \leqq n)$. Then if $|p(z)|$ attains its maximum value on the circle $|z| = r < R$ at a point $z_0$, we have

\begin{equation}
\frac{z_0 p'(z_0)}{p(z_0)} \geqq n + 1.
\end{equation}

Applying the above lemma, we derive

Lemma 1.2. Let $p(z)$ be analytic in $U, p(0) = 1, p'(0) = 0$, and let $p(z) \neq 0 (z \in U)$. If there exists a point $z_0 \in U$ such that

\begin{equation}
|\arg p(z)| < \frac{\pi}{2} \alpha \quad (|z| < |z_0|),
\end{equation}

and

\begin{equation}
|\arg p(z_0)| = \frac{\pi}{2} \alpha
\end{equation}

for some $\alpha > 0$, then we have

\begin{equation}
\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k,
\end{equation}

where

\begin{equation}
k \geqq \left( a + \frac{1}{a} \right) \geqq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \alpha
\end{equation}

and

\begin{equation}
k \leqq - \left( a + \frac{1}{a} \right) \leqq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \alpha
\end{equation}

where $p(z_0)^{1/\alpha} = \pm ia$ and $a > 0$.

Proof. We use the same manner which was used by Nunokawa [4] for the proof of the lemma. Let us put

\begin{equation}
q(z) = p(z)^{1/\alpha}.
\end{equation}

Then we see that $\text{Re}q(z) > 0 (|z| < |z_0|), \text{Re}q(z_0) = 0, q(0) = 1$ and $q'(0) = 0$. Defining the function $\phi(z)$ by

\begin{equation}
\phi(z) = \frac{1 - q(z)}{1 + q(z)},
\end{equation}

we have that $\phi(0) = 0, |\phi(z)| < 1 (|z| < |z_0|)$, and $|\phi(z_0)| = 1$. In view of Lemma 1.1, we know that

\begin{equation}
\frac{z_0 \phi'(z_0)}{\phi(z_0)} = \frac{-2z_0 q'(z_0)}{1 - q(z_0)^2}
\end{equation}
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\[ \frac{-2z_0q'(z_0)}{1 + |q(z_0)|^2} \geq 2. \]

It follows from (1.5) that

(1.6) \[ -z_0q'(z_0) \geq (1 + |q(z_0)|^2) \]

and \( z_0q'(z_0) \) is a negative real number. Since \( q(z_0) \) is a non-vanishing pure imaginary number, we can put \( q(z_0) = ia \), where \( a \) is a non-vanishing real number.

We have, for \( a > 0 \),

(1.7) \[ \text{Im}\left( \frac{z_0q'(z_0)}{q(z_0)} \right) = \text{Im}\left( -\frac{iz_0q'(z_0)}{|q(z_0)|} \right) \geq \left( \frac{1 + a^2}{a} \right) \geq 2 \]

and, for \( a < 0 \),

(1.8) \[ \text{Im}\left( \frac{z_0q'(z_0)}{q(z_0)} \right) = \text{Im}\left( \frac{iz_0q'(z_0)}{|q(z_0)|} \right) \leq -\left( \frac{1 + a^2}{a} \right) \leq -2 \]

On the other hand, it follows that

(1.9) \[ \frac{z_0q'(z_0)}{q(z_0)} = \frac{1}{\alpha} \left( \frac{z_0p'(z_0)}{p(z_0)} \right). \]

This completes the proof of Lemma 1.2. \( \square \)

2. ARGUMENT ESTIMATES

Our first property for argument estimates of analytic function \( p(z) \) is contained in

**Theorem 2.1.** Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \) and \( p'(0) = 0 \). If \( p(z) \) satisfies

(2.1) \[ |\arg (zp'(z) + p(z)^2 + \alpha p(z))| < \pi \rho \quad (z \in U) \]

for some \( \alpha (\alpha > 0) \), \( \rho (0 < \rho \leq \rho_0) \), where \( \rho_0 (0 < \rho_0 < 1) \) is given by

\[ \tan\left( \frac{\pi}{2} \rho_0 \right) = \frac{2}{\alpha} \rho_0, \]

then

(2.2) \[ |\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in U). \]

**Proof.** Let a function \( p(z) \) satisfy the conditions of the theorem. If there exists a point \( z_0 \in U \) such that

\[ |\arg p(z)| < \frac{\pi}{2} \rho \quad (|z| < |z_0|) \]

and

\[ |\arg p(z_0)| = \frac{\pi}{2} \rho, \]

then applying Lemma 1.2, we have that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i\rho k,
\]

where

\[ k \geq a + \frac{1}{a} \geq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \rho \]

and

\[ k \leq - \left( a + \frac{1}{a} \right) \leq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \rho \]

with \( p(z_0)^{1/\rho} = \pm ia \quad (a > 0) \). It follows that, for \( \arg p(z_0) = \frac{\pi}{2} \rho \) and \( k \geq a + \frac{1}{a} \geq 2 \),

\[
\arg (z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)) = \arg p(z_0) \left( \frac{z_0 p'(z_0)}{p(z_0)} + p(z_0) + \alpha \right)
\]

\[ = \frac{\pi}{2} \rho + \arg (i\rho k + a^\rho e^{i\frac{\pi}{2} \rho} + \alpha) = \frac{\pi}{2} \rho + \tan^{-1} \left( \frac{\rho k + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) \]

