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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2003), 1341: 77-84</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2003-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/43473">http://hdl.handle.net/2433/43473</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
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ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. Let \( p(z) \) be analytic in the open unit disk \( \mathbb{D} \) with \( p(0) = 1 \) and \( p'(0) = 0 \). S.S. Miller and P.T. Mocanu (J. Math. Anal. Appl. 276(2002)) have shown some interesting subordination theorems for such functions \( p(z) \). The object of the present paper is to discuss some sufficient conditions for arguments of \( p(z) \) to be \( |\arg p(z)| < \frac{\pi}{2} \rho \) for \( z \in \mathbb{D} \).

1. Introduction

Let \( p(z) \) be analytic in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) with \( p(0) = 1 \) and \( p'(0) = 0 \). For such functions \( p(z) \), Miller and Mocanu [3] have shown some interesting subordination theorems.

Theorem A. ([3]) For \( \frac{1}{2} < \rho \leq 1 \) define the function \( q(z) \) by

\[
q(z) = q_\rho(z) = \left( \frac{1+z}{1-z} \right)^\rho,
\]

and let \( t_0 \in (0,1) \) be the unique solution of

\[
t^\rho \left\{ (1-\rho)t^2 \cos \left( \frac{\pi}{2} \rho \right) + t \sin \left( \frac{\pi}{2} \rho \right) - (1-\rho) \cos \left( \frac{\pi}{2} \rho \right) \right\} + t^2 - 1 = 0.
\]

If \( p(z) \) is analytic in \( \mathbb{D} \), with \( p(0) = 1, p'(0) = 0 \) and

\[
\left| \arg \left( zp'(z) + p(z)^2 + p(z) \right) \right| < \frac{\pi}{2} (\rho + 1) - \tan^{-1} \left( \frac{t_0}{1 + \rho - (1-\rho)t_0^2} \right),
\]

then \( p(z) \prec q_\rho(z) \), where the symbol "\( \prec \)" means the subordinations.

To discuss our problems for functions \( p(z) \), we need the following lemma due to Hal- lenbeck and Ruscheweyh [2] which is the same as one by Fukui and Sakaguchi [1].

2000 Mathematics Subject Classification. Primary 30C45.
Key Words and Phrases. analytic function, argument estimate, subordination.
Lemma 1.1. Let $p(z)$ be analytic in $|z| < R$ and $p^{(k)}(0) = 0 (0 \leq k \leq n)$. Then if $|p(z)|$ attains its maximum value on the circle $|z| = r < R$ at a point $z_0$, we have

\begin{equation}
\frac{z_0 p'(z_0)}{p(z_0)} \geq n + 1.
\end{equation}

Applying the above lemma, we derive

Lemma 1.2. Let $p(z)$ be analytic in $\mathbb{U}$, $p(0) = 1$, $p'(0) = 0$, and let $p(z) \neq 0 (z \in \mathbb{U})$. If there exists a point $z_0 \in \mathbb{U}$ such that

\[ |\arg p(z)| < \frac{\pi}{2} \alpha \] (\[ |z| < |z_0| \])

and

\[ |\arg p(z_0)| = \frac{\pi}{2} \alpha \]

for some $\alpha > 0$, then we have

\begin{equation}
\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k,
\end{equation}

where

\[ k \geq \left( a + \frac{1}{a} \right) \geq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \alpha \]

and

\[ k \leq - \left( a + \frac{1}{a} \right) \leq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \alpha, \]

where $p(z_0)^{1/\alpha} = \pm ia$ and $a > 0$.

Proof. We use the same manner which was used by Nunokawa [4] for the proof of the lemma. Let us put

\begin{equation}
q(z) = p(z)^{1/\alpha}.
\end{equation}

Then we see that $\text{Re} q(z) > 0$ ($|z| < |z_0|$), $\text{Re} q(z_0) = 0$, $q(0) = 1$ and $q'(0) = 0$. Defining the function $\phi(z)$ by

\begin{equation}
\phi(z) = \frac{1 - q(z)}{1 + q(z)},
\end{equation}

we have that $\phi(0) = 0$, $|\phi(z)| < 1$ ($|z| < |z_0|$), and $|\phi(z_0)| = 1$.

In view of Lemma 1.1, we know that

\begin{equation}
\frac{z_0 \phi'(z_0)}{\phi(z_0)} = \frac{-2z_0 q'(z_0)}{1 - q(z_0)^2}
\end{equation}
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\[ \frac{-2z_0 q'(z_0)}{1 + |q(z_0)|^2} \geq 2. \]

It follows from (1.5) that

(1.6) \[ -z_0 q'(z_0) \geq (1 + |q(z_0)|^2) \]

and \( z_0 q'(z_0) \) is a negative real number. Since \( q(z_0) \) is a non-vanishing pure imaginary number, we can put \( q(z_0) = ia \), where \( a \) is a non-vanishing real number.

We have, for \( a > 0 \),

(1.7) \[ \text{Im} \left( \frac{z_0 q'(z_0)}{q(z_0)} \right) = \text{Im} \left( -\frac{iz_0 q'(z_0)}{|q(z_0)|} \right) \geq \left( \frac{1 + a^2}{a} \right) \geq 2 \]

and, for \( a < 0 \),

(1.8) \[ \text{Im} \left( \frac{z_0 q'(z_0)}{q(z_0)} \right) = \text{Im} \left( \frac{iz_0 q'(z_0)}{|q(z_0)|} \right) \leq -\left( \frac{1 + a^2}{a} \right) \leq -2 \]

On the other hand, it follows that

(1.9) \[ \frac{z_0 q'(z_0)}{q(z_0)} = \frac{1}{\alpha} \left( \frac{z_0 p'(z_0)}{p(z_0)} \right). \]

This completes the proof of Lemma 1.2.

\( \square \)

2. ARGUMENT ESTIMATES

Our first property for argument estimates of analytic function \( p(z) \) is contained in

**Theorem 2.1.** Let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \) and \( p'(0) = 0 \). If \( p(z) \) satisfies

(2.1) \[ |\arg (zp'(z) + p(z)^2 + \alpha p(z))| < \pi \rho \quad (z \in U) \]

for some \( \alpha (\alpha > 0), \rho (0 < \rho \leq \rho_0) \), where \( \rho_0 (0 < \rho_0 < 1) \) is given by

\[ \tan \left( \frac{\pi}{2} \rho_0 \right) = \frac{2}{\alpha \rho_0}, \]

then

(2.2) \[ |\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in U). \]

**Proof.** Let a function \( p(z) \) satisfy the conditions of the theorem. If there exists a point \( z_0 \in U \) such that

|argp(z)| < \( \frac{\pi}{2} \rho \) \quad (|z| < |z_0|)

and

|argp(z_0)| = \( \frac{\pi}{2} \rho \),

then applying Lemma 1.2, we have that
\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \rho k,
\]

where

\[
k \geqq a + \frac{1}{a} \geqq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2} \rho
\]

and

\[
k \leqq - a - \frac{1}{a} \leqq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2} \rho
\]

with \( p(z_0)^{1/\rho} = \pm \rho \alpha \). It follows that, for \( \arg p(z_0) = \frac{\pi}{2} \rho \) and \( k \geqq a + \frac{1}{a} \geqq 2 \),

\[
\arg (z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)) = \arg p(z_0) \left( \frac{z_0 p'(z_0)}{p(z_0)} + p(z_0) + \alpha \right)
\]

\[
= \frac{\pi}{2} \rho + \arg (i \rho k + a^\rho e^{i \frac{\pi}{2} \rho} + \alpha) = \frac{\pi}{2} \rho + \tan^{-1} \left( \frac{\rho k + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right)
\]

Since, by \( 0 < \rho \leqq \rho_0 < 1 \) and \( k \geqq 2 \),

\[
\tan^{-1} \left( \frac{\rho k + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) \geqq \tan^{-1} \left( \frac{2 \rho + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) > 0,
\]

we define \( g(a) \) by

\[
g(a) = \frac{2 \rho + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \quad (a > 0).
\]

Noting that

\[
g'(a) = \frac{\alpha \rho a^{\rho-1} \cos \left( \frac{\pi}{2} \rho \right) \left( \tan \left( \frac{\pi}{2} \rho \right) - \frac{2 \rho}{\alpha} \right)}{(\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right))^2},
\]

we define \( h(\rho) \) by

\[
h(\rho) = \tan \left( \frac{\pi}{2} \rho \right) - \frac{2 \rho}{\alpha} \quad (0 < \rho \leqq \rho_0 < 1).
\]