Since, by \( 0 < \rho \leq \rho_0 < 1 \) and \( k \geq 2 \),

\[
\tan^{-1} \left( \frac{\rho k + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) \geq \tan^{-1} \left( \frac{2\rho + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) > 0,
\]

we define \( g(a) \) by

\[
g(a) = \frac{2\rho + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \quad (a > 0).\]

Noting that

\[
g'(a) = \frac{a^\rho a^\rho \cos \left( \frac{\pi}{2} \rho \right) (\tan \left( \frac{\pi}{2} \rho \right) - \frac{2k}{a})}{(\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right))^2},
\]

we define \( h(\rho) \) by

\[
h(\rho) = \tan \left( \frac{\pi}{2} \rho \right) - \frac{2\rho}{\alpha} \quad (0 < \rho \leq \rho_0 < 1).\]

Then \( h(0) = 0, h(\rho_0) = 0, \) and

\[
h''(\rho) = \frac{\pi^2}{2} \sec^2 \left( \frac{\pi}{2} \rho \right) \tan \left( \frac{\pi}{2} \rho \right) > 0.
\]

This shows that \( g'(a) \leq 0 \) for \( a > 0 \), that is, that

\[
\tan^{-1} \left( \frac{\rho k + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) \geq \tan^{-1} \left( \tan \left( \frac{\pi}{2} \rho \right) \right) = \frac{\pi}{2} \rho.
\]
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Therefore, we conclude that

\begin{equation}
\text{arg} \left( z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0) \right) \geq \pi \rho \tag{2.11}
\end{equation}

when \( \text{arg} p(z_0) = \pi \rho \).

Similarly, for \( \text{arg} p(z_0) = -\frac{\pi}{2} \rho \) and \( k \leq -(a + \frac{1}{a}) \leq -2 \), we have that

\begin{equation}
\text{arg} \left( z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0) \right) = -\frac{\pi}{2} \rho + \text{arg} \left( i \rho k + a^\rho e^{-i \frac{\pi}{2} \rho} + \alpha \right)
\end{equation}

\begin{align*}
\leq & -\frac{\pi}{2} \rho - \text{Tan}^{-1} \left( \frac{-2 \rho - a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) \\
\leq & -\frac{\pi}{2} \rho - \text{Tan}^{-1} \left( \frac{2 \rho + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) \\
\end{align*}

Thus, for such a point \( z_0 \in \mathbb{U} \). we see that

\begin{equation}
\text{arg} \left( z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0) \right) \geq \pi \rho \tag{2.13}
\end{equation}

which contradicts our condition for \( p(z) \).

Consequently, we conclude that

\[ \text{arg} p(z) < \frac{\pi}{2} \rho \quad (z \in \mathbb{U}). \]

\[ \square \]

**Example 2.1.** Let us consider the function \( p(z) \) defined by

\[ p(z) = 1 + \frac{1}{5} z^2. \]

Then we see that

\[ z p'(z) + p(z)^2 + \frac{1}{2} p(z) = \frac{3}{2} + \frac{9}{10} z^2 + \frac{1}{25} z^4. \]

Letting \( \alpha = \frac{1}{2} \) and

\[ \rho = \frac{1}{\pi} \sin^{-1} \left( \frac{19}{30} \right) \]

in Theorem 2.1, we have that

\[ \left| \text{arg} \left( z p'(z) + p(z)^2 + \frac{1}{2} p(z) \right) \right| < \pi \rho = \sin^{-1} \left( \frac{19}{30} \right) \]
and
\[|\arg p(z)| < \sin^{-1}\left(\frac{1}{5}\right) < \frac{\pi}{2}\rho.\]

If we take \(\alpha = 1\) in Theorem 2.1, then

**Corollary 2.1.** Let \(p(z)\) be analytic in \(U\) with \(p(0) = 1\) and \(p'(0) = 0\). If \(p(z)\) satisfies
\[(2.14) \quad |\arg(zp'(z) + p(z)^2 + p(z))| < \pi\rho \quad (z \in U)\]
for some \(\rho \left(0 < \rho \leq \frac{1}{2}\right)\), then
\[(2.15) \quad |\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in U).\]

**Remark 2.1.**
1. If \(\alpha = \frac{4}{5}\), then \(0 < \rho \leq \rho_0\) and \(0.647873 < \rho_0 < 0.647874\).
2. If \(\alpha = \frac{1}{2}\), then \(0 < \rho \leq \rho_0\) and \(0.809251 < \rho_0 < 0.809252\).
3. If \(\alpha = \frac{1}{3}\), then \(0 < \rho \leq \rho_0\) and \(0.880666 < \rho_0 < 0.88067\).
4. If \(\alpha = \frac{1}{4}\), then \(0 < \rho \leq \rho_0\) and \(0.913417 < \rho_0 < 0.913418\).
5. If \(\alpha = 1.1\), then \(0 < \rho \leq \rho_0\) and \(0.401247 < \rho_0 < 0.491248\).
6. If \(\alpha = 1.2\), then \(0 < \rho \leq \rho_0\) and \(0.262943 < \rho_0 < 0.262944\).
7. If \(\alpha = 1.3\), then there is no \(\rho_0 > 0\) such that \(\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho\). Thus we see that \(0 < \alpha < 1.3\) in Theorem 2.1.

Next, we derive

**Theorem 2.2.** Let \(p(z)\) be analytic in \(U\) with \(p(0) = 1\) and \(p'(0) = 0\). If \(p(z)\) satisfies
\[(2.16) \quad |\arg(zp'(z) + p(z)^2 + \alpha p(z))| < \frac{\pi}{2}\rho + \tan^{-1}\left(\frac{2\rho}{\alpha}\right) \quad (z \in U)\]
for some \(\alpha > 0\), \(\rho \left(0 < \rho \leq \rho_0 < 1\right)\), where \(\rho_0\) \((0 < \rho_0 < 1)\) is given by \(\tan\left(\frac{\pi}{2}\rho_0\right) = \frac{2}{\alpha}\rho_0\), then
\[(1.7) \quad |\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in U).\]

**Proof.** Using the same technique as in the proof of Theorem 2.1, we know that
\[
\tan^{-1}\left(\frac{2\rho + a^\rho\sin\left(\frac{\pi}{2}\rho\right)}{a^\rho\cos\left(\frac{\pi}{2}\rho\right)}\right)
\]
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is increasing for $a > 0$. Thus, we obtain

\[(2.18) \quad |\arg (z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0))| \geq \frac{\pi}{2} \rho + \tan^{-1} \left( \frac{2\rho}{\alpha} \right)\]

for $z_0 \in U$ such that

\[|\arg p(z)| < \frac{\pi}{2} \rho \quad (|z| < |z_0|)\]

and

\[|\arg p(z_0)| = \frac{\pi}{2} \rho.\]

This contradicts our condition of the theorem. Therefore,

\[|\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in U).\]

\[\square\]

Letting $\alpha = 1$ in Theorem 2.2, we obtain

**Corollary 2.2.** Let $p(z)$ be analytic in $U$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies

\[(2.19) \quad |\arg (zp'(z) + p(z)^2 + p(z))| < \frac{\pi}{2} \rho + \tan^{-1}(2\rho) \quad (z \in U)\]

for some $\rho \left( \frac{1}{2} \leq \rho < 1 \right)$, then

\[(2.20) \quad |\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in U).\]

Finally, we note that

**Remark 2.2.**

1. If $\alpha = \frac{4}{5}$, then $0 < \rho \leq \rho_0$ and $0.647873 < \rho_0 < 0.647874$.

2. If $\alpha = \frac{1}{2}$, then $0 < \rho \leq \rho_0$ and $0.809251 < \rho_0 < 0.809252$.

3. If $\alpha = \frac{1}{3}$, then $0 < \rho \leq \rho_0$ and $0.880966 < \rho_0 < 0.880967$.

4. If $\alpha = \frac{1}{4}$, then $0 < \rho \leq \rho_0$ and $0.913417 < \rho_0 < 0.913418$.

5. If $\alpha = 1.1$, then $0 < \rho \leq \rho_0$ and $0.401247 < \rho_0 < 0.491248$.

6. If $\alpha = 1.2$, then $0 < \rho \leq \rho_0$ and $0.262943 < \rho_0 < 0.262944$.

7. If $\alpha = 1.3$, then there is no $\rho_0 > 0$ such that $\tan \left( \frac{\pi}{2} \rho_0 \right) = \frac{2}{\alpha} \rho$. Thus we see that $0 < \alpha < 1.3$ in Theorem 2.2.
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Mamoru Nunokawa
Emeritus Professor
Department of Mathematics
University of Gunma
Aramaki, Maebashi, Gunma 371-8510
Japan

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan

Hitoshi Saitoh
Department of Mathematics
National Gunma College of Technology
Toriba, Maebashi, Gunma 371-8530
Japan

Nicolae N. Pascu
Department of Mathematics
Transilvania University of Brasov
R-2200 Brasov