Then \( h(0) = 0, h(\rho_0) = 0 \), and

\[
h''(\rho) = \frac{\pi^2}{2} \tan^2 \left( \frac{\pi}{2} \rho \right) \tan \left( \frac{\pi}{2} \rho \right) > 0.
\]

This shows that \( g'(a) \leqq 0 \) for \( a > 0 \), that is, that

\[
\tan^{-1} \left( \frac{\rho k + a^\rho \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right) \geqq \tan^{-1} \left( \tan \left( \frac{\pi}{2} \rho \right) \right) = \frac{\pi}{2} \rho.
\]
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Therefore, we conclude that

\[(2.11) \quad \arg (z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)) \geq \pi \rho\]

when \(\arg p(z_0) = \frac{\pi}{2} \rho\).

Similarly, for \(\arg p(z_0) = -\frac{\pi}{2} \rho\) and \(k \leq - \left( a + \frac{1}{a} \right) \leq -2\), we have that

\[(2.12) \quad \arg (z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0)) = -\frac{\pi}{2} \rho + \arg (i \rho k + a^\rho e^{-i \frac{\pi}{2} \rho} + \alpha)
\]

\[\leq -\frac{\pi}{2} \rho + \tan^{-1} \left( \frac{-2 \rho - a^\rho \sin \left( \frac{\pi}{2} \rho \right) \alpha}{\alpha + a^\rho \cos \left( \frac{\pi}{2} \rho \right)} \right)\]

\[\leq -\frac{\pi}{2} \rho - \frac{\pi}{2} \rho = -\pi \rho.\]

Thus, for such a point \(z_0 \in \mathbb{U}\). we see that

\[(2.13) \quad |\arg (z_0 p'(z_0) + p(z_0)^2 + \alpha p(z_0))| \geq \pi \rho,\]

which contradicts our condition for \(p(z)\).

Consequently, we conclude that

\[|\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in \mathbb{U}).\]

\[\square\]

**Example 2.1.** Let us consider the function \(p(z)\) defined by

\[p(z) = 1 + \frac{1}{5} z^2.\]

Then we see that

\[zp'(z) + p(z)^2 + \frac{1}{2} p(z) = \frac{3}{2} + \frac{9}{10} z^2 + \frac{1}{25} z^4.\]

Letting \(\alpha = \frac{1}{2}\) and

\[\rho = \frac{1}{\pi} \sin^{-1} \left( \frac{19}{30} \right)\]

in Theorem 2.1, we have that

\[\left| \arg \left( zp'(z) + p(z)^2 + \frac{1}{2} p(z) \right) \right| < \pi \rho = \sin^{-1} \left( \frac{19}{30} \right)\]
and
\[ |\arg p(z)| < \sin^{-1} \left( \frac{1}{5} \right) < \frac{\pi}{2} \rho. \]

If we take \( \alpha = 1 \) in Theorem 2.1, then

**Corollary 2.1.** Let \( p(z) \) be analytic in \( \mathbb{U} \) with \( p(0) = 1 \) and \( p'(0) = 0 \). If \( p(z) \) satisfies
\[ |\arg (zp'(z) + p(z)^2 + p(z))| < \pi \rho \quad (z \in \mathbb{U}) \]
for some \( \rho \left( 0 < \rho \leq \frac{1}{2} \right) \), then
\[ |\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in \mathbb{U}). \]

**Remark 2.1.**
1. If \( \alpha = \frac{4}{5} \), then \( 0 < \rho \leq \rho_0 \) and \( 0.647873 < \rho_0 < 0.647874 \).
2. If \( \alpha = \frac{1}{2} \), then \( 0 < \rho \leq \rho_0 \) and \( 0.809251 < \rho_0 < 0.809252 \).
3. If \( \alpha = \frac{1}{3} \), then \( 0 < \rho \leq \rho_0 \) and \( 0.880966 < \rho_0 < 0.880967 \).
4. If \( \alpha = \frac{1}{4} \), then \( 0 < \rho \leq \rho_0 \) and \( 0.913417 < \rho_0 < 0.913418 \).
5. If \( \alpha = 1.1 \), then \( 0 < \rho \leq \rho_0 \) and \( 0.401247 < \rho_0 < 0.491248 \).
6. If \( \alpha = 1.2 \), then \( 0 < \rho \leq \rho_0 \) and \( 0.262943 < \rho_0 < 0.262944 \).
7. If \( \alpha = 1.3 \), then there is no \( \rho_0 > 0 \) such that \( \tan \left( \frac{\pi}{2} \rho_0 \right) = \frac{2}{\alpha} \rho \). Thus we see that \( 0 < \alpha < 1.3 \) in Theorem 2.1.

Next, we derive

**Theorem 2.2.** Let \( p(z) \) be analytic in \( \mathbb{U} \) with \( p(0) = 1 \) and \( p'(0) = 0 \). If \( p(z) \) satisfies
\[ |\arg (zp'(z) + p(z)^2 + \alpha p(z))| < \frac{\pi}{2} \rho + \tan^{-1} \left( \frac{2\rho}{\alpha} \right) \quad (z \in \mathbb{U}) \]
for some \( \alpha \ (\alpha > 0) \), \( \rho \ (\rho_0 \leq \rho < 1) \), where \( \rho_0 \ (0 < \rho_0 < 1) \) is given by \( \tan \left( \frac{\pi}{2} \rho_0 \right) = \frac{2}{\alpha} \rho_0 \), then
\[ |\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in \mathbb{U}). \]

**Proof.** Using the same technique as in the proof of Theorem 2.1, we know that
\[ \tan^{-1} \left( \frac{2\rho + \alpha \sin \left( \frac{\pi}{2} \rho \right)}{\alpha + \alpha \cos \left( \frac{\pi}{2} \rho \right)} \right). \]
is increasing for $a > 0$. Thus, we obtain

\begin{equation}
|\arg (z_0p'(z_0) + p(z_0)^2 + \alpha p(z_0))| \geq \frac{\pi}{2} \rho + \tan^{-1}\left(\frac{2\rho}{\alpha}\right)
\end{equation}

for $z_0 \in U$ such that

$|\arg p(z)| < \frac{\pi}{2} \rho$ \quad (|z| < |z_0|)

and

$|\arg p(z_0)| = \frac{\pi}{2} \rho$.

This contradicts our condition of the theorem. Therefore,

$|\arg p(z)| < \frac{\pi}{2} \rho$ \quad (z \in U).

\[\square\]

Letting $\alpha = 1$ in Theorem 2.2, we obtain

**Corollary 2.2.** Let $p(z)$ be analytic in $U$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies

\begin{equation}
|\arg (zp'(z) + p(z)^2 + p(z))| < \frac{\pi}{2} \rho + \tan^{-1}(2\rho) \quad (z \in U)
\end{equation}

for some $\rho \left(\frac{1}{2} \leq \rho < 1\right)$, then

\begin{equation}
|\arg p(z)| < \frac{\pi}{2} \rho \quad (z \in U).
\end{equation}

Finally, we note that

**Remark 2.2.**

1. If $\alpha = \frac{4}{5}$, then $0 < \rho \leq \rho_0$ and $0.647873 < \rho_0 < 0.647874$.

2. If $\alpha = \frac{1}{2}$, then $0 < \rho \leq \rho_0$ and $0.809251 < \rho_0 < 0.809252$.

3. If $\alpha = \frac{1}{3}$, then $0 < \rho \leq \rho_0$ and $0.880966 < \rho_0 < 0.880967$.

4. If $\alpha = \frac{1}{4}$, then $0 < \rho \leq \rho_0$ and $0.913417 < \rho_0 < 0.913418$.

5. If $\alpha = 1.1$, then $0 < \rho \leq \rho_0$ and $0.401247 < \rho_0 < 0.491248$.

6. If $\alpha = 1.2$, then $0 < \rho \leq \rho_0$ and $0.262943 < \rho_0 < 0.262944$.

7. If $\alpha = 1.3$, then there is no $\rho_0 > 0$ such that $\tan\left(\frac{\pi}{2} \rho_0\right) = \frac{2}{\alpha} \rho$. Thus we see that $0 < \alpha < 1.3$ in Theorem 2.2.
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